Part F

Advanced Topics in Abstract Model Theory

Abstract model theory is the attempt to systematize the study of logics by studying the relationships between them and between various of their properties. The perspective taken in abstract model theory is discussed in Section 2 of Chapter I. The basic definitions and results of the subject were presented in Part A. Other results are scattered throughout the book. This final part of the book is devoted to more advanced topics in abstract model theory.

Chapter XVII views part of our experience with concrete logics in an abstract light. A concrete logic is presented by describing a class of structures, telling how the formulas are built up, and how formulas are interpreted in structures. Since formulas can be viewed as well-founded trees, they can be represented as set-theoretical objects. Similarly, structures are usually thought of as certain kinds of set-theoretical objects. Thus, we can think of a logic $\mathscr{L}$ as given by two predicates of sets: "x is a sentence of $\mathscr{L}$" and "the structure x satisfies the sentence y of $\mathscr{L}$." Chapter XVII deals with the following general problem: What can we say about the model-theoretic properties of $\mathscr{L}$ if we have information about how these predicates can be defined? Two forms of definitions are considered, implicit (Section 1) and explicit (the rest). The usual style of the inductive definition of truth is of the first kind, with its set-theoretical explanation being of the second kind.

When the inductive clauses for a logic $\mathscr{L}'$ can be written down in a logic $\mathscr{L}$, in a suitable precise sense, one says that $\mathscr{L}$ is *adequate to truth in* $\mathscr{L}'$. This gives a useful "effective" relation between logics which, in certain cases, agrees with the relation $\mathscr{L}' \leq_{\mathrm{RPC}} \mathscr{L}$, though not in general. Of special interest are logics which are adequate to truth in themselves.

On the explicit side, one may consider the complexity of the definition of a logic in terms of the Levy hierarchy of set-theoretic predicates, and in terms of the strength of the meta-theory T needed for the definitions. Particularly significant are the cases where the satisfaction relation for $\mathscr{L}$ is Δ_1 relative to a set theory T, which is the same as its being *absolute* relative to models of T. This insures that the meaning of a sentence is not sensitive to which universe of set theory is being considered. Absoluteness has a number of applications to the characterization of the infinitary logics $\mathscr{L}_{\infty\omega}$, $\mathscr{L}_{\infty G}$, and $\mathscr{L}_{\infty V}$ discussed in Chapters VIII and X. The discussions of the implicit and explicit approaches in this chapter are largely independent.

Chapter XVIII explores the relation between certain compactness, embedding, and definability properties. Refinements and generalizations of compactness are presented and treated at the outset. Analogues of various well-known properties from first-order model theory, such as amalgamation Robinson consistency and Beth definability are introduced and related to the various notions of compactness. Striking results emerge, such as the equivalence under certain conditions of full compactness and an abstract version of amalgamation. Also surprising is the appearance of large cardinals in both hypotheses and conclusions of many of the results in this chapter.

Chapter XIX studies the relationship between abstract equivalence relations on structures and logics. Each logic $\mathscr{L}$ determines an equivalence relation $\equiv_{\mathscr{L}}$ on $\mathscr{L}$-structures, that of being $\mathscr{L}$-equivalent. Isomorphic structures are always $\mathscr{L}$-equivalent. Many properties of $\mathscr{L}$ can be stated in terms of these equivalence relations, but it often happens that two quite different logics can give rise to the same equivalence relation.

The primary emphasis in Chapter XIX is on the relation between the equivalence relations for logics and the Robinson consistency property for logics. In Chapter I we discussed the relationship between the interpolation property and the Robinson consistency property. In Chapter XIX quite general results are obtained in an abstract setting on the relationship between compactness, interpolation and the Robinson property. There is also an extensive abstract treatment of (projective) embedding relations and the amalgamation property. Certain dualities are established between logics, equivalence relations, and embedding relations. The chapter concludes with a general study of back-and-forth systems for equivalence relations.

Chapter XVII

Set-Theoretic Definability of Logics

by J. Väänänen

Simply put, an abstract logic is determined by two predicates of set theory, "$x \in \mathcal{L}$" and "$y \models_{\mathcal{L}} x$." The general problem to be considered in this chapter is as follows: What can we say of the model-theoretic properties of $\mathcal{L}$ if we known how the predicates "$x \in \mathcal{L}$" and "$y \models_{\mathcal{L}} x$" behave as predicates of set theory?

Typical model-theoretic properties that are relevant here are Löwenheim–Skolem-type properties, various interpolation properties, completeness and compactness properties, and the conditions that are related to inductive definability of truth. Typical set-theoretic conditions that can be imposed on "$x \in \mathcal{L}$" and "$y \models_{\mathcal{L}} x$" are various forms of absoluteness. A simple example of the use of set theory in abstract model theory is the following result (see Corollary 2.2.3): *If the predicates* "$y \in \mathcal{L}$" *and* "$x \models_{\mathcal{L}} y$" *are* Σ_1 *in set theory, then every* $\varphi \in \mathcal{L}$ *such that* $\varphi \in \mathrm{HC}$ *and* φ *has a model, has a countable model,* where HC denotes the set of hereditarily countable sets.

An important tool throughout this chapter will be the notion of adequacy to truth, a concept that is due to S. Feferman. This notion provides an analysis of implicit definability of the actual truth-definition of a logic and is, therefore, naturally connected with the explicit set-theoretical definability of "$x \in \mathcal{L}$" and "$y \models_{\mathcal{L}} x$." A study of adequacy to truth is presented in Section 1.

As opposed to the model-theoretic approach taken in Section 1, Section 2 is devoted to set-theoretic criteria. The simplest and best known example in this direction is the notion of absoluteness of a logic, due to J. Barwise. Set theoretic methods have shown themselves to be more fruitful in connection with absolute logics than anywhere else. When we pass to non-absolute logics, the various independence results of set theory blur the picture. The developments in Section 3 establish the exact relationships between model-theoretic and set-theoretic definability of truth. This is, in effect, the main part of the chapter. We will obtain set-theoretical characterizations of logics such as $\mathcal{L}_{\omega\omega}$ and $\mathcal{L}_A$ and characterize definability in the Δ-extensions of various logics.

The results of Section 4 apply the methods of the previous sections and present some new examples of the interplay between model-theoretic and set-theoretic definability. We will conclude the section by making some remarks on possible further work in the area.

1. Model-Theoretic Definability Criteria

The sole purpose of this section is to introduce the notion of adequacy to truth together with its main properties and applications. This notion was first defined by Feferman [1974a] and has its origins in generalized recursion theory. Essentially, it is part of an entire program whose aim is to bring recursion-theoretic notions to bear in abstract model theory.

1.1. Adequacy to Truth

The definitions of most logics, at least of those we would call "syntactic", are given by a recursive definition: For non-atomic φ,

(∗) $\mathfrak{M} \models_{\mathscr{L}} \varphi$ if and only if $\mathfrak{M}$ and the subformulae φ_i $(i \in I)$ of φ have the property ...,

where the property ... is expressed in terms of the sequence of assertions $\mathfrak{M} \models_{\mathscr{L}} \varphi_i$ $(i \in I)$. Although (∗) is usually written in plain English, it may also be formalizable in another logic, a logic which we would then call "adequate to truth" in $\mathscr{L}$. Before we examine the exact definition, we will give careful consideration to a special case.

1.1.1 Preliminary Example. Consider the logic $\mathscr{L}_{\omega\omega}$. Let us think of formulae of $\mathscr{L}_{\omega\omega}$ as elements of HF, where HF denotes the collection of hereditarily finite sets. A set $a \in \mathrm{HF}$ is an $\mathscr{L}_{\omega\omega}$-formula if it has one of the forms

(1) atomic, $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\exists v_n \varphi$, $\forall v_n \varphi$,

where φ and ψ are $\mathscr{L}_{\omega\omega}$-formulae and v_n is a variable symbol. We can write out an $\mathscr{L}_{\omega\omega}$-formula Form($x$) such that

$$\mathrm{HF} \models \mathrm{Form}(a) \quad \text{if and only if} \quad a \text{ is an } \mathscr{L}_{\omega\omega}\text{-formula.}$$

The truth-relation $\models$ of $\mathscr{L}_{\omega\omega}$ is a relation between HF and the model under consideration. Let $\mathfrak{N}$ be the structure $(M^{<\omega}, P)$, where P maps $s \in M^{<\omega}$ and $n \in \omega$ onto the nth element s_n of s. By writing out the usual clauses of the inductive truth definition, we obtain a formula η in $\mathscr{L}_{\omega\omega}$ containing a new binary predicate $S(x, y)$ such that

(2) If $(\mathfrak{M}, \mathrm{HF}, S, \mathfrak{N}) \models \eta$, then for each formula φ with free variables among $x_1, \ldots, x_n$ we have $S(\varphi, s)$ iff $\mathfrak{M} \models \varphi(s_1, \ldots, s_n)$.

Thus we can formalize the truth of $\mathscr{L}_{\omega\omega}$ in $\mathscr{L}_{\omega\omega}$ up to the definition of HF. But now comes the crucial observation: In contrast to $\mathfrak{M}$, all of HF is not needed in (2)—we can replace HF by any set-theoretical structure which is standard as far as subformulas of $\varphi(x_1, \ldots, x_n)$ are concerned.

Let $\mathfrak{B}$ be a set-theoretical structure, $\mathfrak{B} = (B, E)$. Let π_φ be a sentence in the language of $\mathfrak{B}$ which says that $\mathfrak{B}$ contains $\varphi(x_1, \ldots, x_n)$. ($\varphi$ with free variables $\subseteq \{x_1, \ldots, x_n\}$.) That is, inside $\mathfrak{B}$, regarded as a set-theoretical object, $\varphi(x_1, \ldots, x_n)$ has the same set-theoretical structure as it has in the real world. Then, of course, HF $\models \pi_\varphi$. But, moreover, for any $\mathfrak{B}$

(3) *If* $(\mathfrak{M}, \mathfrak{B}, S', \mathfrak{N}) \models \eta \wedge \pi_\varphi$, then $S'(\varphi, s)$ if and only if $\mathfrak{M} \models \varphi(s_1, \ldots, s_n)$.

Let θ be the $\mathscr{L}_{\omega\omega}$-sentence

$$\eta \wedge \forall x(\mathrm{Th}(x) \leftrightarrow \exists s\, S(x, s)),$$

where Th is a new unary predicate symbol. If we merge S into $\mathfrak{N}$, we then have, for any $\varphi \in \mathscr{L}_{\omega\omega}$:

(4) If $(\mathfrak{M}, \mathfrak{B}, T, \mathfrak{N}) \models \theta \wedge \pi_\varphi$, then $\varphi \in T$ if and only if $\mathfrak{M} \models \varphi$.

We thus have an implicit definition of truth of $\mathscr{L}_{\omega\omega}$ inside $\mathscr{L}_{\omega\omega}$ using extra symbols and the infinitary sentences π_φ. This is what adequacy of $\mathscr{L}_{\omega\omega}$ to truth means in itself.

Before proceeding to the definition of adequacy to truth in general, we need some conventions concerning representation of syntax and the definition of the formulas π_φ.

For any set a, let $\mu_a(z)$ be the following (possibly) infinitary formula in the vocabulary $\tau_{\mathrm{set}} = \{\in\}$:

$$\mu_a(x) = \forall y(y \in x \leftrightarrow \bigvee_{b \in a} \mu_b(y)).$$

This recursive definition has the intuitive content $\mu_a(x) \leftrightarrow x = a$, which indeed takes place in any transitive set containing $\mathrm{TC}(\{a\})$. For example, $\mu_{\{a_1, \ldots, a_n\}}(x)$ is

$$\forall y(y \in x \leftrightarrow \mu_{a_1}(y) \vee \cdots \vee \mu_{a_n}(y)).$$

Now, let

$$\pi_a(x) = \mu_a(x) \wedge \bigwedge_{b \in \mathrm{TC}(a)} \exists y \mu_b(y).$$

If $\mathfrak{B} = (B, E)$ is a model of the axiom of extensionality, $\mathfrak{B}_0$ the well-founded part of $\mathfrak{B}$, and $\mathfrak{A}$ the transitive collapse of $\mathfrak{B}_0$ via $i\colon \mathfrak{B}_0 \to \mathfrak{A}$, then:

$$\mathfrak{B} \models \pi_a(x) \quad \text{if and only if} \quad x \in B_0,\, a \in A \text{ and } i(x) = a.$$

1.1.2 Convention. We have made no requirements on the way the syntax of various logics is defined. Henceforth, we will assume that associated with the logic $\mathscr{L}$ is a transitive set A such that $\mathscr{L}(\tau) \subseteq A$, for all τ considered. Moreover, it is assumed that

$$\mathrm{Mod}(\pi_a) \in \mathrm{EC}_{\mathscr{L}[\tau_{\mathrm{set}}]} \quad \text{for} \quad a \in A.$$

In other words, $\mathscr{L}$ is supposed to be strong enough to fix—or "pin down", as it were—each element of A. Finally, A is assumed to be closed under primitive recursive set functions. In this case, we say that the *syntax of $\mathscr{L}$ is represented on A*. As a standing piece of notation, $\mathscr{L}$ is represented on a set denoted by A, $\mathscr{L}'$ on A', $\mathscr{L}''$ on A'' etc. Clearly, the syntax of the logics

$$\mathscr{L}_A,\ \mathscr{L}_A(Q),\ \mathscr{L}_A^2,\ \mathscr{L}_A(\mathrm{aa}),\ \mathscr{L}_A(\mathrm{pos})$$

is represented on A, the syntax of $\mathscr{L}_{\omega_1\omega_1}$ on HC, *etc.* In this chapter, "$\mathscr{L}(Q)$" means $\mathscr{L}(Q_1)$. The logic $\Delta(\mathscr{L})$ is more problematic. However, we may identify sentences of $\Delta(\mathscr{L})$ with triples $\langle \tau, \varphi, \varphi' \rangle$ where the reductions of $\mathrm{Mod}(\varphi)$ and the complement of $\mathrm{Mod}(\varphi')$ to the vocabulary τ coincide. Understood in this way, $\Delta(\mathscr{L})$ has a canonical representation of syntax on A. We use $\mathfrak{A}$, $\mathfrak{A}'$, $\mathfrak{A}''$, etc. to denote the set-theoretical structures $(A, \in|_A)$, $(A', \in|_{A'})$, etc.

1.1.3 Definition. We say that a logic $\mathscr{L}$ is *adequate to truth in* a logic $\mathscr{L}'$ if for every τ there is $\tau^+ = [\tau, \tau_{\mathrm{set}}, \mathrm{Th}, \tau']$ and $\theta \in \mathscr{L}[\tau^+]$ such that for every $\mathfrak{M} \in \mathrm{Str}[\tau]$, the following conditions hold:

(AT1) $(\mathfrak{M}, \mathfrak{A}', \mathrm{Th}_{\mathscr{L}'}(\mathfrak{M}), \mathfrak{N}) \models_{\mathscr{L}} \theta$ for some $\mathfrak{N}$.

(AT2) If $(\mathfrak{M}, \mathfrak{B}, T, \mathfrak{N}) \models_{\mathscr{L}} \theta \wedge \pi_\varphi(b)$, then $b \in T$ if and only if $\mathfrak{M} \models_{\mathscr{L}'} \varphi$, whatever $\varphi \in A'$ and $b \in B$.

Compare (AT2) with (4) above. The role of τ' is to provide the auxiliary tools (such as the pairing function and $S(x, y)$ in Example 1.1.1) that are mainly needed for coding.

1.1.4 Example. The logic $\mathscr{L}_{\omega\omega}$ is adequate to truth in $\mathscr{L}_A$. To prove this, we need only make some additions to Example 1.1.1. There the sentence η is supposed to conjoin the different cases of the inductive truth-definition of $\mathscr{L}_{\omega\omega}$. To extend this to $\mathscr{L}_A$, we simply conjoin η with something like

$$S\Big(\bigwedge_{i \in I} \varphi_i, s\Big) \leftrightarrow \forall i (i \in I \to S(\varphi_i, s)),$$

$$S\Big(\bigvee_{i \in I} \varphi_i, s\Big) \leftrightarrow \exists i (i \in I \wedge S(\varphi_i, s)).$$

Note here that π_a will not be in $\mathscr{L}_{\omega\omega}$ unless $a \in \mathrm{HF}$.

1.1.5 Example. The logic $\mathscr{L}_{\omega\omega}(Q)$ is adequate to truth in $\mathscr{L}_A(Q)$, whatever Q. This time, we extend Example 1.1.1 by a case for Q. Suppose, for the sake of simplicity, that Q is of signature $\langle 2\rangle$. Then we add the following case to η:

$$\begin{aligned}&S(Qx_1x_2\,\varphi(x_1, x_2), s)\\ &\quad\leftrightarrow Qx_1x_2\,\exists s'(s'_1 = x_1 \wedge s'_2 = x_2 \wedge (s'_n = s_n \text{ for } n > 2)\\ &\quad\wedge S(\varphi(x_1, x_2), s')).\end{aligned}$$

Then η will contain Q and will no longer be a sentence of $\mathscr{L}_{\omega\omega}$ but rather of $\mathscr{L}_{\omega\omega}(Q)$.

1.1.6 Example. The logic $\mathscr{L}^2_{\omega\omega}$ is adequate to truth in $\mathscr{L}^2_A$. This case needs somewhat more changes to Example 1.1.1 than the previous ones. In Example 1.1.1 $\mathfrak{N}$ contained a new sort for sequences of elements of $\mathfrak{M}$. Now we add to $\mathfrak{N}$ a new sort N_0 for subsets of the domains of $\mathfrak{M}$ and a new sort N_1 for finite sequences of such subsets as well as the projection function for N_1. With these new sorts at hand, we can easily extend the implicit truth-definition, coded in η, to $\mathscr{L}^2_A$. At the same time, we must add the obvious axioms for N_0 and N_1 to η as well. Similarly, we see that $\mathscr{L}^2_{\omega\omega}$ is also adequate to truth in a variety of higher-order logics.

1.1.7 Example. (i) $\mathscr{L}_{\omega\omega}(\text{aa})$ is adequate to truth in $\mathscr{L}_A(\text{aa})$.
(ii) $\mathscr{L}_{\lambda\lambda}$ is adequate to truth in $\mathscr{L}_{\kappa\lambda}$ for all κ.
(iii) $\mathscr{L}_{\omega_1 G}$ and $\mathscr{L}_{\omega_1 V}$ are adequate to truth in themselves. The reader should see Chapter X for the definition of these game logics.

1.1.8 Example. Let $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_n)_{n<\omega}$, where Q_n is the quantifier "there exists at least ω_n." This logic is not adequate to truth in itself, if represented in the canonical way on HF. The proof of this is simple enough. The Löwenheim–Skolem theorem of $\mathscr{L}$ shows that no θ in $\mathscr{L}_{\omega\omega}(Q_n)_{n<m}$ $(m < \omega)$ can capture $Q_m x(x = x)$, for example.

As we proceed, we will meet other examples of the failure of adequacy to truth. The failure of $\mathscr{L}_{\omega\omega}(Q_n)_{n<\omega}$ to be self-adequate follows intuitively from the fact that the inductive truth-definition has an infinity of genuinely different cases (in fact, one for each Q_n) and there is no way of putting them all together. A similar situation occurs in $\mathscr{L}^2_{\omega\omega}$—here, there is one case for each arity of predicate-variables—but the expressive power of $\mathscr{L}^2_{\omega\omega}$ allows us to take the long conjunction.

1.1.9 Remark. Suppose that $\mathscr{L}$ is a logic and $T\colon \text{Str}[\tau] \to \mathscr{P}(A)$. Feferman [1975] calls T *#-uiid$_x$ in $\mathscr{L}$ if there is* $\tau^+ = [\tau, \tau_{\text{set}}, \text{Th}, \tau']$ and $\theta \in \mathscr{L}[\tau^+]$ such that if $\mathfrak{M} \in \text{Str}[\tau]$, then:

(#1) $[\mathfrak{M}, \mathfrak{A}, T, \mathfrak{N}] \models \theta$ for some $\mathfrak{N}$;

and

(#2) If $[\mathfrak{M}, \mathfrak{B}, T', \mathfrak{N}] \models \theta \wedge \pi_a(a')$, then $a \in T$ if and only if $a' \in T'$.

This is a notion which arises naturally from analogous notions in generalized recursion theory, such as the invariant implicit definability of Kunen [1968]. The ("uiid" is short for "uniformly invariantly implicitly definable," and the "x" is used to indicate the possibility of extra sorts in $\mathscr{L}'$). With this notion at hand, we could define adequacy of $\mathscr{L}$ to truth in $\mathscr{L}'$ by simply saying that the mapping $T(\mathfrak{M}) = \mathrm{Th}_{\mathscr{L}'}(\mathfrak{M})$ is $\#$-uiid$_x$ in $\mathscr{L}$. The notion $\#$-uiid$_x$ permits many variations, such as $\#$-usiid$_x$ ("s" for "semi") which replaces "if and only if" by "only if" in ($\#$2). The corresponding weaker form of adequacy to truth could be called *semi-adequacy to truth.*

The notion of adequacy to truth bears a special relation to the Δ-operation defined in Chapter II. The rest of this section is devoted to a study of this. Also recall from Chapter II the notion $\mathrm{RPC}_{\mathscr{L}}$ of relational projective class in $\mathscr{L}$.

1.1.10 Lemma. *Suppose that $\mathscr{L}$ is adequate to truth in $\mathscr{L}'$, $\varphi \in \mathscr{L}'$ and π_φ is* $\mathrm{RPC}_{\mathscr{L}}$*-definable. Then* $\mathrm{Mod}(\varphi)$ *is $\Delta(\mathscr{L})$-definable.*

Proof. Suppose $\varphi \in \mathscr{L}'[\tau]$. Let $\theta \in \mathscr{L}[\tau^+]$ be as in Definition 1.1.3. The following conditions are equivalent for any $\mathfrak{M} \in \mathrm{Str}[\tau]$:

(a) $\mathfrak{M} \models_{\mathscr{L}'} \varphi$;
(b) $[\mathfrak{M}, \mathfrak{B}, \mathfrak{N}] \models \theta \wedge \pi_\varphi(b) \wedge \mathrm{Th}(b)$ for some $\mathfrak{B}$, $\mathfrak{N}$, and b;
(c) $[\mathfrak{M}, \mathfrak{B}, \mathfrak{N}] \models \theta \wedge \pi_\varphi(b) \to \mathrm{Th}(b)$ for all $\mathfrak{B}$, $\mathfrak{N}$, and b.

By substituting the $\mathrm{RPC}_{\mathscr{L}}$-definition of π_φ into (b) and (c), we obtain a $\Delta(\mathscr{L})$-definition of φ. ☐

1.1.11 Remarks. (i) If $\mathscr{L}$ is only semi-adequate to truth in $\mathscr{L}'$ as given in Lemma 1.1.10, we can still obtain a co-RPC($\mathscr{L}$)-definition for φ from the proof.

(ii) Lemma 1.1.10 has interesting consequences for logics which have more power than their syntax suggests. Take, for example, $\mathscr{L}^2_{\omega\omega}$. Many set-theoretically definable $\varphi \in \mathscr{L}^2_{\infty\omega}$ satisfy the assumption that π_φ is $\mathrm{RPC}_{\mathscr{L}^2_{\omega\omega}}$. Whence, $\mathrm{Mod}(\varphi)$ is $\Delta(\mathscr{L}^2_{\omega\omega})$-definable. This shows clearly the infinitary nature of $\Delta(\mathscr{L}^2_{\omega\omega})$.

1.1.12 Corrollary. *If $\mathscr{L}$ is adequate to truth in $\mathscr{L}'$ and $A' \subseteq A$, then $\mathscr{L}' \leq_{\mathrm{RPC}} \mathscr{L}$.* ☐

1.1.13 Lemma. *If $\mathscr{L}'' \leq_{\mathrm{RPC}} \mathscr{L}$ and $\mathscr{L}''$ is adequate to truth in $\mathscr{L}'$, then $\mathscr{L}$ is adequate to truth in $\mathscr{L}'$.*

Proof. Let τ be a vocabulary and let $\theta' \in \mathscr{L}''[\tau^+]$ witness the adequacy of $\mathscr{L}''$ to truth in $\mathscr{L}'$. Let $\tau_1 \supseteq \tau^+$ and $\theta \in \mathscr{L}[\tau_1]$ such that

$$\mathfrak{M} \models \theta_1 \quad \text{if and only if} \quad (\mathfrak{M}, \mathfrak{N}) \models \theta \text{ for some } \mathfrak{N}.$$

Clearly, θ satisfies (AT1) for $\mathscr{L}$ and $\mathscr{L}'$. For (AT2), suppose that

$$(\mathfrak{M}, \mathfrak{B}, T, \mathfrak{N}', \mathfrak{N}) \models \theta \wedge \pi_\varphi(b).$$

Then $(\mathfrak{M}, \mathfrak{B}, T, \mathfrak{N}') \models \theta_1 \wedge \pi_\varphi(b)$, whence

$$b \in T \quad \text{if and only if} \quad \mathfrak{M} \models \varphi,$$

as required. ☐

1.1.14 Proposition. *Suppose that $\mathscr{L}'$ is adequate to truth in itself and $A' \subseteq A$. Then the following are equivalent:*

(a) *$\mathscr{L}$ is adequate to truth in $\mathscr{L}'$.*
(b) *$\mathscr{L}' \leq_{\mathrm{RPC}} \mathscr{L}$.* ☐

Discussion. The proposition shows that for syntacticly natural logics, adequacy to truth reduces to the familiar and much simpler concept of $\leq_{\mathrm{RPC}}$. However, this does not take place in general. Rather, we may construe the relation of adequacy to truth as an effective version of $\leq_{\mathrm{RPC}}$. This effectivity can be demonstrated by examples. Thus, unlike $\leq_{\mathrm{RPC}}$, adequacy to truth preserves Σ_1-compactness and Σ_1-definability of validity (see Section 4.3).

1.1.15 Proposition. *Suppose that $\mathscr{L}$ and $\mathscr{L}'$ are logics such that $A' \subseteq A$. Then the following are equivalent:*

(a) *$\Delta(\mathscr{L}) \leq \mathscr{L}'$.*
(b) *Whenever $\mathscr{L}$ is adequate to truth in $\mathscr{L}''$, with $A'' \subseteq A$, then $\mathscr{L}'' \leq \mathscr{L}'$.*

Proof. The argument for (a) implies (b) follows from Corollary 1.1.12. To prove the converse, suppose that $\mathscr{K}$ is a $\Delta(\mathscr{L})$-definable model class. Let Q be the generalized quantifier associated with $\mathscr{K}$. By Proposition 1.1.14, we have that $\mathscr{L}$ is adequate to truth in $\mathscr{L}_A(Q)$. Thus, by letting $\mathscr{L}'' = \mathscr{L}_A(Q)$ in (b), we get $\mathscr{L}_A(Q) \leq \mathscr{L}'$. Whence, Q is $\mathscr{L}'$-definable. ☐

1.1.16 Definition. A logic $\mathscr{L}$ is *truth maximal* if $\mathscr{L}' \leq \mathscr{L}$ whenever $\mathscr{L}$ is adequate to truth in $\mathscr{L}'$ with $A' \subseteq A$. If, in addition, $\mathscr{L}$ is adequate to truth in itself, we say that $\mathscr{L}$ is *truth complete.*

1.1.17 Corollary. *$\mathscr{L}$ has the Δ-interpolation property if and only if $\mathscr{L}$ is truth maximal.* ☐

1.1.18 Examples. If $A \subseteq \mathrm{HC}$ is admissible, then $\mathscr{L}_A$ is truth complete. If A is the union of countable admissible sets, then $\mathscr{L}_A$ is truth maximal but not necessarily truth complete, in the case where A is not admissible. The Δ-extension of any logic is truth maximal.

The concepts of truth maximality and truth completeness were introduced by Feferman [1974a] and Corollary 1.1.17 was also proven there. Feferman's paper was among the first to discuss Souslin–Kleene interpolation in an abstract setting, and it provided strong support for further work on the Δ-operation.

1.2. Definability of Syntax Set

We have assumed that there is associated with every logic $\mathscr{L}$ a syntax set A on which the syntax of $\mathscr{L}$ is represented. Part of this convention is that every element of A is definable in $\mathscr{L}$. For some $\mathscr{L}$, it happens that A itself is in one form or another definable in $\mathscr{L}$. The results below suggest that such $\mathscr{L}$ have been defined without proper concern to the balance between syntax and semantics. Recall that we use $\mathfrak{A}$ for $(A, \varepsilon|_A)$. Along these same lines, let us use $\mathscr{I}(A)$ for the isomorphism class of $\mathfrak{A}$, $\mathscr{E}(A)$ for the class of structures isomorphic to an end-extension of $\mathfrak{A}$. That is,

$$\mathscr{I}(A) = \{\mathfrak{B} \in \mathrm{Str}[\tau_{\mathrm{set}}] \mid \mathfrak{B} \cong \mathfrak{A}\}$$

$$\mathscr{E}(A) = \{\mathfrak{B} \in \mathrm{Str}[\tau_{\mathrm{set}}] \mid \mathscr{C} \subseteq_e \mathfrak{B}, \text{ for some } \mathscr{C} \in \mathscr{I}(A)\}.$$

We will now consider definability of $\mathscr{I}(A)$ and $\mathscr{E}(A)$.

1.2.1 Examples. (i) $\mathscr{E}(\mathrm{HF}) \in \mathrm{EC}_{\mathscr{L}_{\omega\omega}}$.
(ii) $\mathscr{I}(\mathrm{HF}) \in \mathrm{EC}_{\mathscr{L}_{\omega\omega}(Q_0)}$.
(iii) $\mathscr{E}(A) \in \mathrm{PC}_{\mathscr{L}_A}$ if $A = B^+$, B admissible (see Barwise [1975, V. 3.9]).
(iv) $\mathscr{I}(\mathrm{HC}) \in \mathrm{EC}_{\mathscr{L}_{\omega_1\omega_1}}$.

1.2.2 Proposition. *Suppose $\mathscr{L}$ is adequate to truth in $\mathscr{L}'$ and $\Phi \subset \mathscr{L}'$. Then* Mod($\Phi$) *is* $\mathrm{RPC}_{\mathscr{L}}$ *if either of the following conditions holds*:

(a) *$\mathscr{E}(A')$ is $\mathrm{RPC}_{\mathscr{L}}$ and Φ is a Σ_1 subset of A.*
(b) *$\mathscr{I}(A')$ is $\mathrm{RPC}_{\mathscr{L}}$ and Φ is a Π_1^1 subset of A.* □

Remarks. The method of proving this proposition is similar to that used in the proof of Lemma 1.1.10. In (b) we only need to know that Φ is definable by a co-$\mathrm{RPC}_{\mathscr{L}}$-formula over A'. If $A' \subseteq A$, then Σ_1 can be replaced by $\boldsymbol{\Sigma}_1$ and Π_1^1 by $\boldsymbol{\Pi}_1^1$. If $\mathscr{I}(A')$ is $\Delta(\mathscr{L})$-definable in (b), and if Φ is Δ_1^1, then Mod(Φ) is $\Delta(\mathscr{L})$-definable.

Applications. The Kleene–Craig–Vaught theorem says that recursively axiomatizable theories in $\mathscr{L}_{\omega\omega}$ can be finitely axiomatized using extra predicates (see Craig–Vaught [1958]). This is exactly what Proposition 1.2.2(a) says if $\mathscr{L} = \mathscr{L}' = \mathscr{L}_{\omega\omega}$ and $A = A' = \mathrm{HF}$. By letting $\mathscr{L} = \mathscr{L}' = \mathscr{L}_A(Q)$, we get the same theorem for $\mathscr{L}_A(Q)$.

An element of paradox is always near when we speak about definability of truth. The following application of this paradoxical element has a long history:

1.2.3 Proposition. *Suppose that $\mathscr{L}$ is adequate to truth in $\mathscr{L}'$ and $\mathscr{I}(A')$ is $\Delta(\mathscr{L})$-definable. Then $\Delta(\mathscr{L}) \not\leq \mathscr{L}'$.*

Proof. Let τ be a vocabulary, and let $\theta \in \mathscr{L}[\tau^+]$ witness the adequacy of $\mathscr{L}$ to truth in $\mathscr{L}'$. Also, let θ' be the conjunction of θ and the $\mathrm{RPC}_{\mathscr{L}}$-definition of $\mathscr{I}(A)$, and let

$$\mathscr{K} = \{(\mathfrak{M}, \mathfrak{B}, b) \mid \exists \mathfrak{N}(\mathfrak{M}, \mathfrak{B}, \mathfrak{N}) \models \theta' \wedge \mathrm{Th}(b))\}.$$

By its definition, $\mathscr{K}$ is $\mathrm{RPC}_{\mathscr{L}}$-definable. On the other hand, we claim that

$$(*) \qquad (\mathfrak{M}, \mathfrak{B}, b) \notin \mathscr{K} \quad \text{if and only if} \quad \mathfrak{B} \notin \mathscr{I}(A') \quad \text{or} \quad \exists \mathfrak{N}((\mathfrak{M}, \mathfrak{B}, \mathfrak{N}) \models \theta' \wedge \neg \mathrm{Th}(b)).$$

Suppose first that $(\mathfrak{M}, \mathfrak{B}, b) \notin \mathscr{K}$ but $\mathfrak{B} \in \mathscr{I}(A')$. By (AT1), $(\mathfrak{M}, \mathfrak{B}, \mathfrak{N}) \models \theta'$ holds, for some $\mathfrak{N}$. By the definition of $\mathscr{K}$, $(\mathfrak{M}, \mathfrak{B}, \mathfrak{N}) \models \neg\mathrm{Th}(b)$. Now, for the converse, we suppose that $(\mathfrak{M}, \mathfrak{B}, b) \in \mathscr{K}$. By the definition of $\mathscr{K}$, $\mathfrak{B} \in \mathscr{I}(A')$ and $(\mathfrak{M}, \mathfrak{B}, \mathfrak{N}) \models \theta' \wedge \mathrm{Th}(b)$, for some $\mathfrak{N}$. Now if $(\mathfrak{M}, \mathfrak{B}, \mathfrak{N}') \models \theta' \wedge \neg\mathrm{Th}(b)$, for some $\mathfrak{N}'$, then by (AT2), $\mathfrak{M} \models_{\mathscr{L}'} b$ and $\mathfrak{M} \not\models_{\mathscr{L}'} b$, which is absurd. This ends the proof of (*). It follows that $\mathscr{K}$ is $\Delta(\mathscr{L})$-definable. To conclude the proof we show that $\mathscr{K}$ is not definable in $\mathscr{L}'$. To this end, suppose that $\mathscr{K} = \mathrm{Mod}(\varphi)$, for some

$$\varphi \in \mathscr{L}'[\tau \cup \tau_{\mathrm{set}} \cup \{c\}].$$

For any $\mathfrak{M} \in \mathrm{Str}[\tau]$ and $\psi \in \mathscr{L}'[\tau]$, we thus have:

$$(\#) \qquad \mathfrak{M} \models_{\mathscr{L}'} \psi \quad \text{if and only if} \quad (\mathfrak{M}, \mathfrak{A}', \psi) \models_{\mathscr{L}'} \varphi.$$

Now we choose $\tau = \tau_{\mathrm{set}} \cup \{c\}$, $\psi = \neg\varphi$ and $\mathfrak{M} = (\mathfrak{A}', \psi)$. Then (#) gives

$$\begin{aligned} \mathfrak{M} \models_{\mathscr{L}'} \psi \quad & \text{if and only if} \quad \mathfrak{M} \models_{\mathscr{L}'} \varphi \\ & \text{if and only if} \quad \mathfrak{M} \not\models_{\mathscr{L}'} \psi. \end{aligned}$$

This contradiction completes the proof. □

1.2.4 Corollary. *If $\mathscr{I}(A)$ is $\Delta(\mathscr{L})$-definable, then $\Delta(\mathscr{L})$ is not adequate to truth in itself.* □

Applications. $\Delta(\mathscr{L})$ is not selfadequate if $\mathscr{L}$ happens to be one of $\mathscr{L}_{\omega\omega}(Q_0)$, $\mathscr{L}^2_{\omega\omega}$, $\mathscr{L}_{\omega_1\omega_1}$ among many others.

Remark. We will later prove that $\Delta(\mathscr{L}_{\omega\omega}(Q_0)) \equiv \mathscr{L}_A$ for $A = (\mathrm{HF})^+$, the smallest admissible set containing HF. Thus, $\Delta(\mathscr{L}_{\omega\omega}(Q_0))$ is selfadequate if represented on $(\mathrm{HF})^+$ rather than on HF. This provides an example of the importance of the exact manner in which the syntax is defined.

Some Refinements. The notion of adequacy to truth is based very heavily on the use of extra predicates. Our prime example (see Example 1.1.1) uses the extra symbols

$M^{<\omega}$—finite sequences of elements of the model,

$P(s, n)$—the nth element of the sequence s,

$S(x, y)$—the sequence y satisfies the formula x.

The use of the new sort $M^{<\omega}$ is actually unnecessary if the model $\mathfrak{M}$ is infinite. Let us say that $\mathscr{L}$ is *simply adequate to truth* in $\mathscr{L}'$ if Definition 1.1.3 can be satisfied (for infinite models) with no new sorts of τ' over those of $\tau \cup \tau_{\text{set}}$. The following are examples of simply selfadequate logics:

$$\mathscr{L}_A,\ \mathscr{L}_A(Q), L_{AG}, \mathscr{L}_{\kappa\omega}.$$

The above results connecting adequacy to truth and RPC carry over to simple adequacy to truth if RPC is replaced by PC and all models are infinite. As $\mathrm{PC}_{\mathscr{L}} = \mathrm{EC}_{\mathscr{L}}$, for $\mathscr{L} = \mathscr{L}^2_{\omega\omega}$, we see from the analogue of Proposition 1.2.3 that $\mathscr{L}^2_{\omega\omega}$ is not simply adequate to truth in itself.

Another refinement arises in the following way. Looking again at Example 1.1.1, we notice that the only really new symbol one needs is S. That is, we can allow $\mathfrak{M}$ in Example 1.1.1(4) to contain the symbols $M^{<\omega}$ and P. On the other hand, S is implicitly defined by θ as soon as there are no non-standard formulae. This observation motivates the following definition. We say that $\mathscr{L}$ is *uniquely adequate to truth* in $\mathscr{L}'$ if there is a vocabulary τ_{code} such that Lemma 1.1.13 can be satisfied with $\mathfrak{M} \in \mathrm{Str}[\tau \cup \tau_{\text{code}}]$; and, moreover, the relations of $\tau' - \tau_{\text{code}}$ are implicitly defined by θ. The following are examples of uniquely selfadequate logics:

$$\mathscr{L}_{\omega\omega}(Q_0), \mathscr{L}^{2,w}_{\omega\omega}, \mathscr{L}^2_{\omega\omega}, \mathscr{L}_{\omega_1 G}, \mathscr{L}_{\omega_1\omega_1}.$$

If $\mathscr{L}$ is uniquely adequate to truth in $\mathscr{L}'$ and $\mathscr{I}(A')$ is $\mathrm{WB}(\mathscr{L})$-definable, then it can be proven as in Proposition 1.2.3 that $\mathrm{WB}(\mathscr{L}) \not\leq \mathscr{L}'$. Thus, $\mathrm{WB}(\mathscr{L})$ is not uniquely adequate to truth in itself for $\mathscr{L}$ as above.

Another uniform feature in the examples we have is the following: The new type τ^+ is obtained from τ effectively. This gives rise to the following refinement. $\mathscr{L}$ is *effectively adequate to truth in* $\mathscr{L}'$ if 1.1.3 can be satisfied in such a way that τ^+ is obtained from τ via a Σ_1 operation on A.

Historical and Bibliographical Remarks. The notion of adequacy to truth was introduced in Feferman [1974a] and has been further developed in Feferman [1975]. Indeed, Corollary 1.1.17 is from Feferman [1974a]. Definability of syntax set is discussed in Paulos [1976] where Proposition 1.2.3 is (essentially) proven. The roots of Proposition 1.2.3 go back to Craig [1965] and Kreisel [1967]. While Craig only considered higher-order logics, it was Mostowski [1968] who first explicitly proved the failure of interpolation and Beth-definability (see Section 4.1 for this) for logics which can define their own syntax set. On the other hand, we may trace the roots of the application of self-reference and the Liar Paradox in logic back to K. Gödel and A. Tarski. The Kleene–Craig–Vaught theorem is proven in Craig–Vaught [1958] and its generalization to $\mathscr{L}_{\omega\omega}(Q)$ was used in Lindström [1969]. Its generalization (see Proposition 1.2.2(a)) to abstract model theory was remarked in Feferman [1974a]. The reader is referred to Barwise [1975] for a proof of Example 1.2.1(iii).

2. *Set-Theoretic Definability Criteria*

Suppose that we are given a logic $\mathscr{L}$. The predicates $\varphi \in \mathscr{L}$ and $\mathfrak{M} \models_{\mathscr{L}} \varphi$ of φ and $\mathfrak{M}$ are certain set-theoretic predicates, and we may raise the following question: What is the set-theoretic complexity of these predicates? In this section we will study logics with a fixed upper bound for these complexities. Moreover, we will be particularly interested in those definitions of the predicates whose meaning does not depend on the particular interpretation given to set-theoretical axioms.

2.1. Absolute Logics

The idea of absoluteness of a logic is that the truth or falsity of the predicate $\mathfrak{M} \models_{\mathscr{L}} \varphi$ should not depend on the entire set-theoretical universe but rather should depend on the sets that are required to exist (in addition to $\mathfrak{M}$ and φ) by the axioms of a fixed set theory T only. An important candidate for such a set theory T is the Kripke–Platek axioms KP (with the axiom of infinity included). For details on KP and the set-theoretic notion of absoluteness the reader is referred to Barwise [1975], where the following crucial characterization (due to Feferman and Kreisel) can also be found on page 35: For any T, a predicate is absolute in models of T if and only if it is Δ_1 with respect to T (see Feferman [1974a] for a proof of this result). Absolute logics were first studied systematically by Barwise [1972a].

2.1.1 Definition. Let $\mathscr{L}$ be a logic and T a set theory. We say that $\mathscr{L}$ is *absolute relative to T* if there is a predicate $S(x, y)$, Δ_1 with respect to T, such that for $\varphi \in A$ and for any $\mathfrak{M}$

(A) $\quad S(\mathfrak{M}, \varphi) \leftrightarrow \varphi \in \mathscr{L} \quad \text{and} \quad \mathfrak{M} \models_{\mathscr{L}} \varphi,$

and the syntactic operations of $\mathscr{L}$ are Δ_1 with respect to T. The logic $\mathscr{L}$ is (*strictly*) *absolute* if it is absolute relative to some T (relative to KP) which is true in the real world.

Explanations. By "syntactic operations" we mean finitary conjunction, disjunction, permutation, $\pi_a(x)$, *etc*., which are built into the definition of an abstract logic. By "true in the real world" we mean that T is a consequence of the axioms of our meta-set-theory. It would make little sense to allow T to be, for instance, inconsistent! The most important consequence of T being a true set theory is that if $(A, \ )$ is a transitive model of T and $\mathscr{L}$ is absolute relative to T, and if $\varphi \in \mathscr{L}$ and $\mathfrak{M}$, $\varphi \in A$, then we have

$$(A,\in) \models \text{“}\mathfrak{M} \models_{\mathscr{L}} \varphi\text{”} \quad \text{if and only if} \quad \mathfrak{M} \models_{\mathscr{L}} \varphi.$$

2.1.2 Example. The infinitary logic $\mathscr{L}_A$ is strictly absolute. The fact that the satisfaction relation of $\mathscr{L}_A$ is Δ_1 in KP-Inf-(= Axiom of Infinity) is proven in Barwise [1975]. The crucial property of KP-Inf is that it allows the definition of Δ-predicates by recursion. All the syntactic and semantic notions of $\mathscr{L}_A$ can be defined in KP-Inf by set-recursion using Δ-predicates.

2.1.3 Example. The logics $\mathscr{L}_{\omega\omega}(Q_0)$ and $\mathscr{L}_A^{2,w}$ are strictly absolute. This can be seen by reducing these logics to $\mathscr{L}_A$ or by considering the proof of the selfadequacy of these logics. The point to note here is that the predicate "x is finite" is Δ_1 in KP. However, the predicate "x is countable" is not Δ_1 in any first-order set-theory; and, indeed, the logic $\mathscr{L}_{\omega\omega}(Q_1)$ turns out to be non-absolute.

2.1.4 Example. The game logics $\mathscr{L}_{AG}$, $\mathscr{L}_{AV}$, and $\mathscr{L}_{AS}$ are absolute relative to KP + Σ_1-separation + DC (= Axiom of Dependent Choices) (see Chapter X). Burgess [1977] introduced the Borel game logic $\mathscr{L}_{\infty B}$ which extends $\mathscr{L}_{\infty V}$ by allowing the operation

$$\bigwedge_{i_0 \in I} \forall v_0 \bigvee_{i_1 \in I} \exists v_1 \ldots \{n \mid \varphi_{i_0 \ldots i_n}(v_0, \ldots, v_{2n+1}) \text{ true}\} \in B,$$

where B is any Borel set of sets of natural numbers and I is a set. It follows from Martin's Borel Determinacy Theorem that $\mathscr{L}_{AB}$ is absolute relative to ZFC.

2.2. *Some Properties of Absolute Logics*

The two principal properties of absolute logics are the downward Löwenheim–Skolem theorem (see Theorem 2.2.2) and the approximation theorem (see Theorem 2.2.8). Many useful implications can be drawn from these two results. Interestingly enough, both have the form of an approximation result, although the two notions of approximation are unrelated. The notion of countable approximation, to be studied first, is due to Kueker [1972, 1977]. It leads to a very strong formulation of the downward Löwenheim–Skolem theorem. The second notion of approximation is the result of gradual development starting from Moschovakis.

The Löwenheim–Skolem Theorem. In order to obtain a particularly simple formulation of the Löwenheim–Skolem theorem, we will assume for a moment that there is a proper class of urelements and that the elements of all models are urelements. This is not an essential restriction, because every model is isomorphic to one consisting of urelements. Moreover, urelements could be avoided by using a more cumbersome notation.

For any sets a and s, let

$$a^s = a \text{ if } a \text{ urelement}, \quad a^s = \{x^s \mid x \in s \cap a\} \text{ otherwise.}$$

If s is countable, then a^s is called a *countable approximation of* a. Note that if $\alpha \in On$, then $\alpha^s < \omega_1$, for all countable s. If $\mathfrak{M}$ is a (relational) structure, then $\mathfrak{M}^s$ is a countable substructure of $\mathfrak{M}$.

If $P(x_1, \ldots, x_n)$ is a predicate of set theory, then we say that

$$P(a_1^s, \ldots, a_n^s) \text{ holds } \textit{almost everywhere} \text{ (abbreviated by a.e.)}$$

if $P(a_1^s, \ldots, a_n^s)$ holds for all s in a closed unbounded (cub) set of countable subsets of $\mathrm{TC}(\{a_1, \ldots, a_n\})$. See Chapter II, and Chapter IV, Section 4 for more on "almost all countable sets".

2.2.1 Lemma. *If $P(x_1, \ldots, x_n)$ is a Σ_1-predicate and $P(a_1, \ldots, a_n)$ holds, then $P(a_1^s, \ldots, a_n^s)$ holds almost everywhere.* □

2.2.2 Downward Löwenheim–Skolem Theorem. *Suppose that $\mathscr{L}$ is an absolute logic, $\varphi \in \mathscr{L}$ and $\mathfrak{M}$ is a model. Then $\varphi^s \in \mathscr{L}$ almost everywhere and*

$$\mathfrak{M} \models_{\mathscr{L}} \varphi \quad \textit{if and only if} \quad \mathfrak{M}^s \models_{\mathscr{L}} \varphi^s \textit{ almost everywhere.}$$

Proof. The predicate $\varphi \in \mathscr{L}$ is Σ_1, and hence $\varphi^s \in \mathscr{L}$ a.e., by Lemma 2.2.1. Similarly, the predicate $\mathfrak{M} \models_{\mathscr{L}} \varphi$ is Σ_1 and we get $\mathfrak{M}^s \models_{\mathscr{L}} \varphi^s$ a.e. from Lemma 2.2.1. □

If $a \in \mathrm{HC}$, then $a^s = a$ a.e., since the set $\mathrm{TC}(\{a\})$ is countable. Hence, we have

2.2.3 Corollary. *Suppose $\mathscr{L}$ is an absolute logic, $\varphi \in \mathscr{L}$ and $\varphi \in \mathrm{HC}$. If φ has a model, then φ has a countable model.* □

Application. $\mathscr{L}_A(Q_1)$ and $\mathscr{L}_A^2$ are not absolute as they do not satisfy Corollary 2.2.3.

For a sharper application of Theorem 2.2.2, we need a sharper cub set calculation. The proof of the following lemma is not hard.

2.2.4 Lemma. *Suppose that X is a* cub *set of countable subsets of A. Suppose further that $I \subseteq A$ and κ is an infinite cardinal such that $|I| \leq \kappa < |A|$. Then there is a $B \subset A$ such that $I \subseteq B$, $|B| = \kappa$ and the set of countable subsets of B in X form a* cub *set on B.* □

2.2.5 Corollary. *Suppose $\mathscr{L}$ is an absolute logic, $\varphi \in \mathscr{L}$, $\mathfrak{M} \models_{\mathscr{L}} \varphi$, $N_0 \subset M$ has cardinality at most κ and $\varphi \in H_{\kappa^+}$. Then there is an $\mathfrak{N} \subseteq \mathfrak{M}$ of cardinality κ such that $N_0 \subset N$ and $\mathfrak{N} \models_{\mathscr{L}} \varphi$.* □

Proof. Let $A = \mathrm{TC}(\{\varphi, \mathfrak{M}, \kappa^+\})$, and let X be a cub set of countable $s \subset A$ such that $\mathfrak{M}^s \models_{\mathscr{L}} \varphi^s$. Furthermore, let $I = \mathrm{TC}(\{N_0, \varphi\})$. Finally, let $B \subset A$ be given by Lemma 2.2.4 and define $\mathfrak{N}$ to be the restriction of $\mathfrak{M}$ to $B \cap M$. If s is in the cub set of countable subsets of B that are in X, then $\mathfrak{M}^s \models_{\mathscr{L}} \varphi^s$. But $\mathfrak{M}^s = \mathfrak{N}$ a.e. on B and $\varphi^s = \varphi$ a.e. on B. Hence, $\mathfrak{N} \models_{\mathscr{L}} \varphi$. □

The Approximation Theorem

The countable approximations φ^s that we studied in the above discussions were defined from a set-theoretical point of view. We will now associate with every formula φ of an absolute logic approximations $A(\varphi, \alpha)$ ($\alpha \in \mathrm{On}$) which are formulae of $\mathscr{L}_{\infty\omega}$ and which are logically related to φ. It is instructive to first examine the approximations of game formulae. This is the historical order of events: The approximations were developed by Moschovakis and others for game formulae, and it was only later that Burgess [1977] presented the general case (Theorem 2.2.8).

Let us consider a disjunctive game formula

$$(*) \qquad \forall x_0 \bigwedge_{i_0 \in I} \exists y_0 \bigvee_{j_0 \in I} \cdots \bigvee_{n<\omega} \varphi^{i_0 j_0 \ldots i_n j_n}(x_0, y_0, \ldots, x_n, y_n).$$

In order to better understand the idea of approximation, it is useful to write $(*)$ in a new form. Recall that the truth of $(*)$ is determined according to whether player I or II has a winning strategy in the (determinate) infinite game in which each play consists of running through $(*)$ from left to right, with $\exists x_n$ and $\bigvee_{i_n \in I}$ moves of I (pick an x_n or an i_n) and $\forall y_n$ and $\bigwedge_{j_n \in I}$ moves of II (pick a y_n or a j_n) and I wins if one of $\varphi^{i_0 \cdots j_n}(x_0, \ldots, y_n)$ is true in the end. As the truth of $\varphi^{i_0 \cdots j_n}(x_0, \ldots, y_n)$ does not depend on the moves number $n + 1$, $n + 2$, etc., we may as well construe the sense of $(*)$ as

$$(**) \qquad \forall x_0 \bigwedge_{i_0 \in I} \exists y_0 \bigvee_{j_0 \in I}$$

$$\left(\varphi^{i_0 j_0}(x_0, y_0) \vee \forall x_1 \bigwedge_{i_1 \in I} \exists y_1 \bigvee_{j_1 \in I} (\varphi^{i_0 \cdots j_1}(x_0, \ldots, y_1) \vee \ldots)\right).$$

Let us now define approximations $A(\alpha, \varphi)$ for formulae φ obtained from atomic formulae using $\wedge$, $\vee$, $\neg$, $\exists$, $\forall$, $\bigvee$, $\bigwedge$ and (**):

$$A(0, \varphi) = \bot \ (= \text{false}),$$

$$A(\alpha + 1, \varphi) = \varphi \text{ if } \varphi \text{ is atomic},$$

$$A(\alpha + 1, \neg\varphi) = \neg A(\alpha, \varphi),$$

$$A(\alpha + 1, \varphi \wedge \psi) = A(\alpha, \varphi) \wedge A(\alpha, \psi),$$

$$A(\alpha + 1, \varphi \vee \psi) = A(\alpha, \varphi) \vee A(\alpha, \psi),$$

$$A(\alpha + 1, \exists x\varphi(x)) = \exists x A(\alpha, \varphi(x)),$$

$$A(\alpha + 1, \forall x\varphi(x)) = \forall x A(\alpha, \varphi(x)),$$

$$A\left(\alpha + 1, \bigvee_{i \in I} \varphi_i\right) = \bigvee_{i \in I} A(\alpha, \varphi_i),$$

$$A\left(\alpha + 1, \bigwedge_{i \in I} \varphi_i\right) = \bigwedge_{i \in I} A(\alpha, \varphi_i),$$

$$A(\nu, \varphi) = \bigvee_{\alpha < \nu} A(\alpha, \varphi) \text{ for limit } \nu.$$

In order to see what happens to $A(\alpha, \varphi)$ for various α and for φ as in (**) above, we will assume that $\forall x_0 \bigwedge_{i_0 \in I} \exists y_0 \bigvee_{j_0 \in I} \varphi^{i_0 j_0}(x_0, y_0)$ is true and every $\varphi^{i_0 j_0}(x_0, y_0)$ is atomic. Then $A(6, \varphi)$ is true. If the formulae $\varphi^{i_0 j_0}(x_0, y_0)$ are not atomic but are still in $\mathscr{L}_{\infty\omega}$, then $A(\omega + 5, \varphi)$ is true. We can now prove that φ is true in a model $\mathfrak{M}$ if and only if $A(\alpha, \varphi)$ is true in $\mathfrak{M}$, for some $\alpha \in \mathrm{On}$. Observe that this would not be true if the syntax (*) were used as no approximation would have reached to the long disjunction at the end. If we start with a conjuctive game formula φ, we can define $A(\alpha, \varphi) = \neg A(\alpha, \varphi\neg)$, where $\varphi\neg$ is the dual of φ obtained by everywhere interchanging $\wedge$ and $\vee$, $\exists$ and $\forall$, $\bigvee$ and $\bigwedge$, and an atomic formula and its negation.

A trivial induction on α shows that if $A(\alpha, \varphi)$ is true, then $A(\beta, \varphi)$ is true for all $\beta \geq \alpha$. On the other hand, one need not study very large α: the first α as above is below $|\mathfrak{M}|^+$. The reader should see Chapter X for more on approximations.

2.2.6 Definition. An *approximation function* for a logic $\mathscr{L}$ is a mapping $A\colon \mathrm{On} \times \mathscr{L} \to \mathscr{L}_{\infty\omega}$ such that for all $\varphi \in \mathscr{L}$ and $\mathfrak{M}$:

$$\mathfrak{M} \models_{\mathscr{L}} \varphi \quad \text{if and only if} \quad \mathfrak{M} \models A(\alpha, \varphi) \text{ for some } \alpha \in \mathrm{On}.$$

2.2.7 Example. $\mathscr{L}_{AV}$ has a Δ_1 approximation function. This function was defined above. It is easily proven by induction on α that $A(\alpha, \varphi) \in \mathscr{L}_{\infty\omega}$, for all $\alpha \in \mathrm{On}$ and $\varphi \in \mathscr{L}_{AV}$.

2.2.8 Approximation Theorem. *Every absolute logic has a Δ_1 approximation function.*

Idea of Proof. It will be shown how to define a Δ_1 approximation function on countable $\mathfrak{M}$. The general case is based on forcing and is omitted here see Burgess [1977] for the details). As $\mathscr{L}$ is absolute, there is a Σ_1 predicate $S(x, y)$ such that for $x, y \subset \omega$ we have

$S(x, y)$ if and only if x codes a model $\mathfrak{M}$, y codes a $\varphi \in \mathscr{L}$ and $\mathfrak{M} \models_{\mathscr{L}} \varphi$.

But Σ_1-properties of reals are Σ_2^1 over ω. By using the standard tree representation of Π_2^1 sets, we find a recursive F such that

$S(x, y)$ if and only if $F(x, y, z)$ wellorders ω in some type $< \omega_1$, for some $z \subset \omega$.

Thus, we have a Σ_1^1 property $\psi(x, y, u)$ of reals such that

$$S(x, y) \leftrightarrow \exists \alpha < \omega_1 \, \exists u((\omega, u) \cong (\alpha, \in|_\alpha) \wedge \psi(x, y, u)).$$

Recall that a Σ_1^1 property of reals can be defined by a game formula. Let $A'(\alpha, \varphi)$ be a formula of $\mathscr{L}_{\infty V}$ which says that for some $u \subset \omega$, $(\omega, u) \cong (\alpha, \in|_\alpha)$ and $\psi(x, y, n)$ holds for the code y of φ and for the code x of the model we are considering. It thus follows that

$$S(\mathfrak{M}, \varphi) \leftrightarrow \exists \alpha < \omega_1 A'(\alpha, \varphi).$$

To get an approximation $A(\alpha, \varphi) \in \mathscr{L}_{\infty\omega}$, we use the fact that $\mathscr{L}_{\infty V}$ permits approximation (see Example 2.2.7). For more details, consult Burgess [1977]. ◻

Remark. A logic with a Δ_1 approximation function is, in fact, absolute if its syntactic operations are Δ_1. In particular, every logic with a Δ_1 approximation function has the downward Löwenheim–Skolem property of Theorem 2.2.2. ◻

2.2.9 Corollary. *Every absolute logic has the Karp property.*

Proof. This is such a basic property of absolute logics that we indicate two proofs here, one using countable approximations and the other approximations in $\mathscr{L}_{\infty\omega}$. Let $\mathscr{L}$ be an absolute logic.

First Proof. Suppose that $\mathfrak{M} \cong_p \mathfrak{N}$ but not $\mathfrak{M} \equiv_{\mathscr{L}} \mathfrak{N}$. The previous sentence is a Σ_1-property of $\mathfrak{M}$ and $\mathfrak{N}$. By Lemma 2.2.1 there are countable approximations

$\mathfrak{M}^s$ and $\mathfrak{N}^s$ of $\mathfrak{M}$ and $\mathfrak{N}$ with the same property. But then $\mathfrak{M}^s \cong \mathfrak{N}^s$, which implies that $\mathfrak{M}^s \equiv_{\mathscr{L}} \mathfrak{N}^s$.

Second Proof. If $\mathfrak{M} \equiv_{\infty\omega} \mathfrak{N}$, then $\mathfrak{M}$ and $\mathfrak{N}$ satisfy the same approximations of $\mathscr{L}$-sentences. Hence, by the approximation theorem (see Theorem 2.2.8), $\mathfrak{M} \equiv_{\mathscr{L}} \mathfrak{N}$. □

2.2.10 Corollary. *If $\mathscr{L}$ is an absolute logic, then $\Delta(\mathscr{L}_{\omega_2\omega}) \nleq \mathscr{L}$.*

Proof. Consider the structures $\mathfrak{M}$ and $\mathfrak{N}$ over the empty vocabulary such that $|\mathfrak{M}| = \aleph_0$ and $|\mathfrak{N}| = \aleph_1$. Now, $\mathfrak{M} \cong_p \mathfrak{N}$, and the model classes $\{\mathfrak{A} \mid \mathfrak{A} \cong \mathfrak{M}\}$, $\{\mathfrak{A} \mid \mathfrak{A} \cong \mathfrak{N}\}$ are $\Delta(\mathscr{L}_{\omega_2\omega})$-definable. If $\Delta(\mathscr{L}_{\omega_2\omega}) \leq \mathscr{L}$, then $\mathfrak{M} \not\equiv_{\mathscr{L}} \mathfrak{N}$, which is contrary to Karp property (see Theorem 2.2.8). □

The rather simple observation given in Corollary 2.2.10 has the following immediate but important consequence: There is no way of extending $\mathscr{L}_{\infty\omega}$ to a logic which obeys the Craig interpolation theorem and which would still be absolute.

2.2.11 Corollary. (i) *Let $\mathscr{L}$ be an absolute logic and $\varphi \in \mathscr{L}$ such that $\varphi \in H_\kappa$ ($\kappa > \omega$). There are $\varphi_\alpha \in \mathscr{L}_{\kappa\omega}$ ($\alpha < \kappa$) such that for any $\mathfrak{M}$ of cardinality $< \kappa$:*

$$\mathfrak{M} \models \varphi \leftrightarrow \bigvee_{\alpha<\kappa} \varphi_\alpha.$$

(ii) *If $\mathscr{L}$ is absolute and $\varphi \in \mathscr{L}$ such that $\varphi \in$ HC, then the number of non-isomorphic countable models of φ is either $\leq \aleph_1$ or $2^{\aleph_0}$.*

Proof. The proof of (i) follows from Theorem 2.2.8 and Levy's reflection principle. We take $\varphi_\alpha = A(\alpha, \varphi)$. The proof of (ii) follows from (i) and the similar result for $\mathscr{L}_{\omega_1\omega}$. □

Definability of Well-Order

2.2.12 Definition. A sentence $\varphi(M, <, \ldots)$ *pins down* an ordinal α if $(M, <)$ is well-ordered in every model of $\varphi(M, <, \ldots)$ and $\varphi(M, <, \ldots)$ has at least one model with $(M, <)$ of order type $\geq \alpha$. A logic $\mathscr{L}$ *pins down* α if some $\varphi \in \mathscr{L}$ does. A logic $\mathscr{L}$ is *strong* if some $\varphi \in \mathscr{L}$ pins down every countable ordinal. A logic $\mathscr{L}$ is *bounded* if no $\varphi \in \mathscr{L}$ is able to pin down every ordinal, otherwise it is *unbounded*.

2.2.13 Examples. (i) $\mathscr{L}_{\omega\omega}(Q_0)$ and $\mathscr{L}_{(\mathrm{HF})^+}$ pin down every $\alpha < \omega_1^{\mathrm{CK}}$.

(ii) $\mathscr{L}_{\omega_1\omega}$ pins down every $\alpha < \omega_1$.

(iii) If $\mathrm{cf}(\kappa) > \omega$, then $\mathscr{L}_{\kappa^+\omega}$ pins down κ^+.

(iv) $\mathscr{L}_{\omega_1 S}$ is strong.

(v) $\mathscr{L}_{\omega_1 G}$ is unbounded.

Remarks on Proofs. Recall that ω_1^{CK} (the Church–Kleene ω_1) is the smallest ordinal which is not the order type of a recursive well-ordering of ω. We have used $(\mathrm{HF})^+$ for the set $L_{\omega_1^{\mathrm{CK}}}$. If $\alpha < \omega_1^{\mathrm{CK}}$, then π_α (as defined in Section 1.1) is in $\mathscr{L}_{(\mathrm{HF})^+}$

and pins down α. It will be proven in Chapter VIII that $\mathscr{L}_{(\mathrm{HF})^+}$ cannot pin down ω_1^{CK}. Similarly, π_α will pin down any $\alpha < \omega_1$ in $\mathscr{L}_{\omega_1\omega}$. To pin down an $\alpha < \omega_1^{\mathrm{CK}}$ in $\mathscr{L}_{\omega\omega}(Q_0)$, we simply write down the standard definition of $(\mathbb{N}, +, \cdot, 0, 1, <)$ in $\mathscr{L}_{\omega\omega}(Q_0)$ and then use the recursive definition of α to define α. Since $\mathbb{N}$ is standard, this will really define α. For a proof of (iii) see Barwise–Kunen [1971]. The example (iv) is based on the observation that a linear ordering $<$ of ω is a well-ordering if and only if

$$\bigvee_{i_0} \bigvee_{i_1} \cdots \bigwedge_{n<\omega} i_{n+1} < i_n.$$

It is known that $\mathscr{L}_{\omega_1 S}$ does not pin down ω_2 but pins down every $\alpha < \omega_2$ if $MA + 2^\omega > \omega_1 + \omega_1 = \omega_1^L$. ☐

In Chapter III it was proven (Theorem III.3.6) that every bounded logic with the downward Löwenheim–Skolem property as in Theorem 2.2.2, is a sublogic of $\mathscr{L}_{\infty\omega}$. The result is interesting enough to be rephrased here as

2.2.14 Theorem. *Suppose that $\mathscr{L}$ is a regular, absolute, and bounded logic. Then $\mathscr{L} \leq \mathscr{L}_{\infty\omega}$.* ☐

2.3. Relative Absoluteness and Generalized Quantifiers

We have observed that $\mathscr{L}_{\omega\omega}(Q_1)$ is not absolute, the reason being that the predicate "x is countable" is itself not absolute. However, even $\mathscr{L}_{\omega\omega}(Q_1^{\mathrm{E}})$ is absolute if $\aleph_1$ is preserved. More generally, by suitably relativizing the notion of absoluteness, we will be able to examine non-absolute logics.

In the following definition, R is an arbitrary predicate of set theory. Recall the characterization of absoluteness as stated earlier in Definition 2.1.1. This characterization is valid in extended languages as well.

2.3.1 Definition. Let $\mathscr{L}$ be a logic, R a predicate of set theory, and T a set theory. We say that $\mathscr{L}$ is *absolute relative to R (and T)* if it is absolute (relative to T) in the extended language $\{\in, R\}$.

2.3.2 Examples. (i) The logic $\mathscr{L}_A(Q)$ is absolute relative to Q and KP(Q) ($=$ KP in the language $\{\in, Q\}$).

(ii) The logic $\mathscr{L}_A^2$ is absolute relative to Pw and KP(Pw) $+$ axiom of power set, where $Pw(x, y) \leftrightarrow y = \mathscr{P}(x)$.

There is a difficulty in proceeding with relative absoluteness in the same way as with absoluteness. The basic method in the theory of absolute logics is to appeal to transitive models of set theory. In the case of relative absoluteness, however, the analogue of transitivity of a model of set theory is the property of being of the form $(M, \in, R \cap M^n)$, where n is the arity of the predicate R. Very little is known of models of this form; and, accordingly, there are very few general results about relatively absolute logics.

In special cases, more specific results obtain. Let Cbl(x) be the predicate of set theory expressing that x is countable (that is, mappable one to one into ω). Clearly, $\mathscr{L}_{\omega\omega}(Q_1)$ is absolute relative to Cbl. But so is $\mathscr{L}_{\omega\omega}(Q_1^{\mathrm{E}})$, where Q_1^{E} says that an equivalence relation has $\aleph_1$ classes. Furthermore, combinations such as $\mathscr{L}_{AG}(Q_1^{\mathrm{E}})$ are absolute relative to Cbl.

2.3.3 Proposition. *Suppose that $\mathscr{L}$ is absolute relative to* Cbl *and $\varphi \in \mathscr{L}$* such that $\varphi \in H_{\omega_2}$. *If φ has a model, then φ has a model of power at most $\aleph_1$.*

Idea of Proof. Let $S(x, y)$ be a predicate Σ_1 relative to Cbl, which defines the truth of $\mathscr{L}$. Then there is a Σ_1-predicate $S'(x, y, z)$ such that $S(x, y) \leftrightarrow S'(x, y, \aleph_1)$ holds in ZFC$^-$ (= ZFC-power set axiom). If $\exists x S(x, \varphi)$, then by Levy's reflection principle $H_{\omega_2} \models \exists x S'(x, \varphi, \aleph_1)$. Whence,

$$S(\mathfrak{M}, \varphi) \quad \text{for some} \quad \mathfrak{M} \in H_{\omega_2}. \quad \square$$

We can improve Proposition 2.3.3 in the direction of Theorem 2.2.2 by studying Kueker's uncountable approximations (see Kueker [1977]).

Hutchinson [1976] showed that the axiomatizability and countable compactness of $\mathscr{L}_{\omega\omega}(Q_1)$ follow from properties of countable models of set theory. Although we will not go into the details, these set-theoretical methods extend naturally to logics that are absolute relative to Cbl.

2.3.4 Example. Let us define

$$\mathrm{Cd}(x) \leftrightarrow \text{"}x \text{ is a cardinal.}$$

Then $\mathscr{L}_A(I)$ is absolute relative to Cd, where I is the equicardinality quantifier $IxyA(x)B(y) \leftrightarrow |A(\cdot)| = |B(\cdot)|$.

To get a result analogous to Proposition 2.3.3 for $\mathscr{L}_A(I)$ we would have to start with an H_κ having the following rather strong reflection property: *If $a \in H_\kappa$, $\varphi(x)$ is Σ_1 relative to* Cd *and $\varphi(a)$ holds, then $H_\kappa \models \varphi(a)$.* Such κ exist, but how large are they? From a standpoint of consistency κ could be 2^ω or κ could be bigger than a measurable cardinal (see Väänänen [1978]).

2.3.5 Proposition. *If $V = L$, then $\mathscr{L}_A^2$ is absolute relative to* Cd.

Proof. If $V = L$, then Pw is Σ_1 relative to Cd:

$$Pw(x, y) \leftrightarrow \exists \kappa \, \exists \alpha \in \kappa (x \in L_\alpha \wedge \mathrm{Cd}(\kappa) \wedge L_\kappa \models Pw(x, y)). \quad \square$$

2.4. Absoluteness and Boolean Extensions

In this discussion we will assume that the reader is familiar with Boolean-valued models of set theory and forcing.

2.4.1 Definition. Let $\mathbb{B}$ be a complete Boolean algebra. A logic $\mathscr{L}$ is *absolute for* $\mathbb{B}$, if for all $\mathfrak{M}$:

$$\mathfrak{M} \models_{\mathscr{L}} \varphi \quad \text{if and only if} \quad [\check{\mathfrak{M}} \models_{\mathscr{L}} \check{\varphi}]^{\mathbb{B}} = 1. \quad \square$$

Remarks. It may be that $[\check{\varphi} \in \mathscr{L}]^{\mathbb{B}} = 1$ although $\varphi \notin \mathscr{L}$, for instance, if $\mathscr{L} = \mathscr{L}_{\omega_1\omega}$ and $\mathbb{B}$ collapses φ to a countable set. It may also happen that $[\check{\mathfrak{M}} \models_{\mathscr{L}} \check{\varphi}]^{\mathbb{B}}$ is neither 0 nor 1. However, for homogenous $\mathbb{B}$ this never happens.

2.4.2 Example. If $\mathscr{L}$ is absolute relative to T and $V^{\mathbb{B}} \models T$, then $\mathscr{L}$ is absolute for $\mathbb{B}$. In particular, $\mathscr{L}_A$, $\mathscr{L}_{AG}$, $\mathscr{L}_{AB}$ are all absolute for any $\mathbb{B}$.

2.4.3 Example. $\mathscr{L}_A(I)$ is absolute relative to all Boolean algebras with c.c.c. This is because every c.c.c. algebra preserves the predicate Cd.

2.4.4 Example. $\mathscr{L}_A(Q_1)$ and $\mathscr{L}_{\omega_1\omega_1}$are absolute for countably closed forcing, since such extensions preserve the predicate Cbl and do not add new countable subsets.

2.4.5 Example. $\mathscr{L}_A$(aa) is absolute for proper forcing (a notion of forcing is *proper* if it does not destroy stationary subset of ω_1; this condition is, of course, weaker than both countable closure and c.c.c.).

2.4.6 Proposition. *There is no extension of $\mathscr{L}_{\omega\omega}(Q_1)$ which provably in* ZFC *satisfies the Craig interpolation property and is provably absolute for* c.c.c. *forcing.*

Idea of Proof. We shall consider tree-like partially ordered structures as defined in Baumgartner *et al.* [1970]. Let $\mathscr{K}_1$ be the class of tree-like structures with an uncountable branch and $\mathscr{K}_2$ the class of tree-like structures homomorphic to the ordering of the rationals. Then $\mathscr{K}_1$ and $\mathscr{K}_2$ are disjoint PC-classes of $\mathscr{L}_{\omega\omega}(Q_1)$. Suppose that θ is a sentence in a logic absolute for c.c.c. forcing, such that $\mathscr{K}_1 \subseteq \text{Mod}(\theta)$ and $\text{Mod}(\theta) \cap \mathscr{K}_2 = \varnothing$. Let T be a Souslin tree (if there is none, we can obtain one by c.c.c. forcing). Suppose that $T \models \theta$. Let $\mathbb{B}$ be a c.c.c. algebra which embeds T homomorphically into the rationals (see *ibid.*). Then $[T \models \theta]^{\mathbb{B}} = 1$—a contradiction. If $T \not\models \theta$, let $\mathbb{B}$ be a c.c.c. algebra which produces a long branch through T. Then $[T \in \mathscr{K}_1]^{\mathbb{B}} = 1$—a contradiction again. $\square$

Considering the great interest in extensions of $\mathscr{L}_{\omega\omega}(Q_1)$—especially those satisfying Craig—the above result is most useful. It shows that such an extension has to be based on something more complicated than cardinality, cofinality, or stationary sets. In this sense, Proposition 2.4.6 is analogous to Corollary 2.2.9.

The following result shows another direction in the applications of forcing to absolute logics. It is but one— and a simple one, at that—in the range of independence results concerning strong abstract logics.

2.4.7 Proposition. *If* CON(ZF), *then it is consistent that every logic* $\mathscr{L}_{\omega\omega}(Q)$, *provably absolute for* c.c.c. *forcing, has Löwenheim number* $<2^{\omega}$.

Proof. We shall construct a c.c.c. algebra $\mathbb{B}$ such that $[\mathscr{L}_{\omega\omega}(I)$ has Löwenheim number $< 2^{\omega}]^{\mathbb{B}} = 1$. The more general result will then follow by compactness.

Let $\mathscr{L}_{\omega\omega}(I)[\tau] = \{\varphi_n | n < \omega\}$, where τ is a vocabulary general enough to give the right Löwenheim number. Let $\mathbb{B}_0 = \{0, 1\}$. If $\mathbb{B}_n$ is defined, let $\mathbb{B}_{n+1} \supseteq \mathbb{B}_n$ be a c.c.c. algebra such that if $[\check{\varphi}$ has a model$]^{\mathbb{B}} > 0$ for some c.c.c. $\mathbb{B} \supseteq \mathbb{B}_n$, then $[\check{\varphi}_n$ has a model of power $< 2^{\omega}]^{\mathbb{B}_{n+1}} = 1$. This is possible in view of the unlimited size of 2^{ω} in c.c.c. extensions. If $\mathbb{B}_0 \subseteq \cdots \subseteq \mathbb{B}_n \cdots$ is defined for $n < \omega$, let $\mathbb{B}$ be the direct limit of $(\mathbb{B}_n)_{n<\omega}$. Then $\mathbb{B}$ has c.c.c. Suppose now that $[\check{\varphi}_n$ has a model$]^{\mathbb{B}} > 0$. Then also $[\check{\varphi}_n$ has a model of power $<2^{\omega}]^{\mathbb{B}_{n+1}} = 1$, by construction. Hence, $[\check{\varphi}_n$ has a model of power $<2^{\omega}]^{\mathbb{B}} = 1$, by the absoluteness of $\mathscr{L}_{\omega\omega}(I)$ for $\mathbb{B}_{n+1}$ and $\mathbb{B}$. ☐

Historical and Bibliographical Remarks. The definition of an absolute logic goes back to Barwise [1972a]. The Löwenheim–Skolem theorem for absolute logics was first proven in the weaker form (see Corollary 2.2.3) in Barwise [1972a], and the full result Theorem 2.2.2 appeared in Barwise [1974b] using ideas from Kueker [1972]. The results Lemma 2.2.4 and Corollary 2.2.5 are from Kueker [1977]. The notion of approximation was developed for game and Vaught formulae by Vaught [1973b] and has been since generalized for all absolute logics in Burgess [1977], where Theorem 2.2.8 and its corollary is proved. Corollaries 2.2.9 and 2.2.10 were already proven by Barwise [1972a]. The reader is referred to Ellentuck [1975] and Burgess [1978] for results on Souslin logic $\mathscr{L}_{AS}$. The characterization given in Theorem 2.2.13 of $\mathscr{L}_{\infty\omega}$ is due to Barwise [1972a]. Proposition 2.4.6 is due to S. Shelah, while Proposition 2.4.7 is from Väänänen [1980b], where other related results are also proven.

3. *Characterizations of Abstract Logics*

In this section we shall relate the model-theoretic notion of adequacy to truth and the set-theoretic notion of relative absoluteness. In rough terms, we show that if a logic $\mathscr{L}$ is sufficiently strong and is sufficiently absolute, then $\mathscr{L}$ is adequate to truth in $\mathscr{L}$. As applications, we get rather strong results on Δ-extensions of various logics—the main results of this chapter.

3.1. A General Framework

In order that a logic be adequate to truth in a given logic, it must have enough expressive power to enable it to capture the truth definition of the logic. In our set-theoretical approach, this leads to the following new notion:

3.1.1 Definition. Let $\mathscr{L}$ be a logic and R a predicate of set theory. We say that $\mathscr{L}$ *captures* R if there is an $\mathrm{RPC}_{\mathscr{L}}$-class $\mathscr{K}$ of set-theoretical structures such that

(C1) For any set a there is a transitive set M such that $a \in M$, and $(M, \in|_M) \in \mathscr{K}$.
(C2) If $\mathfrak{M} \in \mathscr{K}$ and $\mathfrak{M} \models \pi_{a_i}(m_i)$ $(i = 1 \dots n)$, then $R(a_1, \dots, a_n)$ if and only if $\mathfrak{M} \models R(m_1, \dots, m_n)$.

Explanations. Intuitively, in the above $\mathscr{K}$ is a class of transitive models of set theory. Condition (C1) says only that $\mathscr{K}$ is non-trivial. (C2) is the critical condition and asserts that models of $\mathscr{K}$ preserve R upwards and downwards.

3.1.2 Example. Let $R(x)$ be a predicate which is Δ_1 in KP-Inf. Then $\mathscr{L}_{\omega\omega}$ captures R. To see this, let $\mathscr{K}$ be the $\mathrm{EC}_{\mathscr{L}_{\omega\omega}}$-class of models of a large finite part of KP-Inf. Condition (C1) is then true, since $H_\kappa \in \mathscr{K}$ for all κ. In order to verify (C2), we let $\mathfrak{M} \in \mathscr{K}$ and $\mathfrak{M} \models \pi_a(m)$. We may assume that the well-founded part $\mathfrak{N}$ of $\mathfrak{M}$ is a standard $\in$-structure and thus that $m = a$ also. By the truncation lemma (see Barwise [1975], p. 73), $\mathfrak{N} \in \mathscr{K}$ holds. And, by the absoluteness of Δ_1-predicates we have

$$\begin{aligned} R(a) \quad &\text{if and only if} \quad \mathfrak{N} \models R(a) \\ &\text{if and only if} \quad \mathfrak{M} \models R(a). \end{aligned}$$

Before we examine more examples of capture, let us prove the main result of this section. In this theorem, we will again assume that the elements of all models considered are urelements. In fact, we do not need this convention before Theorem 3.4.15 except in the proof of

3.1.3 Theorem. *Suppose that $\mathscr{L}$ and $\mathscr{L}'$ are logics. Suppose also that $\mathscr{L}$ captures the predicate $S(x, y)$ such that*

$$S(\mathfrak{M}, \varphi) \quad \textit{if and only if} \quad \varphi \in \mathscr{L}' \quad \textit{and} \quad \mathfrak{M} \models_{\mathscr{L}'} \varphi,$$

for all φ and all $\mathfrak{M}$. Then $\mathscr{L}$ is adequate to truth in $\mathscr{L}'$.

Proof. Suppose that $\mathscr{K}$ witnesses the capture of $S(x, y)$. Let τ'_{set} (disjoint form τ_{set}) be the vocabulary of $\mathscr{K}$. In order to demonstrate the adequacy of $\mathscr{L}$ to truth in $\mathscr{L}'$, we shall begin with an arbitrary type τ. As a means of simplifying notation, we will assume that τ contains only one binary predicate symbol R and one sort s. We let $\tau^+ = [\tau, \tau_{\mathrm{set}}, T, \tau']$, where τ' contains τ'_{set} and three constant symbols m, n, and r of the sort of τ'_{set}. Let $S'(x, y)$ be the predicate $S(x, y)$ in vocabulary τ'_{set}, and let $\mathscr{K}'$ be the class of τ^+-structures $\mathfrak{M}' = [\mathfrak{M}, \mathfrak{B}, T, \mathfrak{N}, m, n, r, f]$ such that all the following hold:

(a) $\mathfrak{N} \in \mathscr{K}$.
(b) $\mathfrak{B} \subseteq_{\text{end}} \mathfrak{N}$.
(c) $\mathfrak{N} \models$ "m is a structure (n, r) of type $\langle 2 \rangle$ and n is a set of urelements."
(d) $\forall x(x \in M \leftrightarrow \mathfrak{N} \models f(x) \in n)$ & "f is 1–1 on M."
(e) $\forall x, y \in M(R(x, y) \leftrightarrow \mathfrak{N} \models (f(x), f(y)) \in r$.
(f) $\forall x \in B(T(x) \leftrightarrow S'(m, x))$.

Intuitively, the following idea is behind $\mathscr{K}'$. $\mathfrak{B}$ is the syntax set of $\mathscr{L}'$, and $\mathfrak{N}$ is a larger set-theoretical universe within which $S(x, y)$ is captured by $\mathscr{K}$. In view of the choice of $S(x, y)$, this essentially entails that $\models_{\mathscr{L}'}$ be captured within $\mathfrak{N}$. Inside the universe $\mathfrak{N}$ m is a structure (n, r) of the same type as the structure $\mathfrak{M}$, the true sentences of which we try to define. Conditions (d) and (e) assert that m looks exactly like $\mathfrak{M}$. Finally, condition (f) defines the truth-predicate T in the obvious way.

Clearly, $\mathscr{K}'$ is an $\text{RPC}_{\mathscr{L}}$-class, so that it is RPC-defined by some $\theta \in \mathscr{L}$. Now, in order to prove (AT1), we let $\mathfrak{M} \in \text{Str}[\tau]$ be given. By (C1), there is a transitive set N such that $A', \mathfrak{M} \in N$ and $\mathfrak{N} = (N, \in|_N) \in \mathscr{K}$. Let us examine the structure

$$\mathfrak{M}' = [\mathfrak{M}, \mathfrak{A}, \text{Th}_{\mathscr{L}'}(\mathfrak{M}), \mathfrak{N}, n, m, r, f],$$

where n, m, r, and f are defined so as to make conditions (a) through (e) true. Now, also condition (f) holds, since by (C2) we have

$$\begin{aligned} \mathfrak{N} \models S'(\mathfrak{M}, \varphi) \quad &\text{if and only if} \quad S(\mathfrak{M}, \varphi) \\ &\text{if and only if} \quad \varphi \in \text{Th}_{\mathscr{L}'}(\mathfrak{M}). \end{aligned}$$

Thus, $\mathfrak{M}' \in \mathscr{K}'$ and therefore expands to a model of θ. This ends the proof of (AT1).

As to the proof of (AT2), we suppose that

$$[\mathfrak{M}, \mathfrak{B}, T, \mathfrak{N}, m, n, r, f, \ldots] \models \theta \wedge \pi_\varphi(b),$$

where $\varphi \in A'$ and $b \in B$. Furthermore, let $\mathfrak{N}'$ be the well-founded part of $\mathfrak{N}$ and i a transitive collapse of $\mathfrak{N}$, $i: \mathfrak{N} \to (N, \in)$. As $\mathfrak{B} \models \pi_\varphi(b)$, we have $b \in N'$ and $i(b) = \varphi$. Since n is a set of urelements in $\mathfrak{N}$, $m \in N'$ and $i(m)$ is a structure $\mathfrak{M}'$ isomorphic to $\mathfrak{M}$. Now we may reason as follows:

$$\begin{aligned} b \in T \quad &\text{if and only if} \quad \mathfrak{N} \models S'(m, b), \\ &\text{if and only if} \quad S(\mathfrak{M}', \varphi), \\ &\text{if and only if} \quad \varphi \in \mathscr{L}' \quad \text{and} \quad \mathfrak{M}' \models_{\mathscr{L}'} \varphi, \\ &\text{if and only if} \quad \varphi \in \mathscr{L}' \quad \text{and} \quad \mathfrak{M} \models_{\mathscr{L}'} \varphi. \end{aligned}$$

This ends the proof of (AT2). □

3.1.4 Corollary. *Suppose $\mathscr{L}$ captures the predicate $x \in \mathscr{K}$, where $\mathscr{K}$ is a model class. Then $\mathscr{K}$ is $\Delta(\mathscr{L})$-definable.*

3.1.5 Corollary (Characterization of $\mathscr{L}_{\omega\omega}$). *$\mathscr{L}_{\omega\omega}$ is the only logic which is represented on* HF *and is absolute relative to* KP-Inf.

Proof. $\mathscr{L}_{\omega\omega}$ certainly has the stated property by Example 2.1.2. On the other hand, if $\mathscr{L}$ is represented on HF and absolute relative to KP-Inf, then by Example 3.1.2 and Theorem 3.1.3, we have that $\mathscr{L}_{\omega\omega}$ is adequate to truth in $\mathscr{L}$. Whence, by Corollary 1.1.12, it follows that $\mathscr{L} \leq \Delta(\mathscr{L}_{\omega\omega}) = \mathscr{L}_{\omega\omega}$. ☐

Observe that the proof of Corollary 3.1.5 actually gives the stronger conclusion that $\mathscr{L}_{\omega\omega}$ is adequate to truth in any logic that is absolute relative to KP-Inf.

3.1.6 Remarks. (i) The proof of Theorem 3.1.3 also gives a sufficient condition for simple adequacy to truth. All we have to know in addition about capture is that the transitive set M in (C1) can be chosen to be of cardinality at most $|a| \cdot \aleph_0$.

(ii) If $\mathscr{L}$ captures S in the weaker sense that "if" holds in (C2) rather than "if and only if," then we still get $\mathscr{L}$ semi-adequate to the truth in $\mathscr{L}'$ in the sense of Remark 1.1.9, and we still obtain $\mathrm{RPC}_{\mathscr{L}}$-definability of $\mathscr{K}$ in Corollary 3.1.4.

(iii) Some logics may be defined with respect to a parameter (for example α is the parameter for $\mathscr{L}_{\omega\omega}(Q_\alpha)$). There is no essential difficulty in having a parameter p in Theorem 3.1.3 and Corollary 3.1.4, but in Definition 3.1.1(C2), we must assume, of course, that every $\mathfrak{M} \in \mathscr{K}$ contains a set q such that $\mathfrak{M} \models \pi_p(q)$.

(iv) To prove Corollary 3.1.4 we do not need the full strength of "capture." Thus, in (C1), we can restrict to $a = (\mathfrak{M}, p)$, where $\mathfrak{M}$ is an arbitrary model of the type of $\mathscr{K}$ and p is a parameter in the definition of $\mathscr{K}$ (if any).

(v) There is a certain uniformity in the way τ^+ is obtained from τ above. More precisely, the conclusion of Theorem 3.1.3 can be improved to: *$\mathscr{L}$ is effectively adequate to truth in $\mathscr{L}'$.*

3.2. Absolute Logics Revisited

The notion of capture is used to prove the following theorem concerning strict absoluteness (Definition 2.1.1).

3.2.1 Theorem. *$\mathscr{L}_{\omega\omega}(Q_0)$ is adequate to truth in any strictly absolute logic.*

Proof. Suppose that $\mathscr{L}$ is strictly absolute. Thus, *there is a predicate $S(x, y)$ Δ_1 in* KP, *such that*

$$S(\mathfrak{M}, \varphi) \quad \text{if and only if} \quad \varphi \in \mathscr{L} \quad \text{and} \quad \mathfrak{M} \models_{\mathscr{L}} \varphi,$$

for all $\mathfrak{M}$ and φ. An argument similar to the one used in Example 3.1.2 shows that $\mathscr{L}_{\omega\omega}(Q_0)$ captures $S(x, y)$. By Theorem 3.1.3, $\mathscr{L}_{\omega\omega}(Q_0)$ is adequate to truth in $\mathscr{L}$. $\square$

Observe that Theorem 3.2.1 does not allow us to conclude that $\mathscr{L} \leq \Delta(\mathscr{L}_{\omega\omega}(Q_0))$ if $\mathscr{L}$ is strictly absolute, since $\mathscr{L}$ may not be represented on the same syntax set. However, we do have the following important result:

3.2.2 Theorem. $\Delta(\mathscr{L}_{\omega\omega}(Q_0)) \equiv \mathscr{L}_{(\mathrm{HF})^+}$.

Proof. It suffices to prove that $\mathscr{L}_{(\mathrm{HF})^+} \leq \Delta(\mathscr{L}_{\omega\omega}(Q_0))$, as it is well known that $\mathscr{L}_{(\mathrm{HF})^+}$ satisfies the Craig interpolation theorem. As $\mathscr{L}_{(\mathrm{HF})^+}$ is strictly absolute, there is a predicate $S(x, y)$, Δ_1 in KP, such that

$$S(\mathfrak{M}, \varphi) \quad \text{if and only if} \quad \varphi \in \mathscr{L} \quad \text{and} \quad \mathfrak{M} \models_{\mathscr{L}} \varphi.$$

Let $\varphi \in \mathscr{L}_{(\mathrm{HF})^+}$. As an element of $\mathscr{L}_{\omega_1^{\mathrm{CK}}}$, the set φ is definable by a predicate which is Δ_1 in KP (see, for example, Barwise [1975], Section II.5.14). Thus, the model class $\mathrm{Mod}(\varphi)$ is definable by a predicate, Δ_1 in KP. By Example 3.1.2, $\mathscr{L}_{\omega\omega}(Q_0)$ captures this predicate; and, by Corollary 3.1.4, $\mathrm{Mod}(\varphi)$ is $\Delta(\mathscr{L}_{\omega\omega}(Q_0))$-definable, as desired. $\square$

The above theorem can be relativized to a parameter in the following way. Recall what was said about logics defined with respect to a parameter in Remark 3.1.6(iii).

If $X \subseteq \omega$, let Q_X be the generalized quantifier associated with

$$\{\mathfrak{A} \mid \mathfrak{A} \cong (\omega, <, X)\}$$

and HYP(X) the smallest admissible set containing X as an element.

3.2.3 Theorem. *If* $X \subseteq \omega$, *then* $\Delta(\mathscr{L}_{\omega\omega}(Q_X)) \equiv \mathscr{L}_{\mathrm{HYP}(X)}$. *Hence,*

$$\mathscr{L}_{\omega_1\omega} \equiv \Delta(\mathscr{L}_{\omega\omega}(Q_X)_{X \subseteq \omega}).$$

Proof. As $\mathscr{L}_{\mathrm{HYP}(X)}$ satisfies Craig, it is again enough to pick $\varphi \in \mathscr{L}_{\mathrm{HYP}(X)}$ and show that $\mathscr{L}_{\omega\omega}(Q_X)$ captures the predicate $x \in \mathrm{Mod}(\varphi)$, as we did in the proof of Theorem 3.2.2. Note that $\mathscr{L}_{\mathrm{HYP}(X)}$ is strictly absolute and $\mathrm{Mod}(\varphi)$ is therefore definable by a predicate, Δ_1 in KP, with φ as a parameter. As an element of HYP(X), φ is itself definable by a predicate Δ_1 in KP, with X as a parameter (see Barwise [1975], Section IV.1.6). Using $\mathscr{L}_{\omega\omega}(Q_X)$, it is now easy to capture the predicate $x \in \mathrm{Mod}(\varphi)$: We simply proceed as in Example 3.1.2 and use Q_X to capture the parameter X. $\square$

We shall apply Theorem 3.2.1 now to prove the main result of Barwise [1972a]:

3.2.4 Theorem. *Let* A *be an admissible set containing* ω, *and let* $\mathscr{L}$ *be a strictly absolute logic the syntax of which is represented on* A. *Then* $\mathscr{L} \leq \mathscr{L}_A$.

Proof. Let $\mathscr{L}$ be a strictly absolute logic represented on A, and let $\varphi \in \mathscr{L}$. Suppose, for a *reductio ad absurdum*, that $\mathrm{Mod}(\varphi)$ is not definable in $\mathscr{L}_A$. Then the following holds:

$$\exists A_0\ \exists \varphi_0 \in A_0((A_0, \in|_{A_0}) \models KP \wedge \varphi_0 \in \mathscr{L} \wedge \forall \psi \in \mathscr{L}_{A_0} \neg(\varphi_0 \leftrightarrow \psi)).$$

This can be written in Σ-form. Whence, by Levy's reflection lemma, it holds in HC. Thus, we have a countable admissible set A_0 such that some $\varphi_0 \in \mathscr{L}$ is in A_0 but is not definable in $\mathscr{L}_{A_0}$. Let $\mathscr{L}'$ be the strictly absolute sublogic of $\mathscr{L}$ containing those sentences of $\mathscr{L}$ which are in A_0. By Theorem 3.2.1, $\mathscr{L}_{\omega\omega}(Q_0)$, and hence also $\mathscr{L}_{A_0}$, is adequate to truth in $\mathscr{L}'$. By Corollary 1.1.12, $\mathscr{L}' \leq \Delta(\mathscr{L}_{A_0})$. As A_0 is a countable admissible set, $\Delta(\mathscr{L}_{A_0}) \equiv \mathscr{L}_{A_0}$, and hence, $\mathscr{L}' \leq \mathscr{L}_{A_0}$ also. But this contradicts the assumption that $\mathrm{Mod}(\varphi_0)$ is not definable in $\mathscr{L}_{A_0}$. ∎

Application. The logics $\mathscr{L}_{AG}$, $\mathscr{L}_{AV}$, $\mathscr{L}_{AB}$ and other unbounded absolute logics are not strictly absolute.

Theorem 3.2.4 is an important characterization of admissible languages $\mathscr{L}_A$. It uses essentially the Souslin–Kleene property of $\mathscr{L}_A$ for countable A. The lack of this property is the main obstacle to proofs of a similar result for other logics. The following is a local version:

3.2.5 Corollary. *Let A be an admissible set containing ω. A model class is definable in $\mathscr{L}_A$ if and only if it is definable by a predicate Δ_1 in* KP, *with parameters in A.* ∎

3.2.6 Corollary (Characterization of $\mathscr{L}_A$). *If $\omega \in A$, then $\mathscr{L}_A$ is the strongest strictly absolute logic represented on A.* ∎

3.3. Unbounded Logics

Recall that a logic L is unbounded if L contains a sentence which pins down every ordinal or, equivalently, if the notion of well-ordering is RPC in L.

3.3.1 Lemma. *An unbounded logic captures every Δ_1 predicate.*

Proof. The capturing RPC-class $\mathscr{K}$ asserts that $\mathfrak{M}$ is a well-founded model of the sentence expressing the Δ_1-definability of the predicate in question. Condition (C2) then follows from the absoluteness of Δ_1 predicates in transitive domains. ∎

3.3.2 Theorem. *Any unbounded logic is adequate to truth in any absolute logic.*

Proof. The claim follows from the definition of absolute logics (see Lemma 3.3.1 and Theorem 3.1.3). ∎

It would now be in order to search for the simplest possible unbounded logic. Unfortunately, there is no natural choice. The simplest logic in which the notion of well-ordering is EC-definable (rather than RPC-definable) is the logic $\mathscr{L}_{\omega\omega}(W)$, where

$$WxyA(x, y) \leftrightarrow A(\cdot, \cdot)\text{well-orders its field.}$$

But, for example, the unbounded logic $\mathscr{L}_{\omega\omega}(I)$ does not contain $\mathscr{L}_{\omega\omega}(W)$ (see Lindström [1966]).

3.3.3 Corollary. (i) *If $\mathscr{L}$ is absolute, then $\mathscr{L} \leq \Delta(\mathscr{L}_A(W))$.*

(ii) *The logics $\mathscr{L}_A(W)$, $\mathscr{L}_{AG}$, $\mathscr{L}_{AV}$, $\mathscr{L}_{AB}$ and all unbounded absolute logics represented on A are Δ-equivalent.*

(iii) *A model class is definable in $\Delta(\mathscr{L}_A(W))$ if and only if it is definable by a Δ_1 predicate with parameters in A.* □

Remark. We can replace "unbounded" by "strong" in Theorem 3.3.2 and Corollary 3.3.3(ii) if the syntax set A is assumed to be contained in HC. Respectively, if $A \subseteq \text{HC}$, $\mathscr{L}_{AS}$ can be added to the list of logics in Corollary 3.3.3(ii).

It is interesting to observe that there is no strongest absolute logic (this follows from Theorem 3.4.7 below). The family of absolute logics divides into two categories: The first consists of sublogics of $\mathscr{L}_{\infty\omega}$, and the second of Δ-equivalent logics (up to difference of syntax set).

There is an important relation between descriptive set theory and infinitary logic. In order to see this, let us restrict ourselves to countable structures and logics represented on HC for a moment. A class $\mathscr{K}$ of countable models can be viewed as a set of reals, and it thus is meaningful to ask, for example, whether $\mathscr{K}$ is Borel or not. If $\mathscr{K}$ is invariant (that is, closed under isomorphisms), then Diagram 1 shows the equivalence of $\mathscr{K}$ being definable on a level in topology and $\mathscr{K}$ being definable in an infinitary logic. The reader is referred to Vaught [1973] for details on these equivalences. Observe, however, that on the last row, we can replace $\mathscr{L}_{\omega_1 V}$ by any unbounded absolute logic (by Corollary 3.3.3(ii)). Thus, every $\mathscr{K}$ definable in an absolute logic is Δ^1_2. Burgess [1977] showed that the question (posed by Vaught) of whether the converse holds, that is, of whether every Δ^1_2 $\mathscr{K}$ is definable in an absolute logic, is independent of ZFC.

Topology	Infinitary Logic
Borel	$\mathscr{L}_{\omega_1\omega}$
Analytic	PC in $\mathscr{L}_{\omega_1\omega}$
C-set	$\mathscr{L}_{\omega_1 V}$
Σ^1_2	PC in $\mathscr{L}_{\omega_1 V}$

Diagram 1

3.4. Relatively Absolute Logics Revisited

As we remarked earlier, the role that transitive models of set theory play in the theory of absolute logics is taken up by models of the form $(M, \in \cap M^2, R \cap M^n)$ in the theory of relatively absolute logics (see Section 2.3). Getting a hold on $R \cap M^n$ is no easier than making sure that $\in$ is the true $\in$. While unboundedness is a good means for $\in$, we need the following relativized version of pinning down for $R \cap M^n$:

3.4.1 Definition. Let $\mathscr{L}$ be a logic and $R(x_1, \ldots, x_n)$ a predicate. We say that $\mathscr{L}$ *pins down* $R(x_1, \ldots, x_n)$ if there is an $\mathrm{RPC}_{\mathscr{L}}$-class $\mathscr{K}$ such that

$$\mathfrak{N} \in \mathscr{K} \quad \text{if and only if} \quad \mathfrak{N} \cong (N, \in \cap N^2, R \cap N^n), \text{ for some transitive set } N.$$

3.4.2 Examples. (i) $\mathscr{L}_{\omega\omega}(I)$ pins down Cd.

(ii) $\mathscr{L}^2_{\omega\omega}$ pins down *Pw*.

(iii) $\mathscr{L}_{\omega\omega}(H)$ pins down *Pw*.

(iv) If $V = L$, then $\mathscr{L}_{\omega\omega}(I)$ pins down *Pw*.

(v) $\mathscr{L}_{\omega\omega}(Q)$ pins down Q, if unbounded.

3.4.3 Lemma. *If a logic $\mathscr{L}$ pins down a predicate R, then $\mathscr{L}$ is unbounded and captures R. Moreover, $\mathscr{L}$ captures every Δ_1 predicate in the extended language $\{\in, R\}$.* □

Proof. The claim concerning unboundedness and capture is trivial. For the second claim, let $S(x)$ be Δ_1 in the language $\{\in, R\}$. Let $\mathscr{K}$ witness the pinning down of R and let $\mathscr{K}'$ be $\mathscr{K}$ intersected with a statement witnessing the Δ_1 nature of $S(x)$. By reflection, $\mathscr{K}'$ satisfies (C1). Condition (C2) follows from the absoluteness of Δ_1 predicates in end extensions. □

3.4.4 Theorem. *If $\mathscr{L}$ pins down R and $\mathscr{L}'$ is absolute relative to R, then $\mathscr{L}$ is adequate to truth in $\mathscr{L}'$.*

Proof. Let $S(x, y)$ be a predicate, Δ_1 in the extended language $\{\in, R\}$, such that

$$S(\mathfrak{M}, \varphi) \quad \text{if and only if} \quad \varphi \in \mathscr{L}' \quad \text{and} \quad \mathfrak{M} \models_{\mathscr{L}'} \varphi,$$

for all $\mathfrak{M}$ and φ. Now, $\mathscr{L}$ captures $S(x, y)$ by Lemma 3.4.3. Thus, Theorem 3.1.3 gives the desired result. □

3.4.5 Corollary. (i) *If $\mathscr{L}$ pins down R, $\mathscr{L}'$ is absolute relative to R and $A' \subseteq A$, then $\mathscr{L}' \leq \Delta(\mathscr{L})$.*

(ii) *If $\mathscr{L}$ is absolute relative to R and pins down R, then a model class is definable in $\Delta(\mathscr{L})$ ($\mathrm{RPC}_{\mathscr{L}}$) if and only if it is Δ_1 (Σ_1) definable in the extended language $\{\in, R\}$, with parameters in A.* □

3.4.6 Examples. (i) The logic $\mathscr{L}_A(I)$ is absolute relative to Cd and pins down Cd. Therefore:
If $\mathscr{L}$ pins down Cd, then $\mathscr{L}_A(I) \leq \Delta(\mathscr{L})$.
If $\mathscr{L}$ is absolute relative to Cd, then $\mathscr{L} \leq \Delta(\mathscr{L}_A(\mathrm{I}))$.
(ii) The logic $\mathscr{L}_A^2$ is absolute relative to Pw and pins down Pw. Then, using the fact that $\Delta_2 = \Delta_1(Pw)$, we get: A model class is definable in $\Delta(\mathscr{L}_A^2)$ if and only if it is Δ_2 with parameters in A.

The above results lead naturally to the following question: When is $\Delta(\mathscr{L})$ absolute? We can answer this for many unbounded $\mathscr{L}$, but the problem remains unsettled for most bounded $\mathscr{L}$.

3.4.7 Theorem. *If $\mathscr{L}$ pins down R and $\mathscr{L}'$ is absolute relative to R, then $\Delta(\mathscr{L}) \nleq \mathscr{L}'$.*

Proof. Let $S(x, y)$ be a Δ_1 predicate in the extended language $\{\in, R\}$ such that

$$S(\mathfrak{M}, \varphi) \quad \text{if and only if} \quad \varphi \in \mathscr{L}' \quad \text{and} \quad \mathfrak{M} \models_{\mathscr{L}'} \varphi,$$

for all $\mathfrak{M}$ and φ. Let $\mathscr{K}$ be the class of models $\mathfrak{B}$ such that

$$\mathfrak{B} \cong (B, \in \cap B^2), \text{ where } B = \mathrm{TC}(\{a\}) \text{ for some } a \text{ such that } \neg S(\mathfrak{B}, a).$$

$\mathscr{K}$ is clearly, Δ_1 in the language $\{\in, R\}$. By Corollary 3.4.5(ii), $\mathscr{K}$ is definable in $\Delta(\mathscr{L})$. Suppose that $\mathscr{K}$ were definable by some $\varphi \in \mathscr{L}'$ and let $\mathfrak{N} = (N, \in \cap N^2)$, where $N = \mathrm{TC}(\{\varphi\})$. Then

$$\begin{aligned} S(\mathfrak{N}, \varphi) \quad &\text{if and only if} \quad \mathfrak{N} \in \mathscr{K} \\ &\text{if and only if} \quad \neg S(\mathfrak{N}, \varphi). \end{aligned}$$

This contradiction shows that $\mathscr{K}$ is not $\mathrm{EC}_{\mathscr{L}'}$ and the proof is thus completed. □

3.4.8 Examples. (i) $\Delta(\mathscr{L}_{\omega\omega}(W)) \nleq \mathscr{L}_{\infty B}$.
(ii) $\Delta(\mathscr{L}_{\omega\omega}(I)) \nleq \mathscr{L}_{\infty\omega}(I)$.
(iii) $\Delta(\mathscr{L}_{\omega\omega}(H)) \nleq \mathscr{L}^2_{\infty\omega}$.
(iv) $\Delta(\mathscr{L}_{\omega\omega}(Q)) \nleq \mathscr{L}_{\infty\omega}(Q)$, if $\mathscr{L}_{\omega\omega}(Q)$ is unbounded.

Theorem 3.4.7 shows that if $\mathscr{L}$ pins down R, then $\Delta(\mathscr{L})$ cannot be extended to a logic absolute relative to R; even less is $\Delta(\mathscr{L})$ itself absolute relative to R.

3.4.9 Examples. The logic $\Delta(\mathscr{L}_{\omega\omega}(W))$ is not absolute, and neither is $\Delta(\mathscr{L}_{\omega_1 G})$ nor $\Delta(\mathscr{L}_{\omega_1 V})$. Moreover, the logic $\Delta(\mathscr{L}_{\omega\omega}(I))$ is not absolute relative to Cd, nor is the logic $\Delta(\mathscr{L}^2_{\omega\omega})$ absolute relative to Pw.

We observed in Corollary 1.2.4 that $\Delta(\mathscr{L}_{\omega\omega}(Q_0))$ is not selfadequate but is equivalent to one on a larger syntax set. The results we have here are stronger. For example $\Delta(\mathscr{L}_{\omega\omega}(W))$ is not absolute even if represented on a larger syntax set.

Application. If $\mathscr{L}$ is unbounded, then there are no generalized quantifiers $Q_1, \ldots, Q_n$ and no p.r. closed set A such that

$$\Delta(\mathscr{L}) \equiv \mathscr{L}_A(Q_1, \ldots, Q_n).$$

For otherwise $\mathscr{L}_A(Q_1, \ldots, Q_n)$ would be a Δ-logic which pins down and is absolute relative to $Q_1, \ldots, Q_n$.

Iterated Δ-extensions

3.4.10 Definition. Let $\mathscr{L}$ be a logic and let $\Sigma_0(\mathscr{L})$ and $\Pi_0(\mathscr{L})$ mean the same as $EC_{\mathscr{L}}$. Moreover, $\Sigma_{n+1}(\mathscr{L})$ means $RPC_{\Pi_n(\mathscr{L})}$ and $\Pi_{n+1}(\mathscr{L})$ means $RPC_{\Sigma_n(\mathscr{L})}$. Finally,

$$\Delta_{n+1}(\mathscr{L}) \text{ means } \Sigma_{n+1}(\mathscr{L}) \cap \Pi_{n+1}(\mathscr{L}).$$

Explanation. Here we have a hierarchy of RPC-definability defined very much like the hierarchy of Σ_n predicates of set theory or the hierarchy of Σ_n^1-sets in recursion theory. We treat $\Sigma_n(\mathscr{L})$, $\Pi_n(\mathscr{L})$, and $\Delta_n(\mathscr{L})$ as if they were logics, which, in fact, they actually are, as one can easily see. Of course, $\Sigma_n(\mathscr{L})$ and $\Pi_n(\mathscr{L})$ are not closed under negation. However, $\Delta_n(\mathscr{L})$ is closed, if $\mathscr{L}$ is. Moreover, it is easy to see that each $\Delta_n(\mathscr{L})$ is Δ-closed.

3.4.11 Theorem. *Let $n > 1$. A model class is definable in $\Delta_n(\mathscr{L}_A)$ if and only if it is Δ_n-definable in set theory, with parameters in A.* ☐

Remark. If $\mathscr{L}_A$ is replaced by $\mathscr{L}_{AG}$, the result also holds for $n = 1$.

Proof of Theorem 3.4.11. We use induction on n. For $n = 2$, the claim is true since $\mathscr{L}_A^2 \leq \Sigma_2(\mathscr{L}_A)$ implies that $\Delta(\mathscr{L}_A^2) \leq \Delta_2(\mathscr{L}_A)$ holds, and $\Sigma_1(\mathscr{L}_A) \leq \Delta(\mathscr{L}_A^2)$ implies that $\Delta_2(\mathscr{L}_A) \leq \Delta(\mathscr{L}_A^2)$ holds. Assume, then, that the claim holds for n. Let $\mathscr{K}$ be a Σ_{n+1}-definable model class, and let R be a Π_n predicate such that $\mathscr{K}$ is Σ_1 in the extended language $\{\in, R\}$. Moreover, let $\mathscr{L}$ be the logic $\mathscr{L}_A(Q)$, where Q is the quantifier associated with the model class

$$\{\mathfrak{N} \mid \mathfrak{N} \cong (N, \in \cap N^2, R \cap N^m) \text{ for some transitive set } N\}.$$

Then $\mathscr{L}$ is absolute relative to R and pins down R. By Corollary 3.4.5(ii), $\mathscr{K}$ is $RPC_{\mathscr{L}}$-definable. As a Π_n-definable model class, Q is $\Sigma_n(\mathscr{L})$-definable. The converse is similar. ☐

3.4.12 Corollary. $\Delta_{n+1}(\mathscr{L}_A) \equiv \Delta_n(\mathscr{L}_A^2)$, *for $n > 0$.*

The logics $\Delta_n(\mathscr{L}_A)$ are extremely powerful and gradually exhaust all logics definable in set theory. In fact, $\Delta_3(\mathscr{L}_A)$ already contains most familiar logics.

Second-Order Logic

We can construe second-order logic $\mathscr{L}_A^2$ as the result of iteratively closing $\mathscr{L}_A$ under the PC-operation. Therefore, let us examine the extent to which the above results hold for PC in place of RPC. To this purpose, we now consider

3.4.13 Definition. Let $\varphi(x_0, \ldots, x_n)$ be a formula of set theory. The expressions

$$\exists x_0(\mathrm{HC}(x_0) \leq \mathrm{HC}(x_1 \cup \cdots \cup x_n) \wedge \varphi(x_0, \ldots, x_n))$$

and

$$\forall x_0(\mathrm{HC}(x_0) \leq \mathrm{HC}(x_1 \cup \cdots \cup x_n) \rightarrow \varphi(x_0, \ldots, x_n)),$$

where $\mathrm{HC}(x) = \max(\aleph_0, |\mathrm{TC}(x)|)$, are called *flat quantifiers*. The class of *flat formulae* of set theory is the smallest class of formulae which contains Σ_0-formulae and which is closed under $\wedge$, $\vee$, $\neg$ and flat quantification.

The following characterization of second-order logic can be proven by slightly modifying the proof of Theorems 3.3.2 and 3.4.11.

3.4.14 Theorem. (i) *Second-order logic is simply adequate to truth in any logic definable by a flat formula of set theory.*
(ii) *A model class is definable in second order logic $\mathscr{L}_A^2$ if and only if it is definable by a flat formula of set theory with parameters in A.* ☐

Likewise, we may characterize $\mathrm{PC}_{\mathscr{L}}$-definability for a variety of $\mathscr{L}$ by modifying Corollary 3.4.5(ii).

The Logic $\mathscr{L}_{\omega\omega}(Q)$

Let Q be any quantifier. For reasons which will become apparent in the sequel, no characterization of $\mathscr{L}_{\omega\omega}(Q)$ can be proven along the above lines. However, we can say something about $\Delta(\mathscr{L}_{\omega\omega}(Q))$. In particular, we can assert

3.4.15 Theorem. *$\mathscr{L}_{\omega\omega}(Q)$ is adequate to truth in any logic that is absolute relative to Q and* KP(Q)-Inf.

Proof. Suppose that $\mathscr{L}$ is absolute relative to Q and KP(Q)-Inf. Then the predicate "$\varphi \in \mathscr{L} \wedge \mathfrak{M} \models_{\mathscr{L}} \varphi$" is Δ_1 in Q and KP(Q)-Inf. It is easy to show that $\mathscr{L}_{\omega\omega}(Q)$ captures such predicates. Thus, the claim follows from Theorem 3.1.3. ☐

3.4.16 Corollary. *If K is a model class, then* (a) → (b) → (c) *as below holds:*

(a) *K is definable in $\mathscr{L}_A(Q)$.*
(b) *K is Δ_1 in* KP-Inf *in the extended language* $\{\in, Q\}$ *with parameters in A.*
(c) *K is definable in $\Delta(\mathscr{L}_A(Q))$.* ☐

The main obstacle to improving Corollary 3.4.16 to (b) ↔ (c) lies in the fact that certain $\mathscr{K}$ are $\Delta(\mathscr{L}_{\omega\omega}(Q))$-definable in some models of set theory but not in others. For example, if $\mathscr{K}$ is the class of tree-like structures with an uncountable branch, then $MA + \neg\mathrm{CH}$ implies that $\mathscr{K}$ is $\Delta(\mathscr{L}_{\omega\omega}(Q_1))$-definable, but ZFC alone is not enough for this, let alone ZFC-Inf. On the other hand, if the axiom of infinity is added to the picture, much more than $\Delta(\mathscr{L}_{\omega\omega}(Q_1))$ will be Δ_1, for instance, $\mathscr{L}_{\omega\omega}(W)$. These situations manifest the difficulties inherent in trying to prove general set-theoretical characterizations for logics of the form $\mathscr{L}_{\omega\omega}(Q)$.

Historical and Bibliographical Remarks. The first result proven in the direction of this section is the characterization Theorem 3.2.4 of strictly absolute logics, due to Barwise [1972a]. The observation that absolute logics and $\mathscr{L}_A(W)$ are related as in Corollary 3.3.3(i) was made by Swett [1974]. Corollary 3.3.3 was rediscovered independently by Oikkonen [1978]. The relativization to an arbitrary predicate R (see Corollary 3.4.5) was carried out in Oikkonen [1978] and Väänänen [1978]. Finally, the iteration in Theorem 3.4.11 is due to Oikkonen [1978]. The computation of $\Delta(\mathscr{L}_{\omega\omega}(Q_0))$ in Theorem 3.2.2 and its generalization Theorem 3.2.3 are due independently to Barwise [1974a] and Makowsky [1975b]. Burgess [1977] is a good reference to absolute logics. Essentially, it contains Theorem 3.4.7, among other things. Theorem 3.4.14 on second-order logic is from Väänänen [1979a]. The results on first-order logic are due to Manders [1980] and G. Wilmers. In Väänänen [1979a], a logic $\mathscr{L}$ was called symbiotic with a predicate R if $\Delta(\mathscr{L})$-definability coincides with Δ_1-definability in $\{\in, R\}$. The present terminology, centered around absoluteness, capture, and pinning down seems more useful and emphasizes the relation to adequacy to truth. Theorem 3.1.3 is formally new but in fact is really only the codification of the underlying ideas of the above characterization results. The general approach was chosen in an attempt to shed light on these ideas.

4. *Other Topics*

4.1. *The Weak Beth Property Revisited*

Recall the definition of weak Beth property: if a formula $\varphi(R)$ defines the predicate R implicitly (that is, $\varphi(R) \wedge \varphi(R') \models \forall x_1 \cdots x_n(R(x_1, \ldots, x_n) \leftrightarrow R'(x_1, \ldots, x_n))$ and if every model can be expanded to a model of $\varphi(R)$, then some formula $\eta(x_1, \ldots, x_n)$ defines R explicitly (that is, $\varphi(R) \models \forall x_1 \cdots x_n(R(x_1, \ldots, x_n) \leftrightarrow \eta(x_1, \ldots, x_n))$. With every logic $\mathscr{L}$ can be associated the smallest extension of $\mathscr{L}$ to a logic $\mathrm{WB}(\mathscr{L})$ with the weak Beth property.

We have already mentioned the following result in discussing some refinements at the end of Section 1.

4.1.1 Theorem. *If $\mathscr{L}$ is uniquely adequate to truth in $\mathscr{L}'$, $\mathscr{L}$ is closed under negation and $\mathscr{I}(A')$ is definable in* $\mathrm{WB}(\mathscr{L})$, *then* $\mathrm{WB}(\mathscr{L}) \not\leq \mathscr{L}'$. □

Applications. The logics $\mathscr{L}_{\omega\omega}(Q_0)$, $\mathscr{L}^{2,w}_{\omega\omega}$, $\mathscr{L}^{2}_{\omega\omega}$, $\mathscr{L}_{\omega\omega}(H)$, and $\mathscr{L}_{\omega_1\omega_1}$ do not have the weak Beth property.

For unbounded logics (see Corollary 2.2.11) we have the following result of Burgess. The proof uses methods from descriptive set theory, notably the Π^1_1-uniformization property, combined with Theorem 3.4.7.

4.1.2 Theorem. *Suppose that $\mathscr{L}$ is a strong absolute logic closed under countable disjunctions and negations, and $A \subseteq$ HC. Then $\mathscr{L}$ fails to have the weak Beth property.*

Applications. If $\mathscr{L}$ is any of $\mathscr{L}_{\omega\omega}(W)$, $\mathscr{L}_{\omega_1 G}$, $\mathscr{L}_{\omega_1 S}$, $\mathscr{L}_{\omega_1 V}$, or $\mathscr{L}_{\omega_1 B}$, then $\mathscr{L}$ does not have the weak Beth property and WB($\mathscr{L}$) is not absolute.

Particularly strong results on weak Beth closure come from the following theorem of Gostanian–Hrbacek [1976].

4.1.3 Theorem. WB$(\mathscr{L}_{\omega\omega}(W)) \not\leq \mathscr{L}_{\infty\infty}$.

Proof. Let $\mathscr{K}$ be the class of models (A, E, R) such that either (A, E) is non-well-founded and $R = \varnothing$ or (A, E) is well-founded and R is the set of pairs (φ, f) where (A, E) satisfies [$\varphi \in \mathscr{L}_{\infty\infty}$ and f is a function such that the inductive clauses for satisfaction of $\mathscr{L}_{\infty\infty}$-formulae hold]; an example of the inductive clauses here is:

$$(\exists (x_\alpha)_{\alpha<\kappa}\varphi, f) \in R \quad \text{if and only if} \quad \exists g \in A \text{ such that } g(x) = f(x) \text{ for variables } x \neq x_\alpha\ (\alpha < \kappa) \text{ and } (\varphi, g) \in R.$$

If (A, E, R) and (A, E, R') are in $\mathscr{K}$, then we can use induction to prove that $R = R'$. Thus, $\mathscr{K}$ defines R implicitly. Moreover, for all (A, E), there is an R such that $(A, E, R) \in \mathscr{K}$. Suppose that there were a formula $\eta(x, y)$ in $\mathscr{L}_{\infty\infty}$ which defines R explicitly in models of $\mathscr{K}$. Let κ be a regular cardinal such that $\eta \in \mathscr{L}_{\kappa\kappa}$. We shall consider the model $\mathfrak{M} = (H_\kappa, \in \cap H^2)$. The point to notice here is that if $\exists (x_\alpha)_{\alpha<\beta}\psi$ is in H_κ, then $\beta < \kappa$ and every sequence $(x_\alpha)_{\alpha<\beta}$ of elements of H_κ that one might need to satisfy ψ already exists as an element of H_κ. Thus, if R is chosen such that $(H_\kappa, \in \cap H^2, R) \in \mathscr{K}$, then

$$R = \{(\varphi, f) \mid \varphi \in \mathscr{L}_{\kappa\kappa} \text{ and } f \text{ satisfies } \varphi \text{ in } \mathfrak{M}\}.$$

Combining this with the choice of $\eta(x, y)$ yields

$$\mathfrak{M} \models \eta(\varphi, f) \quad \text{if and only if} \quad f \text{ satisfies } \varphi \text{ in } \mathfrak{M}.$$

The standard diagonal argument ends the proof. Hence, let ξ $(= \xi(x))$ be the formula $\neg\eta(x, f)$, where f is a term denoting the function which maps the variable x to x $(f = \{(x, x)\})$. We now have $\xi \in H_\kappa$ and

$$\mathfrak{M} \models \xi(\xi) \leftrightarrow \eta(\xi, \{(x, x)\}) \leftrightarrow \neg\xi(\xi).$$

The contradiction shows that $\mathscr{K}$ does not define R explicitly in $\mathscr{L}_{\infty\infty}$; and this implies the claim, as $\mathscr{K}$ itself is $\mathscr{L}_{\omega\omega}(W)$-definable. $\square$

The above theorem permits several improvements. An immediate observation is that $\mathscr{K}$ need not assert that all its models (with $R \neq \varnothing$) are well-founded. In fact, it is enough that $\mathscr{K}$ pins down the κ such that $\eta \in \mathscr{L}_{\kappa\kappa}$. Thus we have

4.1.4 Theorem. *If $\mathscr{L}$ pins down the regular κ, then* $\mathrm{WB}(\mathscr{L}) \nleq \mathscr{L}_{\kappa\kappa}$. $\square$

4.1.5 Corollary. (i) *If* $\mathrm{cf}(\kappa) > \omega$, *then* $\mathrm{WB}(\mathscr{L}_{\kappa^+\omega}) \nleq \mathscr{L}_{\kappa^+\kappa^+}$.
(ii) $\mathrm{WB}(\mathscr{L}_{\omega_1\omega_1}) \nleq \mathscr{L}_{\infty\infty}$. $\square$

Our second improvement concerns generalized quantifiers. We can make the R in the above proof work for formulae of $\mathscr{L}_{\infty\infty}(Q)$, where Q is an arbitrary generalized quantifier. However, $\mathscr{K}$ is then definable in $\mathscr{L}_{\omega\omega}(W, Q)$ and not in $\mathscr{L}_{\omega\omega}(W)$.

4.1.6 Theorem. *If $\mathscr{L}_{\omega\omega}(Q)$ pins down the regular κ, then* $\mathrm{WB}(\mathscr{L}_{\omega\omega}(Q)) \nleq \mathscr{L}_{\kappa\kappa}(Q)$. $\square$

4.1.7 Corollary. (i) $\mathrm{WB}(\mathscr{L}_{\omega\omega}(I)) \nleq \mathscr{L}_{\infty\infty}(I)$.
(ii) $\mathrm{WB}(\mathscr{L}_{\omega_1 G}) \nleq \mathscr{L}_{\infty\infty}(G)$.
(iii) $\mathrm{WB}(\mathscr{L}_{\omega\omega}(H)) \nleq \mathscr{L}_{\infty\infty}(H)$.
(iv) $\mathrm{WB}(\mathscr{L}_{\kappa^+\omega}(Q)) \nleq \mathscr{L}_{\kappa^+\kappa}(Q)$, if $\mathrm{cf}(\kappa) > \omega$.

What about logics which pin down ordinals but no interesting regular cardinals? Here, we may notice that the only role of regularity of κ in the proof of Theorem 4.1.3 (or of Theorem 4.1.4) is that it gives the correct interpretation for R as the satisfaction relation of $\mathscr{L}_{\kappa\kappa}$. However, if we replace $\mathscr{L}_{\kappa\kappa}$ by $\mathscr{L}_{\kappa\omega}$, any admissible A with $o(A) = \kappa$ can replace H_κ, and we have

4.1.8 Theorem. *If A is an admissible set and $\mathscr{L}'$ pins down $o(A)$, then* $\mathrm{WB}(\mathscr{L}') \nleq \mathscr{L}_A$. $\square$

Again, we can add an arbitrary generalized quantifier Q to this result. In fact, we have

4.1.9 Theorem. *If $\mathscr{L}_{\omega\omega}(Q)$ pins down $o(A)$, where A is an admissible set, then* $\mathrm{WB}(\mathscr{L}_{\omega\omega}(Q)) \nleq \mathscr{L}_A(Q)$. $\square$

4.2. Σ_1-*Compactness*

Recall that a logic $\mathscr{L}$, represented on A, is called Σ_1-*compact* if every $T \subset \mathscr{L}$, Σ_1 over A, which has no models, has an A-finite subset with no models. The following result is thus straightforward.

4.2.1 Proposition. *If $\mathscr{L}$ is effectively adequate to truth in $\mathscr{L}'$ (as explained in the refinement at the end of Section 1), and if $A' = A$ are admissible sets and $\mathscr{L}$ is Σ_1-compact, then $\mathscr{L}'$ is Σ_1-compact.* $\square$

This result, when combined with Theorem 3.3.2, this gives

4.2.2 Corollary. *If $\mathscr{L}_A(W)$, where A admissible, is Σ_1-compact, then so is every absolute logic represented on A.* □

Similarly, for stronger logics we have

4.2.3 Corollary. *If $\mathscr{L}$ is Σ_1-compact and pins down R, then every logic, absolute relative to R and represented on A, is Σ_1-compact.* □

A third kind of consequence of Proposition 4.2.1 is given in

4.2.4 Corollary. *If $\mathscr{L}$ is adequate to truth in itself and Σ_1-compact and Q is $\Delta(\mathscr{L})$-definable, then $\mathscr{L}_A(Q)$ is Σ_1-compact.* □

It is well-known that $\mathscr{L}_A$ is Σ_1-compact if A is a countable admissible set. More generally,

(∗) $\mathscr{L}_A$ is Σ_1-compact if and only if A satisfies s-Π_1^1-reflection.

The reader is referred to Chapter VIII for more on this and other results on $\mathscr{L}_A$. The result given in (∗) above has been generalized to all absolute logics by Cutland–Kaufmann [1980]. In this development, use is made of the notion of a s-Π_1^1-Souslin formula. These formulae are (in their normal form) of the form

$$\forall V_1 \ldots \forall V_m Q_s x_1 \ldots Q_s x_n \exists y_1 \ldots \exists y_2 \psi,$$

where ψ is Σ_0 and Q_s is the Souslin quantifier

$$Q_s x \varphi(x) \leftrightarrow \exists x_0 \exists x_1 \ldots \bigwedge_{n<\omega} \varphi(\langle x_0, \ldots, x_n \rangle).$$

4.2.5 Theorem. *An admissible set A satisfies s-Π_1^1-Souslin reflection if and only if every absolute logic represented on A is Σ_1-compact.* □

4.2.6 Corollary. *If $\mathscr{L}_{\kappa\omega}$ is Σ_1-compact and* $\mathrm{cf}(\kappa) > \omega$, *then $\mathscr{L}_{\kappa\omega}(W)$ is Σ_1-compact.* □

Recall that an admissible set A is *resolvable* if $A = \cup_{\alpha < o(A)} F(\alpha)$ for some A-recursive function F.

4.2.7 Theorem. *If A is a countable resolvable admissible set, then A satisfies Σ_2^1-reflection if and only if every absolute logic represented on A is Σ_1-compact.* □

4.2.8 Theorem. *If A is a resolvable admissible set, then $\mathscr{L}_A(Q)$ is Σ_1-compact if and only if $\mathscr{I}(A)$ is not* RPC-*defined by a Σ_1-theory of $\mathscr{L}_A(Q)$.*

Proof. The argument is similar to Barwise [1975, VIII.4.8]. □

Remark. Let A be a resolvable admissible set and assume that $\mathscr{E}(A)$ is RPC in $\mathscr{L}_A(Q)$. By Proposition 1.2.2, the conjunction of a Σ_1-theory of $\mathscr{L}_A(Q)$ is RPC in $\mathscr{L}_A(Q)$. Thus, if $\mathscr{L}_A(Q)$ is not Σ_1-compact, then $\mathscr{I}(A)$ is RPC in $\mathscr{L}_A(Q)$, which means that $\mathscr{L}_A(A)$ pins down $o(A)$. By Theorem 4.1.9, $\mathscr{L}_A(Q)$ fails to satisfy the weak Beth property. We have, in effect, a proof of

4.2.9 Theorem. *Suppose A is a resolvable admissible set. If $\mathscr{L}_A(Q)$ satisfies the weak Beth property, then $\mathscr{L}_A(Q)$ is Σ_1-compact.* □

Σ_1-compactness is somewhat related to weak compactness. A logic $\mathscr{L}$ is *weakly compact* if it is Σ_1-compact with any $R \subseteq A$ as a parameter. For $A = H_\kappa$, this assumes the more familiar form: If a theory $T \subset \mathscr{L}$ (and $T \subset H_\kappa$) has no models, then some subtheory of power $< \kappa$ has no models. It is well-known that $\mathscr{L}_{\kappa\omega}$ is weakly compact if and only if $\mathscr{L}_{\kappa\kappa}$ is weakly compact if and only if $\kappa \to (\kappa)^2_2$.

4.2.10 Theorem. *Let $\mathscr{L}$ be any logic and κ a measurable cardinal. There is a stationary set of cardinals $\lambda < \kappa$ such that $\mathscr{L}$ restricted to H_λ is weakly compact.*

Proof. Let U be a normal ultrafilter on κ and $i: V \to M$ the associated embedding (see, for example, Jech [1978, p. 305]). The fundamental property of i is that if $\varphi(x, y)$ is any formula of set theory, then

$$(*) \qquad M \models \varphi(\kappa, i(x)) \quad \text{if and only if} \quad \{\lambda < \kappa \mid \varphi(\lambda, x)\} \in U.$$

We let $\varphi(\lambda, x)$ be the formula "If $T \subset \mathscr{L}$, $T \subseteq H_\lambda \cap x$ and T has no models, then some subset $T_0 \in H_\lambda$ of T has no models". In view of $(*)$ it suffices to prove that $M \models \varphi(\kappa, i(x))$ holds, for $x = \{\varphi \in \mathscr{L} \mid \varphi \in H_\kappa\}$. Suppose we have $T = i(T')$, for some T'. By $(*)$ the set

$$A = \{\lambda < \kappa \mid T' \subset \mathscr{L}, T' \subseteq H_\lambda \cap x \text{ and every subset } T_0 \in H_\lambda \text{ of } T' \text{ has a model}\}$$

is in U. Let $\lambda, \mu \in A$ such that $\lambda < \mu$. Then $T' \in H_\mu$, as $\lambda \in A$; and, hence, T' has a model, as $\mu \in A$. Therefore, $M \models T$ has a model, using $(*)$ again. □

4.3. The Problem of Validity

Recall that if $\mathscr{L}$ is a logic represented on A, we say that *validity in $\mathscr{L}$ is Σ_1* if the set

$$\mathrm{Val}_{\mathscr{L}} = \{\varphi \in \mathscr{L} \mid \models_{\mathscr{L}} \varphi\}$$

is Σ_1 over A.

4.3.1 Proposition. *If $\mathscr{L}$ is effectively adequate to truth in $\mathscr{L}'$, $A' = A$ is an admissible set and validity in $\mathscr{L}$ is Σ_1, then the same holds for $\mathscr{L}'$.* □

4.3.2 Corollary. *If validity in $\mathscr{L}_A(W)$, where A admissible, is Σ_1, then the same holds for every absolute logic represented on A.* ∎

Similarly, for stronger logics, we have

4.3.3 Corollary. *If the validity in $\mathscr{L}$ is Σ_1 and $\mathscr{L}$ pins down R, then validity in any logic absolute relative to R and represented on A is Σ_1.* ∎

A third kind of consequence of Proposition 4.3.1 is given in

4.3.4 Corollary. *If $\mathscr{L}$ is effectively adequate to truth in itself, and validity in $\mathscr{L}$ is Σ_1 and Q is $\Delta(\mathscr{L})$-definable, then validity in $\mathscr{L}_A(Q)$ is Σ_1.*

Application. $\mathscr{L}_{\omega\omega}(Q_1^E)$ is axiomatizable, because Q_1^E is $\Delta(\mathscr{L}_{\omega\omega}(Q_1))$-definable.

As it actually turns out, validity in an unbounded logic is hardly ever Σ_1. In order to see this let us first make two simple remarks. In the following, a subset X of A is said to be Π_1 if it has the form $\{x \in A \mid \varphi(x)\}$, where $\varphi(x)$ is Π_1. Observe that Π_1 over A refers to sets of the form $\{x \in A \mid A \models \varphi(x)\}$, $\varphi(x) \in \Pi_1$. A set $X \subseteq A$ is *complete for* Π_1 *on* A if for every Π_1 subset Y of A there is a Σ_1-function f of A such that for $a \in A$

$$a \in Y \leftrightarrow f(a) \in X.$$

4.3.5 Lemma. (i) *If $\mathscr{L}$ is absolute relative to R, then $\mathrm{Val}_{\mathscr{L}}$ is Π_1 in the extended language $\{\in, R\}$.*
(ii) *If $\mathscr{L}$ pins down R, then $\mathrm{Val}_{\mathscr{L}}$ is complete for Π_1 on A in the extended language $\{\in, R\}$.*

Proof. In order to prove (i), we use absoluteness of $\mathscr{L}$ to write the definition

$$a \in \mathrm{Val}_{\mathscr{L}} \leftrightarrow a \in \mathscr{L} \wedge \forall \mathfrak{A}(\mathfrak{A} \models_{\mathscr{L}} a)$$

in Π_1-form. In order to prove (ii), we suppose that Y is a subset of A defined by the Π_1-formula $\varphi(x)$. For $a \in A$, let $g(a)$ be an $\mathscr{L}$-sentence equivalent to

$$\forall x(\pi_a(x) \rightarrow \varphi(x)).$$

If $\mathscr{K}$ is the class of models $(M, \in \cap M^2, R \cap M^n)$, M transitive, then for all $a \in A$

$$a \in Y \leftrightarrow \mathscr{K} \subseteq \mathrm{Mod}(g(a)).$$

Using the fact that $\mathscr{K}$ is $\mathrm{RPC}_{\mathscr{L}}$, we find a Σ_1-function f on A such that for all $a \in A$, we have

$$a \in Y \leftrightarrow f(a) \in \mathrm{Val}_{\mathscr{L}}. \quad ∎$$

In order to be able to apply Lemma 4.3.5 we would like to know that Π_1 coincides with Π_1 over A for subsets of A. In general, this is not true. An equivalent condition is that $A \prec_1 V$ and this is known to hold for $A = H_\kappa$, $\kappa > \omega$, (Levy reflection principle) and for $A = L_\alpha$, where $\alpha \leq \omega_1^L$ is stable, at least (Schoenfield absoluteness lemma). In such a case, Π_1 plus complete for Π_1 coincide with the ordinary notion of complete Π_1, which is never Σ_1 over A (if A is admissible). Thus, we have the proof of

4.3.6 Theorem. *If $\mathscr{L}$ is an unbounded absolute logic represented on $A \prec_1 V$, then* $\mathrm{Val}_{\mathscr{L}}$ *is complete Π_1 over A and validity in $\mathscr{L}$ is not Σ_1.* □

4.3.7 Corollary. *Validity in an unbounded absolute logic represented on $H_\kappa, \kappa > \omega$, is not Σ_1.* □

Cutland–Kaufman [1980] obtained the following improvement of Theorem 4.3.6 in the case of $A = L_\alpha$.

4.3.8 Theorem. *Validity is not Σ_1 in any unbounded absolute logic represented on an admissible set of the form L_α.* □

Corollary. *If $V = L$, then validity in an unbounded absolute logic represented on an admissible set is never Σ_1.* □

Considering these negative results, one might raise the question of whether some more general completeness property would be more tractable. In this direction Cutland and Kaufman proved

4.3.9 Theorem. *If $\mathscr{L}$ is an absolute logic represented on an admissible set A, then* $\mathrm{Val}_{\mathscr{L}}$ *is* s-Π_1^1*-Souslin over A.*

Feferman [1975] proves a more general completeness theorem. Recall the notion of $\#$-siid$_x$ from Remark 1.1.9. In the following theorem $\mathscr{L}$ has to satisfy a property called "join property," a property which most logics do indeed satisfy and which essentially says that $\mathscr{L}$ permits the construction of disjoint unions of structures.

4.3.10 Theorem. *Let $\mathscr{L}$ be adequate to truth in itself. Then for each τ, the set $\{\varphi \in \mathscr{L}[\tau] \mid \models_{\mathscr{L}} \varphi\}$ is $\#$-siid$_x$ in $\mathscr{L}$. Moreover, if $S \subset \mathscr{L}[\tau]$ is siid$_x$ in $\mathscr{L}$, then the same holds for $\{\varphi \in \mathscr{L}[\tau] \mid S \models_{\mathscr{L}} \varphi\}$.* □

This theorem shows that validity and even consequence is "r.e." in any self-adequate logic once we use an appropriate notion of "r.e.". The notion $\#$-siid$_x$ does indeed have many of the characteristics of r.e. on ω and Σ_1 on an admissible set (see Feferman [1975] and Kunen [1968]).

4.4. Löwenheim Numbers and Spectra

The Löwenheim number of a logic is related to the more general problem of spectra. The *spectrum* of a sentence φ of a logic $\mathscr{L}$ is the class of cardinals of models of φ. That is, in symbols the spectrum of φ is

$$\mathrm{Sp}(\varphi) = \{|\mathfrak{A}|\,|\,\mathfrak{A} \models_{\mathscr{L}} \varphi\}.$$

The problem of spectra of such strong logics as $\mathscr{L}^2_{\omega\omega}$ or $\mathscr{L}_{\omega\omega}(I)$ is a difficult subject and remains mostly unsettled. However, even the spectra of $\mathscr{L}_{\omega\omega}$ present open problems. Well-known is the *Finite Spectrum Problem*: *Is the complement of a spectrum of $\mathscr{L}_{\omega\omega}$ also a spectrum of $\mathscr{L}_{\omega\omega}$, if only finite models are considered*? On the other hand, the infinite part of a spectrum of $\mathscr{L}_{\omega\omega}$ is trivial: It is either empty or contains every infinite cardinal. The spectra of $\mathscr{L}_{\omega_1\omega}$ are more complex: Every set of natural numbers is one, as are also $\{\kappa\,|\,\kappa < 2^{\omega_\alpha}\}$, for $\alpha < \omega_1$. Even more complex, however, are spectra of $\mathscr{L}^2_{\omega\omega}$. The strength of $\mathscr{L}^2_{\omega\omega}$ makes it possible to represent every spectrum as the spectrum of an identity sentence. Thus, the spectra of $\mathscr{L}^2_{\omega\omega}$ form a boolean algebra with respect to complementation, union and intersection. In fact, the spectra of $\mathscr{L}^2_{\omega\omega}$ permit the following general characterization, a consequence of Theorem 3.4.14(ii).

4.4.1 Theorem. *A class C of cardinals is a spectrum of $\mathscr{L}^2_A$ if and only if C is defined by a flat formula of set theory with parameters in A.* ☐

For logics such as $\mathscr{L}_{\omega_1\omega}$, $\mathscr{L}_{\omega\omega}(Q_1)$, and $\mathscr{L}_{\omega\omega}(W)$ the complexity of spectra is limited by a strong downward Löwenheim–Skolem theorem. Another limiting factor is the upward Löwenheim–Skolem theorem.

Recall that the *Lowenheim number* of $\mathscr{L}$ is the cardinal

$$\ell(L) = \sup\{\min C\,|\,C \text{ is a spectrum of } \mathscr{L}\}.$$

Despite our occasional reference to logics such as $\mathscr{L}_{\infty\omega}$ and $\mathscr{L}_{\infty\infty}$, every logic is represented on a set and therefore has a Löwenheim number. The explicit computations $\ell(\mathscr{L}_{\kappa^+\omega}) = \kappa$ and $\ell(\mathscr{L}_{\omega\omega}(Q_\alpha)) = \omega_\alpha$ are immediate. Following are two easy preservation results.

4.4.2 Proposition. (i) *If $\mathscr{L} \leq_{\mathrm{RPC}} \mathscr{L}'$, then $\ell(\mathscr{L}) \leq \ell(\mathscr{L}')$.*

(ii) *If $\mathscr{L}$ is absolute relative to R, $A \subseteq A'$ and $\mathscr{L}'$ pins down R, then $\ell(\mathscr{L}) \leq \ell(\mathscr{L}')$.*

In the following theorem we shall estimate Löwenheim numbers in purely set-theoretical terms

4.4.3 Theorem. *Let $\mathscr{L}$ be a logic, R a predicate and*

$\delta = \sup\{\kappa\,|\,\kappa$ *in* Π_1*-definable in the extended language* $\{\in, R\}$ *with parameters in* $A\}$.

(i) *If $\mathscr{L}$ is absolute relative to R, then $\ell(\mathscr{L}) \leq \delta$.*
(ii) *If $\mathscr{L}$ pins down R, then $\delta \leq \ell(\mathscr{L})$.*

Proof. For (i) we suppose that $\lambda = \min C$, where $C = \mathrm{Sp}(\varphi)$ is spectrum of $\mathscr{L}$. The cardinal λ has the following definition:

$$\alpha \in \lambda \leftrightarrow \forall \beta(\beta \leq \alpha \to \beta \notin C).$$

Using absoluteness of $\mathscr{L}$, we can write this in Π_1-form with φ as a parameter.

(ii) We suppose κ that is Π_1-definable with parameters in A. Let $\mathscr{K}$ be the class of well-ordered structures of type $\geq \kappa$. $\mathscr{K}$ is clearly Σ_1-definable with parameters in A. By Theorem 3.4.4 (letting $\mathscr{L}' = \mathscr{L}_{\omega\omega}(Q)$ such that $\mathscr{K}$ is $\mathrm{EC}_{\mathscr{L}'}$ and $\mathscr{L}'$ is absolute relative to R), $\mathscr{K}$ is $\mathrm{RPC}_{\mathscr{L}}$. Let

$$\lambda = \min\{|\mathfrak{A}| \mid \mathfrak{A} \in \mathscr{K}\}.$$

Now $\kappa \leq \lambda \leq \ell(\mathscr{L})$. ☐

Remark. If $\ell(\mathscr{L})$ is a limit cardinal, we can replace κ by α in Theorem 4.4.3.

4.4.4 Corollary. *Suppose that $\mathscr{L}$ is absolute relative to R and pins down R. Then*

$$\ell(\mathscr{L}) = \sup\{\kappa \mid \kappa \textit{ is } \Pi_1\textit{-definable in the extended language } \{\in, R\} \textit{ with parameters in } A\}. \quad \square$$

4.4.5 Examples. (i) $\ell(\mathscr{L}_A(I)) = \sup\{\alpha \mid \alpha$ is Π_1-definable in $\{\in, \mathrm{Cd}\}$ with parameters in $A\}$.

(ii) $\ell(\mathscr{L}_A^2) = \sup\{\alpha \mid \alpha$ is Π_2-definable with parameters in $A\}$.

An inductive argument based on Theorem 4.4.3 can be used to prove:

4.4.6 Theorem. $\ell(\Delta_n(\mathscr{L}_A)) = \sup\{\alpha \mid \alpha$ *is* Π_n*-definable with parameters in* $A\}$ $(n > 1)$. ☐

We can actually replace Π by Δ in the above results. When this is done, we then have

4.4.7 Theorem. (i) $\ell(\mathscr{L}_A^2) = \sup\{\alpha \mid \alpha$ *is* Δ_2*-definable with parameters in* $A\}$.

(ii) $\ell(\Delta_n(\mathscr{L}_A)) = \sup\{\alpha \mid \alpha$ *is* Δ_n*-definable with parameters in* $A\}$, $(n > 1)$. ☐

Following is a third characterization of $\ell(\Delta_n(\mathscr{L}_A))$ in set-theoretical terms.

4.4.8 Theorem. *If* $n > 1$, *then* $\ell(\Delta_n(\mathscr{L}_{\kappa\omega})) = \kappa$ *if and only if* $R_\kappa \prec_n V$. ☐

Combined with the facts that $R_\kappa \prec_2 V$ for κ supercompact and $R_\kappa \prec_3 V$ for κ extendible, this yields

4.4.9 Corollary. (i) *If* κ *supercompact, then* $\ell(\mathscr{L}_{\kappa\omega}^2) = \kappa$.

(ii) *If* κ *is extendible, then* $\ell(\Delta_3(\mathscr{L}_{\kappa\omega})) = \kappa$. ☐

Remark. Magidor [1971] proves a downward Löwenheim–Skolem theorem for $\mathscr{L}^2_{\omega\omega}$ on a supercompact cardinal, a result which is stronger than that given by Corollary 4.4.9(i).

No upper bound to $\ell(\mathscr{L}^2_{\omega\omega})$ or even to $\ell(\mathscr{L}_{\omega\omega}(I))$ is known in terms of large cardinals below supercompact cardinals. However, observe the following

4.4.10 Theorem. (i) *If a spectrum C of $\mathscr{L}^2_{\omega\omega}$ contains a measurable cardinal κ, then $C \cap \kappa$ is stationary on κ.*

(ii) *If a spectrum C of $\mathscr{L}_{\omega\omega}(I)$ contains a weakly inaccessible cardinal κ, then $C \cap \kappa$ is* cub *on κ.* □

4.5. Hanf Numbers

Recall that the Hanf number of a logic $\mathscr{L}$ is the cardinal

$$h(\mathscr{L}) = \sup\{\sup C \mid C \text{ is a bounded spectrum of } \mathscr{L}\}.$$

There are a few explicit Hanf number computations, such as $h(\mathscr{L}_A) = \beth_\alpha$ for countable admissible A of ordinal α and $h(\mathscr{L}_{\omega\omega}(Q_1)) = \beth_\omega$. But for many $\mathscr{L}$, for more than with $\ell(\mathscr{L})$, $h(\mathscr{L})$ is simply unknown. The following estimates are the best known ones.

4.5.1 Examples. (i) $h(\mathscr{L}_{\omega\omega}(W))$ exceeds the first κ such that $\kappa \to (\omega)^{<\omega}$.

(ii) If $\kappa \to (\omega_1)^{<\omega}$, then $h(\mathscr{L}_{\omega\omega}(W)) < \kappa$.

(iii) $h(\mathscr{L}_{\omega_1 S}) \leq \beth_{\omega_2}$.

Remark. As to (iii), Burgess [1978] shows that $h(\mathscr{L}_{\omega_1 S}) = \beth_{\omega_2}$ under $MA + \neg\text{CH} + \omega_1^L = \omega_1$.

When we proceed to set-theoretical characterization or estimation of Hanf numbers, the first point to notice is the failure of Hanf numbers to be preserved—in general—under Δ-operation. Thus, our general results will mostly concern $h(\text{RPC}_{\mathscr{L}})$ rather than $h(\mathscr{L})$. (Note that $\ell(\text{RPC}_{\mathscr{L}}) = \ell(\mathscr{L})$). In particular examples, on the other hand, $h(\text{RPC}_{\mathscr{L}}) = h(\mathscr{L})$ usually holds, as we shall see.

A typical $\text{RPC}_{\mathscr{L}}$-definition has the form

$$(*) \qquad \mathfrak{M} \in \mathscr{K} \leftrightarrow \exists \mathfrak{N}([\mathfrak{M}, \mathfrak{N}] \models_{\mathscr{L}} \varphi).$$

Problems with $h(\mathscr{L})$ arise because there is no upper bound on the size of $\mathfrak{N}$. But suppose that the following holds in addition to (*) above:

$$\forall \mathfrak{M}\, \exists \kappa\, \forall \mathfrak{N}([\mathfrak{M}, \mathfrak{N}] \models_{\mathscr{L}} \varphi \to |\mathfrak{N}| \leq \kappa).$$

In this case, we say that $\mathscr{K}$ is *bounded* $\text{RPC}_{\mathscr{L}}$. This notion clear, we have

4.5.2 Lemma. $\hbar(\text{bounded RPC}_{\mathscr{L}}) = \hbar(\mathscr{L})$.

Proof. The argument for this result is easy. □

4.5.3 Examples. $\text{RPC}_{\mathscr{L}} = \text{bounded RPC}_{\mathscr{L}}$ if $\mathscr{L}$ is one of $\mathscr{L}_{\kappa^+\omega}$, $\mathscr{L}_{\omega\omega}(Q_\alpha)$, $\mathscr{L}_{\omega\omega}(Q_\alpha^{<\omega})$, $\mathscr{L}_{\kappa G}$ or if Bounded $\text{RPC}_{\mathscr{L}}$ contains $\mathscr{L}^2_{\omega\omega}$. In the first four cases this follows from the strong Löwenheim–Skolem theorems of these logics. In the fifth case, we may use the strength of $\mathscr{L}^2_{\omega\omega}$ to make sure that the set-theoretical rank of $\mathfrak{N}$ in (*) is minimal.

4.5.4 Theorem. *If* Con(ZF), *then* $\text{Con}[\text{ZFC} + \hbar(\mathscr{L}_{\omega\omega}(I)) < \hbar(\text{RPC}_{\mathscr{L}_{\omega\omega}(I)})]$.

Proof. Let $A(\alpha)$ be the statement "there is a sequence $(\omega_{\gamma+\beta+1})_{\beta<\alpha}$ of cardinals κ such that $2^\kappa \geq \kappa^{++}$" and let $B(\alpha)$ be the statement $A(\alpha) \wedge \forall\beta(A(\beta) \to \beta \leq \alpha)$. It can be seen without too much trouble that $B(\alpha)$ implies that $\alpha < \hbar(\text{RPC}_{\mathscr{L}_{\omega\omega}(I)})$. Thus, it remains to construct a boolean extension in which $\hbar(\mathscr{L}_{\omega\omega}(I)) \leq \alpha \wedge B(\alpha)$. The idea here is the following. Construct notions of forcing (proper classes) $F_{\alpha\beta}$ such that $F_{\alpha\beta} \Vdash B(\alpha)$, $F_{\alpha\beta}$ is ω_β-closed, $F_{\alpha\beta} \supseteq F_{\alpha\gamma}$ if $\beta < \gamma$, and $F_{\alpha\beta}$ preserves cardinals. Call $F_{\alpha\beta}$ *a failure* if it fails to force $\hbar(\mathscr{L}_{\omega\omega}(I)) \leq \alpha$. Construct a sequence $(\varphi_\alpha)_{\alpha<\omega_1}$ of sentences of $\mathscr{L}_{\omega\omega}(I)$ and sequences $(\lambda_\alpha)_{\alpha<\omega_1}$ and $(\kappa_\alpha)_{\alpha<\omega_1}$ of cardinals such that for $\lambda = \hbar(\mathscr{L}_{\omega\omega}(I))$, $\lambda_\alpha = \sup(\kappa_\beta)_{\beta<\alpha}$, $F_{\lambda\lambda_\alpha}$ is a failure, because it forces φ_α to have a model of power $\geq\lambda$ but none $\geq\kappa_\alpha$ $(>\lambda_\alpha)$. Take $\alpha < \beta < \omega_1$ such that $\varphi_\alpha = \varphi_\beta$. Then $F_{\lambda\lambda_\alpha}$ forces φ_α to have a model $\mathfrak{M}$ of power $\geq\lambda$ such that $|\mathfrak{M}| < \kappa_\alpha$. As $F_{\lambda\lambda_\beta} \subseteq F_{\lambda\lambda_\alpha}$, $F_{\lambda\lambda_\beta}$ forces the same thing. But since $F_{\lambda\lambda_\beta}$ is λ_β-closed and $\kappa_\alpha \leq \lambda_\beta$, we may assume that $\mathfrak{M} \in V$, whence φ_α already has a model of power $\geq\kappa_\alpha$ in V. This is a contradiction of the definition of φ_α. □

We can establish a similar relation between bounded RPC and "bounded Σ_1" as holds between RPC and Σ_1 (Corollary 3.4.5(ii)). A Σ_1-formula $\exists x\varphi(x, y)$ is called "bounded" if for all y, the class $\{x \mid \varphi(x, y)\}$ is a set. Using such "bounded" formulae, we could actually characterize $\hbar(\mathscr{L})$ set-theoretically for a variety of $\mathscr{L}$. As $\hbar(\mathscr{L}) = \hbar(\text{RPC}_{\mathscr{L}})$ in so many practical cases, we confine ourselves to characterizing $\hbar(\text{RPC}_{\mathscr{L}})$. However, we will first make the simple observation given in

4.5.5 Proposition. *If $\mathscr{L}$ is absolute relative to R, $A \subseteq A'$ and $\mathscr{L}'$ pins down R, then* $\hbar(\text{RPC}_{\mathscr{L}}) \leq \hbar(\text{RPC}_{\mathscr{L}'})$.

Proof. See Corollary 3.4.5(i). □

4.5.6 Corollary. *If $\mathscr{L}$ is absolute, then* $\hbar(\mathscr{L}) \leq \hbar(\mathscr{L}_A(W))$. □

4.5.7 Theorem. *Let $\mathscr{L}$ be a logic, R a predicate and*

$$\delta = \sup\{\alpha \mid \alpha \textit{ is } \Sigma_1\textit{-definable in the extended language } \{\in, R\} \textit{ with parameters in } A\}.$$

(i) *If $\mathscr{L}$ is absolute relative to R, then* $\hbar(\text{RPC}_{\mathscr{L}}) \leq \delta$.
(ii) *If $\mathscr{L}$ pins down R, then* $\delta \leq \hbar(\text{RPC}_{\mathscr{L}})$.

Proof. As to (i) suppose that $\lambda = \sup(C)$, for a bounded spectrum C of $\mathrm{RPC}_{\mathscr{L}}$. The cardinal λ has the following definition:

$$\alpha \in \lambda \leftrightarrow \exists \beta(\alpha \leq \beta \wedge \beta \in C).$$

Using absoluteness of $\mathscr{L}$, we can write this in Σ_1-form with a parameter in A. As to (ii), suppose that α is Σ_1-definable with parameters in A. Let $\mathscr{K}$ be the class of well-ordered structures of type $<\alpha$. $\mathscr{K}$ is clearly Σ_1-definable, with parameters in A. By Theorem 3.4.4, $\mathscr{K}$ is $\mathrm{RPC}_{\mathscr{L}}$-definable. Moreover, $\mathrm{Sp}(\mathscr{K})$ is bounded by $|\alpha|^+$. Let $\lambda = \sup \mathrm{Sp}(\mathscr{K})$. Well-known properties of all Hanf numbers imply that $\alpha^+ < h(\mathscr{L})$. Thus, we have that $\lambda < h(\mathscr{L})$. ☐

4.5.8 Corollary. *If $\mathscr{L}$ is absolute relative to R and pins down R, then*

$$h(\mathrm{RPC}_{\mathscr{L}}) = \sup\{\alpha \mid \alpha \textit{ is } \Sigma_1\textit{-definable in } \{\varepsilon, R\} \textit{ with parameters in } A\}. \quad ☐$$

4.5.9 Examples. (i) $h(\mathscr{L}_A(W)) = \sup\{\alpha \mid \alpha$ is Σ_1-definable with parameters in $A\}$.
(ii) $h(\mathscr{L}_A^2) = \sup\{\alpha \mid \alpha$ is Σ_2-definable with parameters in $A\}$.

Theorem 4.5.7 permits an iteration similar to that of Theorem 4.4.3.

4.5.10 Theorem. $h(\Delta_n(\mathscr{L}_A)) = \sup\{\alpha \mid \alpha$ *is* Σ_n*-definable with parameters in* $A\}$ $(n > 1)$.

Up until now, we have characterized the suprema of Σ_n-, Π_n-, and Δ_n-definable ordinals in terms of Hanf and Löwenheim numbers. The logics in question—$\Delta_n(\mathscr{L}_A)$—are so strong that there is little hope of deciding any questions concerning them in ZFC alone. However, one rather curious relation between the different Hanf and Löwenheim numbers is not hard to prove.

4.5.11 Theorem. $\ell(\Delta_n(\mathscr{L}_A)) < h(\Delta_n(\mathscr{L}_A)) = \ell(\Delta_{n+1}(\mathscr{L}_A))$ $(n > 1)$.

4.5.12 Corollary. *If the required large cardinals exist, then*

1st measurable $< \ell(\Delta_2(\mathscr{L}_A)) <$
1st supercompact $< h(\Delta_2(\mathscr{L}_A)) = \ell(\Delta_3(\mathscr{L}_A)) <$
1st extendible $< h(\Delta_3(\mathscr{L}_A))$.

Remark. If we consider Theorem 4.5.10 for $n = 1$, we have to add the quantifier W to $\mathscr{L}_A$ to make the situation non-trivial (for $n > 1$, this would make no difference). The inequality-part remains true then. The equality-part fails, for if κ is the last κ such that $\kappa \to (\omega_1)^{<\omega}$, then $h(\mathscr{L}_{\omega\omega}(W)) < \kappa < \ell(\Delta_2(\mathscr{L}_{\omega\omega}))$. But, of course, there need not exist such a large κ. Indeed, the theorem does hold also for $n = 1$ in L. And we have

4.5.13 Theorem. (i) $h(\mathscr{L}_A(W)) \leq \ell(\mathscr{L}_A^2)$.
(ii) *If* $V = L$, *then* $h(\mathscr{L}_A(W)) = \ell(\mathscr{L}_A^2)$.

Remarks. A number "about the size" of $h(\mathscr{L}_{\omega\omega}(W))$ and $\ell(\mathscr{L}^2_{\omega\omega})$ is $h(\mathscr{L}_{\omega_1\omega_1})$. If $V = L$, then $\ell(\mathscr{L}^2_{\omega\omega}) < h(\mathscr{L}_{\omega_1\omega_1})$, because the former equals $h(\mathscr{L}_{\omega\omega}(W)) < h(\mathscr{L}_{\omega_1\omega_1})$. Observe that $\mathrm{cf}(h(\mathscr{L}_{\omega_1\omega_1})) > \omega$, so that $h(\mathscr{L}_{\omega_1\omega_1})$ can never really equal either $h(\mathscr{L}_{\omega\omega}(W))$ or $\ell(\mathscr{L}^2_{\omega\omega})$. Kunen [1970] showed that if $V = L^\mu$, then $h(\mathscr{L}_{\omega_1\omega_1})$ exceeds the 1st measurable cardinal. On the other hand, there is a model in which $h(\mathscr{L}_{\omega_1\omega_1})$ is below the first weakly compact (Väänänen [1980c]) and, hence, also below $\ell(\mathscr{L}^2_{\omega\omega})$. Another curiosity in this field is that although $\ell(\mathscr{L}) < h(\mathscr{L})$ is true of almost all logics, it is not a rule: The statement $\ell(\mathscr{L}_{\omega\omega}(I)) < h(\mathscr{L}_{\omega\omega}(I))$ is independent of ZFC. Also, all non-trivial claims of relation between $\ell(\mathscr{L}_{\omega\omega}(I))$, $h(\mathscr{L}_{\omega\omega}(I))$ and large cardinals turn out to be independent (Väänänen [1982a]). The numbers can be as small or as large as conceivably possible, if measured by large cardinals. The interrelations of the Hanf and Löwenheim numbers discussed can be visualized in the form of Diagram 2, where an arrow means "less or equal to".

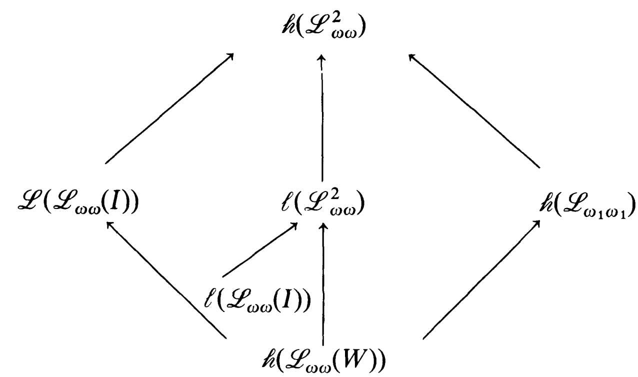

Diagram 2

It is not known whether $h(\mathscr{L}_{\omega_1\omega_1}) < h(\mathscr{L}_{\omega\omega}(I))$ holds absolutely or not, but no arrows are otherwise missing. If $V = L$, the picture collapses (Diagram 3).

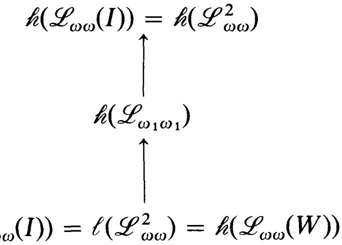

Diagram 3

Historical and Bibliographical Remarks. The main results on the failure of the weak Beth property, Theorems 4.1.2 and 4.1.3 are respectively due to Burgess [1977] and Gostanian–Hrbacek [1976]. They have many precedents in the literature, Mostowski [1968] being perhaps the most notable. Also Theorem 4.1.4, 5, 7(ii) and 8 are from Gostanian–Hrbacek [1976].

The results given in Sections 4.2.5–7 are from Cutland–Kaufman [1980]. Theorem 4.2.10 is from Stavi [1978] which also contains refinements of Theorem 4.2.10. The incompleteness results, Corollary 4.3.7 and Theorem 4.3.8, are respectively due to Barwise [1972a] and Cutland–Kaufman [1980]. The latter is also the best reference to Theorem 4.3.9 and many other results on Σ_1-compactness and validity questions for unbounded absolute logics. The motivation behind Theorem 4.3.10 as well as its proof are given in all details in Feferman [1975].

Theorems 4.4.3–9 are from Väänänen [1979a], while Example 4.4.5(ii) is independently due to Krawczyk–Marek [1977]. The relation between supercompactness, extendibility and $R_\kappa \prec_n V$ are from Solovay *et al.* [1978]. Magidor [1971] establishes important relations between supercompactness, extendibility, and second-order logic. Theorem 4.4.10(i) is proven in a way similar to Theorem 4.2.10. Part (ii) of this result is due to Pinus [1978].

Examples 4.5.1(i) and (ii) are due to Silver [1971]; while (iii) is due to Burgess [1978]. The results given in sections 4.5.2–4 are from Väänänen [1983] where Theorem 4.5.4 is also proven in the stronger form, namely

$$\mathrm{Con}[\hbar(\mathscr{L}_{\omega\omega}(I)) < \hbar(\Delta(\mathscr{L}_{\omega\omega}(I)))].$$

Corollary 4.5.6 is due to Barwise [1972a], and the results in Sections 4.5.7–13 are from Väänänen [1979a]. Example 4.5.9(ii) is independently due to Krawczyk and Marek [1977]. Theorem 4.5.13 is proven in Väänänen [1979b].

Suggestions for Further Work in the Area. It seems likely that further progress can be made in the following parts of this chapter:

1. *The Program Presented in Feferman* [1974b, 1975]. The analysis of adequacy to truth presented here, as well as Theorem 4.3.10 are parts of the program. However, the entire program is much more ambitious.
2. *Relative Absoluteness.* The set-theoretical method is at its best in the context of absolute logics and there are but few results on relatively absolute logics. In view of Hutchinson [1976], it seems possible to develop set-theoretical proofs for compactness theorems. Although, in general we have tended to ignore compact logics in this chapter, it would nevertheless be interesting to extend the scope in their direction.
3. *Canonical Failure of Interpolation.* We have undefinability of truth style proofs for the failure of different forms of interpolation in various logics. These proofs do not apply directly to $\mathscr{L}_{\omega\omega}(Q_1)$ or to $\mathscr{L}_{\omega\omega}(\mathrm{aa})$, for example. Is there a canonical anti-interpolation theorem which applies to these countably compact logics?
4. *Can Validity in an Unbounded Absolute Logic be* Σ_1? The validity problem seems to provide a fruitful framework for further work in abstract model theory.
5. *Löwenheim–Skolem and Hanf Prospects.* One may formulate downward or upward Löwenheim–Skolem theorems which are stronger than those related to Löwenheim and Hanf numbers. Magidor [1971] is an example. The proofs of such theorems tend to depend on large cardinal or combinatorial axioms of set theory.