Chapter XV
Topological Model Theory

by M. Ziegler

1. Topological Structures

A (one-sorted) topological structure $\tilde{\mathfrak{A}} = (\mathfrak{A}, \alpha)$ with vocabulary τ consists of a τ-structure $\mathfrak{A}$ and a topology α on A. Familiar examples are topological spaces ($\tau = \emptyset$), and topological groups and fields. Note that in general we do not assume that the relations and operations of $\mathfrak{A}$ are compatible with α. This in contrast to Robinson [1974].

A *logic* for topological structures is a pair $(\mathscr{L}, \models)$, where $\mathscr{L}[\tau]$ is a class (of "$\mathscr{L}$-sentences") for each vocabulary τ and $\models$ is a relation between topological structures and $\mathscr{L}$-sentences. We will now assume that the axioms of a regular logic hold for topological structures (see Examples 1.1.1 and Discussion 1.2). The relativization axiom is, of course, an exception to this general assumption. The reader should consult Section 2 for a description of the many-sorted case.

1.1. Three Logics for Topological Structures

We first consider *quantification over α and the logic $\mathscr{L}^t_{\text{mon}}$*. We say that an $\mathscr{L}^t_{\text{mon}}[\tau]$-formula is built up from atomic $\mathscr{L}_{\omega\omega}[\tau]$-formulas and atomic formulas

$$t \in X,$$

where t is a τ-term and X a "set variable" (which ranges over α), using $\neg$, $\wedge$, $\vee$, $\forall x$, $\exists x$, $\forall X$, $\exists X$. The semantics are self-explanatory. A logic (for $\tau = \emptyset$) equivalent to $\mathscr{L}^t_{\text{mon}}$ was introduced in Grzegorczyk [1951] and Henson *et al.* [1977].

1.1.1 Examples. (i) $(A, \alpha) \models \forall X\, \forall Y(\exists x\, \exists y(x \in X \wedge y \in Y) \to \exists x((x \in X \wedge x \in Y) \vee (\neg x \in X \wedge \neg x \in Y)))$ or, more briefly, $(A, \alpha) \models \forall X, Y(X \neq \emptyset \wedge Y \neq \emptyset \to (X \cap Y \neq \emptyset \vee X \cup Y \neq \text{universe}))$ which holds iff (A, α) is connected.

(ii) $(A, F, \alpha) \models \forall X\, \exists Y\; Y = f^{-1}(X)$ iff $F: A \to A$ is continuous with respect to α.

(iii) $(A, B, \alpha) \models \exists X\, \forall x(P(x) \leftrightarrow x \in X)$ iff B is open, i.e., $B \in \alpha$.

The next idea is that of restricted quantification over α and the logic $\mathscr{L}^t_{\omega\omega}$. We say that the formulas of $\mathscr{L}^t_{\omega\omega}$ are those $\mathscr{L}^t_{\text{mon}}$-formulas in which quantification over set variables is allowed only in the form

$$\exists X(t \in X \wedge \varphi) \qquad \text{(more briefly, } \exists X \ni t\varphi\text{)},$$

where X (that is, any atomic formula $s \in X$) occurs only negatively in φ, and dually in the form

$$\forall X(t \in X \rightarrow \varphi) \qquad \text{(more briefly, } \forall X \ni t\varphi\text{)},$$

where X occurs only positively in φ. $\mathscr{L}^t_{\omega\omega}$ was introduced by McKee [1975], [1976] and developed in Garavaglia [1978a] and Flum–Ziegler [1980]. Indeed, most of the material in the present chapter is explored in greater detail in Flum–Ziegler [1980a].

1.1.2 Examples. (i) $(A, F, \alpha) \models \forall x\, \forall Y \ni f(x)\, \exists X \ni x\, \forall z(z \in X \rightarrow f(z) \in Y)$ iff F is continuous.

(ii) $(A, B, \alpha) \models \forall x(P(x) \rightarrow \exists X \ni x\, \forall y(y \in X \rightarrow P(y)))$ iff B is open.

(iii) $(A, \alpha) \models \forall x\, \forall X \ni x\, \exists Y \ni x\, \forall y(y \in X \vee \exists Z \ni y \wedge Z \cap Y = \emptyset)$ or, more shortly, iff (A, α) is regular—regular meaning simply that every point has a base of closed neighborhoods.

Finally, we consider the interior operator and the logic $\mathscr{L}_{\omega\omega}(I^n)$ for $n \geq 1$. We pass from $\mathscr{L}_{\omega\omega}$ to $\mathscr{L}_{\omega\omega}(I^n)$, adding the formation rule that if φ is a formula and $x_1 \cdots x_n$ are distinct variables, then

$$I^n x_1 \ldots x_n \varphi$$

is a formula the free variables of which are $x_1 \ldots x_n$ and the free variables of φ. The semantics is given by

$$\mathfrak{A} \models I^n x_1 \ldots x_n\, \varphi(\vec{x}, \vec{y})[a_1 \ldots a_n, \vec{b}] \quad \text{iff}$$
$$\vec{a} \text{ is in the interior of } \{\vec{c} \in A^n \mid \mathfrak{A} \models \varphi(\vec{c}, \vec{b})\}.$$

$\mathscr{L}_{\omega\omega}(I^n)$ was investigated in Sgro [1980a] and Makowsky–Ziegler [1981].

1.1.3 Examples. (i) $(A, B, \alpha) \models \forall x(P(x) \rightarrow I^1 x P(x))$ iff B is open.

(ii) $(A, \alpha) \models \forall x, y(x = y \vee I^2 xy \neg\, x = y)$ iff (A, α) is a Hausdorff space.

1.2. Discussion

From the preceding developments, we clearly have that $\mathscr{L}^t_{\omega\omega} \leq \mathscr{L}^t_{\text{mon}}$. Also, $\mathscr{L}_{\omega\omega}(I^n) \leq \mathscr{L}^t_{\omega\omega}$, since $I^n x_1 \ldots x_n \varphi$ can be expressed by

$$\exists X_1 \ni x_1, \ldots, \exists X_n \ni x_n\, \forall x_1 \ldots x_n \left(\bigwedge_{i=1}^{n} x_i \in X_i \rightarrow \varphi \right).$$

We will now prove that

$$\mathscr{L}_{\omega\omega}(I^n)_{n<\omega} < \mathscr{L}^t_{\omega\omega} < \mathscr{L}^t_{\text{mon}},$$

the first inequality following from

1.2.1 Lemma. *Regularity is not expressible in* $\mathscr{L}_{\omega\omega}(I^n)$.

Proof. By an easy induction on φ, we show that for every $\mathscr{L}_{\omega\omega}(I^n)[\varnothing]$-formula φ there is an quantifier-free $\mathscr{L}_{\omega\omega}[\varnothing]$-formula which is equivalent to φ in any Hausdorff space having no isolated points. Whence, all such spaces are $\mathscr{L}_{\omega\omega}(I^n)$-equivalent. But there are regular and non-regular examples of such spaces. ☐

Remarks. (a) $\mathscr{L}_{\omega\omega}(I^n) < \mathscr{L}_{\omega\omega}(I^{n+1})$.

(b) Continuity is not expressible in $\mathscr{L}_{\omega\omega}(I^n)$.

(c) Sgro [1977a] initiated the study of topological model theory by proving a completeness theorem for $\mathscr{L}_{\omega\omega}(Q)$, where $\mathscr{L}_{\omega\omega}(Q)$ is obtained from $\mathscr{L}_{\omega\omega}$ by adding the quantifier $Qx\ \varphi$ whose meaning is "$\{x \mid \varphi(x)\}$ is open." $\mathscr{L}_{\omega\omega}(Q)$ is weaker than $\mathscr{L}_{\omega\omega}(I^1)$, and does not have the interpolation property even though $\mathscr{L}_{\omega\omega}(I^n)$ does.

To see that $\mathscr{L}^t_{\text{mon}}$ is strictly stronger that $\mathscr{L}^t_{\omega\omega}$, we first observe that $\mathscr{L}^t_{\text{mon}}$ is not $\aleph_0$-compact and does not have the Löwenheim–Skolem property down to $\aleph_0$. (We say that $(\mathfrak{A}, \alpha)$ is *countable* if $\mathfrak{A}$ is countable and α has a countable base.) Moreover, $\mathscr{L}^t_{\text{mon}}$ is not recursively axiomatizable. To see these facts, we will let α be the natural topology on $\mathbb{R}$. Then $(\mathbb{R}, 0, 1, +, -, \cdot, <, \alpha)$ is characterized by the $\mathscr{L}^t_{\text{mon}}$-sentence

$$\theta \equiv \text{"ordered field with connected order topology."}$$

This proves the first two assertions. For the third, we observe that for discrete α $\mathscr{L}^t_{\text{mon}}$ reduces to monadic second-order language, which we can use to characterize $(\mathbb{N}, +, \cdot)$. On the other hand we have:

1.2.2 Theorem. *The logic* $\mathscr{L}^t_{\omega\omega}$

(i) *is compact;*

(ii) *has the Löwenheim–Skolem property down to* $\aleph_0$, *and*

(iii) *is recursively axiomatizable.*

We use the notion of a *weak structure* to prove this result, such a structure being a pair $(\mathfrak{A}, \beta)$, where β is a set of subsets of A. If we consider $\mathscr{L}^t_{\text{mon}}$ as a logic for weak structures, we have—by first-order model theory—compactness, the Löwenheim–Skolem property, and recursive axiomatizability. But the sentences of $\mathscr{L}^t_{\omega\omega}$ are just designed to be *basis-invariant*:

1.2.3 Lemma. *If* $\varphi \in \mathscr{L}^t_{\omega\omega}$, $(\mathfrak{A}, \alpha)$ *is a topological structure and* β *is a base of* α, *then*

$$(\mathfrak{A}, \alpha) \models \varphi \quad \textit{iff} \quad (\mathfrak{A}, \beta) \models \varphi.$$

This is a fact familiar from $\varepsilon - \delta$-calculus. The proof follows immediately from an easy induction on φ.

Finally, consider the $\mathscr{L}^t_{\omega\omega}$-sentence

$$\varphi_{\text{bas}} = \forall x \, \exists X \ni x \wedge \forall x \, \forall X \ni x \, \forall Y \ni x \, \exists Z \ni x \quad Z \subset X \cap Y.$$

Clearly, we have that $(A, \beta) \models \varphi_{\text{bas}}$ iff β is a base of a topology.

1.2.4 Corollary. *$T \subset \mathscr{L}^t_{\omega\omega}$ has a topological model iff $T \cup \{\varphi_{\text{bas}}\}$ has a weak model.*

Remark. This can be rephrased as $T \models_t \varphi$ iff $T \cup \{\varphi_{\text{bas}}\} \models \varphi$. ("$\models$" for weak models).

Theorem 1.2.2 thus follows immediately from Corollary 1.2.4.

In the next section we will prove that for topological structures $\mathscr{L}^t_{\omega\omega}$ is a maximal logic for which is compact and has the Löwenheim–Skolem property. We thus can regard $\mathscr{L}^t_{\omega\omega}$ as the logic which is to topological structure as $\mathscr{L}_{\omega\omega}$ is to ordinary structures. Interestingly enough, Robinson [1973] asked for just such a logic.

The weaker logic $\mathscr{L}_{\omega\omega}(I^n)$ is important because, in some respects at least, it is better behaved than $\mathscr{L}^t_{\omega\omega}$: There is an omitting types theorem—a theorem which is false for $\mathscr{L}^t_{\omega\omega}$, as was shown by Flum–Ziegler [1980, Chapter I, Section 9]—and there is a useful notion of elementary extension.

In subsequent sections we will present results on interpolation, preservation, and definability. That done, we will treat $\mathscr{L}^t_{\omega\omega}$, and, in Section 5, examine the model theory of some special $\mathscr{L}^t_{\omega\omega}$-theories. A series of examples will be given at the end of the chapter, a series that will illustrate how to obtain logics for structures that are similar to topological structures—for example, for uniform structures or for proximity structures. We refer the reader to Flum–Ziegler for more detailed information on these notions.

2. *The Interpolation Theorem*

We discuss the notion of partially isomorphic topological structures and its finite approximations. The methods of Chapter II yield the interpolation theorem and a Lindström theorem for $\mathscr{L}^t_{\omega\omega}$. We will use the interpolation theorem to show that basis-invariant $\mathscr{L}^t_{\text{mon}}$-sentences are equivalent to $\mathscr{L}^t_{\omega\omega}$-sentences. Finally, we will prove that two topological structures are $\mathscr{L}^t_{\omega\omega}$-equivalent iff they have isomorphic ultrapowers. The results stem from Garavaglia [1978a] and Flum–Ziegler [1980].

A many-sorted topological structure is a many-sorted structure with a family of topologies on every sort. Thus, a many-sorted vocabulary for topological structures consists of sort symbols, relation symbols, function symbols, constants, and topology-sort symbols, which are equipped with sort symbols for the universe on which the topology is defined. Thus, we see that the set variables are themselves sorted.

We will often give definitions, theorems, or proofs for only the one-sorted case. However, this is only for the sake of notational simplicity.

2.1. Partial Isomorphisms

We begin our discussion with the notions contained in

2.1.1 Definition. Let $(\mathfrak{A}, \alpha)$ and $(\mathfrak{B}, \beta)$ be topological structures.

(i) A *partial isomorphism* between $(\mathfrak{A}, \alpha)$ and $(\mathfrak{B}, \beta)$ is a triple $\bar{\pi} = (\pi_0, \pi_1, \pi_2)$, where
  (a) $\pi_0 \subset A \times B$ is a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$;
  (b) $\pi_1 \subset \alpha \times \beta$ satisfies $a\, \pi_0\, b$, $U\, \pi_1\, V$ and $a \in U$ imply $b \in V$;
  (c) $\pi_2 \subset \alpha \times \beta$ satisfies $a\, \pi_0\, b$, $U\, \pi_2\, V$ and $b \in V$ imply $a \in U$;

(ii) $(\mathfrak{A}, \alpha)$ and $(\mathfrak{B}, \beta)$ are n-isomorphic $(\simeq_p^n)$, if there is a sequence $I_0 \ldots I_n$ of non-empty sets of partial isomorphisms such that for all $\bar{\rho} \in I_{i+1}$ $(i < n)$ the following holds
  (a) For all $b \in B$ there is an extension $\bar{\pi} \in I_i$ of $\bar{\rho}$ that is, $\pi_i \supseteq \rho_i$, for $i = 0, 1, 2$ such that $b \in \operatorname{Rng} \pi_0$;
  (b) For all $a \in A$ there is an extension $\bar{\pi} \in I_i$ of $\bar{\rho}$ such that $a \in \operatorname{Dom} \pi_0$.
  Furthermore, for all $(a, b) \in \rho_0$, we have
  (c) For all neighborhoods V' of b, there is an extension $\bar{\pi} \in I_i$ of $\bar{\rho}$ and a pair $(U, V) \in \pi_1$ such that $a \in U$ and $b \in V \subset V'$.
  (d) For all neighborhoods U' of a, there is an extension $\bar{\pi} \in I_i$ of $\bar{\rho}$ and a pair $(U, V) \in \pi_2$ such that $b \in V$ and $a \in U \subset U'$.

(iii) $(\mathfrak{A}, \alpha)$ and $(\mathfrak{B}, \beta)$ are partially isomorphic $(\simeq_p)$, if they are 1-isomorphic with $I_0 = I_1$.

2.1.2 Proposition. *Isomorphic topological structures are partially isomorphic. The converse is true for countable topological structures.*

Proof. If $f: \mathfrak{A} \to \mathfrak{B}$ is an isomorphism, set $I = \{(f, \pi, \pi)\}$, where

$$\pi = \{(U, f(U)) \mid U \in \alpha\}.$$

Then $\mathfrak{A} \simeq_p \mathfrak{B}$ via I.

If, conversely, $\mathfrak{A}$ and $\mathfrak{B}$ are countable and partially isomorphic via I, we construct an ascending sequence $\bar{\pi}^i$ $(i \in \omega)$ of elements of I such that $\mathfrak{A} \simeq_p \mathfrak{B}$ via $\{\bar{\pi}^i \mid i \in \omega\}$. (Note that in Definition 2.1.1((ii)(c), (d)) it is enough to let U' and V' range over a countable base of α and β.) But now $\bigcup \{\pi_0^i \mid i \in \omega\}$ is an isomorphism of $\mathfrak{A}$ and $\mathfrak{B}$. □

2.1.3 Proposition. *Suppose τ is finite. Then for every n and every topological τ-structure $\bar{\mathfrak{B}}$ there is an $\mathscr{L}^t_{\omega\omega}[\tau]$-sentence $\psi^n_{\mathfrak{B}}$ such that*

$$\bar{\mathfrak{A}} \simeq^n_p \bar{\mathfrak{B}} \quad \textit{iff} \quad \bar{\mathfrak{A}} \models \psi^n_{\mathfrak{B}}.$$

Proof. Let $\bar{\mathfrak{B}} = (\mathfrak{B}, \beta)$. We define for $b_0 \dots b_{k-1} \in B$ and $V_0 \dots V_{l-1}, V'_0 \dots V'_{m-1} \in \beta$ the formula

$$\psi^0_{\vec{b};\vec{V};\vec{V}'}(x_0 \dots x_{k-1}; X_0 \dots X_{l-1}; Y_0 \dots Y_{m-1})$$

as the conjunction of all reduced basic $\vartheta(\vec{x})$, where $\bar{\mathfrak{B}} \models \vartheta(\vec{b})$, $\neg x_i \in X_j$, where $b_i \notin V_j$, and $x_i \in Y_j$, where $b_i \in V'_j$.

Using induction, we define

$$\psi^{i+1}_{\vec{b};\vec{V};\vec{V}'}(\vec{x}; \vec{X}; \vec{Y})$$

to be the conjunction of the following four formulas which correspond to Definition 2.1.1(ii)(a), (b), (c), (d):

$$\bigwedge_{b \in B} \exists x\, \psi^i_{\vec{b},b;\vec{V};\vec{V}'}(\vec{x}, x; \vec{X}; \vec{Y}),$$

$$\forall x \bigvee_{b \in B} \psi^i_{\vec{b},b;\vec{V};\vec{V}'}(\vec{x}, x; \vec{X}; \vec{Y}),$$

$$\bigwedge_{j<m} \bigwedge_{b_j \in V \in \beta} \exists X \ni x_j \psi^i_{\vec{b};\vec{V};V;\vec{V}'}(\vec{x}; \vec{X}, X; \vec{Y}),$$

and

$$\bigwedge_{j<m} \forall Y \ni x_j \bigvee_{b_j \in V' \in \beta} \psi^i_{\vec{b};\vec{V};\vec{V}',V'}(\vec{x}; \vec{X}; \vec{Y}, Y).$$

Note that we can prove by induction that all conjunctions and disjunctions are in fact finite and that the X_j (Y_j) occur only negatively (positively) in $\psi^i \dots$. We set

$$\psi^n_{\mathfrak{B}} = \psi^n_{\varnothing;\varnothing;\varnothing}.$$

If $\bar{\mathfrak{A}} \simeq^n_p \bar{\mathfrak{B}}$ via $I_0 \dots I_n$, then we show by induction on i that

$$\bar{\mathfrak{A}} \models \psi^i_{\vec{b};\vec{V};\vec{V}'}(\vec{a}; \vec{U}; \vec{U}'),$$

whenever $a_j \pi_0 b_j$, $U_j \pi_1 V_j$ and $U'_j \pi_2 V'_j$, for some $\bar{\pi} \in I_i$.

For the converse, for $\bar{\mathfrak{A}} = (\mathfrak{A}, \alpha)$ define

$$I_i = \{(\{(a_0, b_0) \cdots (a_{k-1}, b_{k-1})\}, \{(U_0, V_0), \dots\}, \{\dots (U'_{m-1}, V'_{m-1})\}) \mid \bar{\mathfrak{A}} \models \psi^i_{\vec{b};\vec{V};\vec{V}'}(\vec{a}, \vec{U}, \vec{U}'), U_j \in \alpha, U'_j \in \alpha\}.$$

Then, $\mathfrak{A} \simeq_p^n \mathfrak{B}$ via $I_0 \dots I_n$, to see that I_n is not empty we notice that $\mathfrak{A} \models \psi_{\mathfrak{B}}^n$ implies $(\emptyset, \emptyset, \emptyset) \in I_n$. □

Remark. In fact, $\mathfrak{A} \equiv_{\mathscr{L}^t_{\omega\omega}} \mathfrak{B}$ iff $\mathfrak{A} \simeq_p^n \mathfrak{B}$ for all n.

2.2. The Interpolation Theorem

2.2.1 Theorem. *$\mathscr{L}^t_{\omega\omega}$ has the interpolation property.*

To prove this result we need the following

2.2.2 Lemma (See Chapter II, Section 5.5). *For finite τ $\simeq_p$ is an* RPC*-relation with definable approximations $\simeq_p^n$. This, in effect, means that there is an extension τ^* of τ containing a new copy of τ and a new relation symbol $<$ and there is $\Sigma \in \mathscr{L}^t_{\omega\omega}[\tau^*]$ such that for all topological τ-structures $\mathfrak{A}$, $\mathfrak{B}$:*

(i) $\mathfrak{A} \simeq_p \mathfrak{B}$ *iff the pair $(\mathfrak{A}, \mathfrak{B})$ can be expanded to a model of Σ, where $<$ defines a non-well-ordering.*
(ii) $\mathfrak{A} \simeq_p^n \mathfrak{B}$ *iff the pair $(\mathfrak{A}, \mathfrak{B})$ can be expanded to a model of Σ, where $<$ defines a linear ordering of its field with more than n elements.*

We leave the proof to the reader.

To prove Theorem 2.2.1 we let κ_1 and κ_2 be two disjoint RPC-classes in $\mathscr{L}^t_{\omega\omega}[\tau]$. Let the $\psi_{\mathfrak{B}}^n$ be as in Proposition 2.1.3. For every n, we have

$$\kappa_1 \models \bigvee \{\psi_{\mathfrak{B}}^n | \mathfrak{B} \in \kappa_1\}.$$

Thus, by compactness $\kappa_1 \models \chi^n$ for a finite disjunction χ^n of the $\psi_{\mathfrak{B}}^n$ $(\mathfrak{B} \in \kappa_1)$.

We want to show that $\kappa_2 \models \neg\chi^n$, for some n. If not, then there is $\mathfrak{A}_n \in \kappa_2$, $\mathfrak{B}_n \in \kappa_1$ such that $\mathfrak{A}_n \models \psi_{\mathfrak{B}_n}^n$. Whence, $\mathfrak{A}_n \simeq_p^n \mathfrak{B}_n$, for every n. By Lemma 2.2.2, compactness and the Löwenheim–Skolem property, there are countable $\mathfrak{A} \in \kappa_2$, $\mathfrak{B} \in \kappa_1$ such that $\mathfrak{A} \simeq_p \mathfrak{B}$. But then $\mathfrak{A} \cong \mathfrak{B}$ and κ_1 and κ_2 are not disjoint—a contradiction. □

2.2.3 Corollary (Flum–Ziegler [1980]). *$\mathscr{L}^t_{\omega\omega}$ is a maximal logic for (many-sorted) topological structures which is compact and has the Löwenheim–Skolem property down to $\aleph_0$.*

(See Chapter II) *Proof.* Let $\mathscr{L}$ be a compact extension of $\mathscr{L}^t_{\omega\omega}$, with the Löwenheim–Skolem property. The above proof shows how to separate disjoint $\mathrm{EC}_{\mathscr{L}}$-classes by an $\mathrm{EC}_{\mathscr{L}^t_{\omega\omega}}$-class. □

2.2.4 Corollary. *The basis-invariant sentences of $\mathscr{L}^t_{\text{mon}}$ are equivalent to $\mathscr{L}^t_{\omega\omega}$-sentences.*

Proof. This follows directly from Corollary 2.2.3, since invariant sentences form a compact logic with the Löwenheim–Skolem property. Instead of proceeding on the basis of Corollary 2.2.3 we give a derivation which stems from Theorem 2.2.1.

Let $(\mathfrak{A}, \alpha)$ be a topological structure and let β_1, β_2 be two bases of α. We code β_1 and β_2 in the structure

$$(\mathfrak{A}, \alpha, B_1, B_2, E_1, E_2)$$

using two new sorts B_1, B_2 and two relations $E_i \subset A \times B_i$ such that

$$\beta_i = \{E_i b \mid b \in B_i\},$$

where $E_i b = \{a \mid aE_i b\}$. If φ is an $\mathscr{L}^t_{\text{mon}}$-sentence, let φ_i denote the $\mathscr{L}_{\omega\omega}$-sentence obtained by replacing the set variables $X, Y, \ldots$ in φ by $x^i, y^i, \ldots$ of sort i and the atomic sentences $t \in X$ by $t\mathbf{E}_i x^i$, where $\mathbf{E}_i$ is the symbol for E_i.

If φ is basis-invariant (in the vocabulary of $\mathfrak{A}$), then we have

$$\models_t (\mathbf{E}_1 \text{ codes a base} \wedge \varphi_1) \to (\mathbf{E}_2 \text{ codes a base} \to \varphi_2)$$

By Theorem 2.2.1, we find an interpolant ψ in $\mathscr{L}^t_{\omega\omega}$. But then $\models_t \varphi \leftrightarrow \psi$. ☐

The final result in this section makes use of the notion of the *ultrapowers*, in particular the ultrapower $(\mathfrak{A}, \alpha)^I/_U$ of $(\mathfrak{A}, \alpha)$ is $(\mathfrak{A}^I/_U, \gamma)$, where γ is the topology with base $\alpha^I/_U$.

2.2.5 Corollary. *Two topological structures are $\mathscr{L}^t_{\omega\omega}$-equivalent iff they have isomorphic ultrapowers.*

Proof. Since $\mathscr{L}^t_{\omega\omega}$-sentences are basis-invariant, a topological structure is $\mathscr{L}^t_{\omega\omega}$-equivalent to its ultrapower. This proves one direction.

Suppose $(\mathfrak{A}_1, \alpha_1) \cong_{\mathscr{L}^t_{\omega\omega}} (\mathfrak{A}_2, \alpha_2)$. Expand the vocabulary τ by two new sorts and two new relation symbols as in the proof of Corollary 2.2.4. Code a base of α_i in $\mathfrak{C}_i = (\mathfrak{A}_i, B_i, E_i)$. By assumption and Theorem 2.1.1 the $\mathscr{L}^t_{\omega\omega}$-theory

$$T = \text{Th}_{\mathscr{L}_{\omega\omega}}(\mathfrak{C}_1) \cup \text{Th}_{\mathscr{L}_{\omega\omega}}(\mathfrak{C}_2) \cup \{\mathbf{E}_1 \text{ codes a base}\} \cup \{\mathbf{E}_2 \text{ codes a base}\}$$

is consistent. Whence there is a model $(\mathfrak{A}, \alpha, B'_1, B'_2, E'_1, E'_2)$ of T. Moreover, by the Keisler–Shelah theorem (see Chang–Keisler [1977]) there is an ultrafilter U such that

$$(\mathfrak{A}_i, B_i, E_i)^I/_U \cong (\mathfrak{A}, B'_i, E'_i)^I/_U.$$

But this implies that

$$(\mathfrak{A}_i, \alpha_i)^I/_U \cong (\mathfrak{A}, \alpha)^I/_U. \quad ☐$$

Remark. It is easy to construct compact logics for topological models having the Löwenheim–Skolem property and which extend $\mathscr{L}_{\omega\omega}$ but are not contained in $\mathscr{L}^t_{\omega\omega}$. However, these examples are not natural.

3. Preservation and Definability

In Section 3.1 we give some examples which will show how to extend preservation theorems from $\mathscr{L}_{\omega\omega}$ to $\mathscr{L}^t_{\omega\omega}$. Here the classical theorem characterizing the $\mathscr{L}_{\omega\omega}$-sentences which are preserved under substructures as the sentences equivalent to universal formulas splits into two.Thus, in this discussion we will use two notions of topological substructure: the just "substructure" (with the subspace topology) appearing in Theorem 3.1.1 and the "open substructure" in Theorem 3.1.2.

In Section 3.2 we prove the topological Feferman–Vaught theorem by an adaptation of the classical proof. This result asserts, in effect, that $\prod_{i\in I} \bar{\mathfrak{A}}_i$ and $\prod_{i\in I} \bar{\mathfrak{B}}_i$ are $\mathscr{L}^t_{\omega\omega}$-equivalent if, for all $i \in I$, $\bar{\mathfrak{A}}_i$ and $\bar{\mathfrak{B}}_i$ are $\mathscr{L}^t_{\omega\omega}$-equivalent. Interestingly enough, a new feature comes into the picture in the case of Beth's theorem. For, according to Definition 2.1.1 an $\mathscr{L}^t_{\omega\omega}$-theory defines a new relation symbol explicitly (by an $\mathscr{L}^t_{\omega\omega}$-formula), if it defines the relation implicitly. But we can now ask what happens if T defines a topology implicitly. If there is no other topology in the vocabulary, then T defines the topology by an $\mathscr{L}_{\omega\omega}$-formula (see Theorem 3.3.2). If not, then no such theorem exists (see Remark 3.3.4)

3.1. Substructures

$(\mathfrak{A}, \alpha)$ is a *substructure* of $(\mathfrak{B}, \beta)$ if $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and α is the restriction of β to A. If $A \in \beta$, then $\bar{\mathfrak{A}}$ is called an open substructure of $\bar{\mathfrak{B}}$.

An $\mathscr{L}^t_{\omega\omega}$-formula in negational normal form (that is, built up from atomic and negated atomic formulas using $\wedge$, $\vee$, $\forall$, $\exists$) is *universal* if it contains no existential individual quantifier. An example of this is the sentence "regular" in Section 1.1.)

3.1.1 Theorem (Flum–Ziegler [1980], Garavaglia [1978a]). *An $\mathscr{L}^t_{\omega\omega}$-sentence is preserved under substructures iff it is equivalent to an universal sentence.*

Proof. Let $\bar{\mathfrak{A}} \subset^n_p \bar{\mathfrak{B}}$ mean that there is a family $I_0 \cdots I_n$ of non-empty sets of partial isomorphisms between $\bar{\mathfrak{A}}$ and $\bar{\mathfrak{B}}$ such that for all $\bar{p} \in I_{i+1}$ $(i < n)$ assertions (b), (c), (d) of Definition 2.1.1(ii) hold. If the above holds for $I_0 = I_1$, we write $\bar{\mathfrak{A}} \subset_p \bar{\mathfrak{B}}$.

The following facts can be shown as Propositions 2.1.2 and 2.1.3 and Lemma 2.2.2:

(a) If $\bar{\mathfrak{A}}$ is a substructure of $\bar{\mathfrak{B}}$, then $\bar{\mathfrak{A}} \subset_p \bar{\mathfrak{B}}$.
(b) If $\bar{\mathfrak{A}}$ and $\bar{\mathfrak{B}}$ are countable and if $\bar{\mathfrak{A}} \subset_p \bar{\mathfrak{B}}$, then $\bar{\mathfrak{A}}$ is isomorphic to a substructure of $\bar{\mathfrak{B}}$.
(c) For every n and every $\bar{\mathfrak{B}}$, there is an universal $\mathscr{L}^t_{\omega\omega}$-sentence $\varphi^n_{\bar{\mathfrak{B}}}$ such that $\bar{\mathfrak{A}} \models \varphi^n_{\bar{\mathfrak{B}}}$ iff $\bar{\mathfrak{A}} \subset^n_p \bar{\mathfrak{B}}$, where τ is finite.
(d) "$\bar{\mathfrak{A}} \subset_p \bar{\mathfrak{B}}$" is an RPC-relation with definable approximations $\subset^n_p$, where τ is finite.

Suppose now that φ is preserved under substructures, or—even more generally—that φ holds in all substructures of models of ψ. Set $\kappa_1 = \operatorname{Mod}\psi$ and $\kappa_2 = \operatorname{Mod}\neg\varphi$. Now the proof of Theorem 2.2.1 shows that there is a universal χ (equal to a finite disjunction of some $\varphi^n_{\bar{\mathfrak{B}}}$, $\bar{\mathfrak{B}} \in \kappa_1$) such that $\kappa_1 \models_t \chi$, $\kappa_2 \models_t \neg\chi$. $\square$

We next treat open substructures. The $\mathscr{L}^t_{\omega\omega}$-sentences that are preserved here are the Π-sentences: In negation normal form the existential individual quantifier can only occur in *bounded* form: $\exists x(x \in X \wedge \varphi)$. The next theorem is related to the Feferman–Kreisel theorem on end extensions (See Section 6) and clarifies the idea of a "local" property.

3.1.2 Theorem (Flum–Ziegler [1980]). *An $\mathscr{L}^t_{\omega\omega}$-sentence is preserved under open substructures iff it is equivalent to a Π-sentence.*

Proof. As the proof of Theorem 3.1.1, we need the proper definition of "$\subset^p_n$ (open)". Thus, we use in conditions (b), (c), and (d) of part (ii) of Definition along with

(a′) For all $(U, V) \in \rho_2$ and all $b \in V$, there is an extension $\bar{\pi} \in I_i$ of $\bar{\rho}$ such that $b \in \operatorname{Rng} \pi_0$. □

Remark. The $\mathscr{L}^t_{\omega\omega}$-sentences preserved under continuous images are the positive sentences without existential set quantification.

3.2. The Feferman–Vaught Theorem

Let $\bar{\mathfrak{A}}_i = (\mathfrak{A}_i, \alpha_i)$, for $i \in I$ be a family of topological structures. The product

$$\prod_{i \in I} \bar{\mathfrak{A}}_i$$

is $(\prod_{i \in I} \mathfrak{A}_i, \alpha)$, where α is the product topology. Furthermore, let σ be the vocabulary of the structure $(\mathscr{P}(I), \cap, \cup, \mathrm{Fin})$, where Fin is the set of finite subsets of I, we can now consider

Theorem (Flum–Ziegler [1980]). *For every $\mathscr{L}^t_{\omega\omega}$-sentence φ there are $\mathscr{L}^t_{\omega\omega}$-sentences $\vartheta_1 \ldots \vartheta_n$ and an $\mathscr{L}_{\omega\omega}[\sigma]$-formula χ such that for all families $(\bar{\mathfrak{A}}_i)_{i \in I}$*

$$\prod_{i \in I} \bar{\mathfrak{A}}_i \models \varphi \quad \textit{iff} \quad (\mathscr{P}(I), \cap, \cup, \mathrm{Fin}) \models \chi(\{i \mid \bar{\mathfrak{A}}_i \models \vartheta_1\}, \ldots, \{i \mid \bar{\mathfrak{A}}_i \models \vartheta_n\}).$$

Proof. Suppose that the X_i only occur negatively in $\varphi(x, \vec{X}^-, \vec{Y}^+)$ and the Y_i only positively. Then we can show by induction on φ that there are $\vartheta_1(\vec{x}, \vec{X}^-, \vec{Y}^+), \ldots, \vartheta_n(\vec{x}, \vec{X}^-, \vec{Y}^+)$ and $\chi(y_1, \ldots, y_n)$ such that χ is monotone in all variables and

$$\prod_{i \in I} \bar{\mathfrak{A}}_i \models \varphi(\vec{a}, \vec{U}, \vec{V}) \quad \text{iff} \quad (\mathscr{P}(I), \ldots) \models \chi(\{i \mid \bar{\mathfrak{A}}_i \models \vartheta_1(\vec{a}_i, \vec{U}_i, \vec{V}_i)\}, \ldots,),$$

for all $\vec{a} \in (\prod_{i \in I} A_i)^k$ and for all $\vec{U}, \vec{V} \in \alpha^k$. □

3.3. Definability

First of all, we note that interpolation implies the Beth definability theorem:

3.3.1 Theorem. *Let $\tau \subset \tau^*$ be vocabularies, $T \subset \mathscr{L}^t_{\omega\omega}[\tau^*]$, and $R \in \tau^*$. If in all models $\mathfrak{A}$ of T the interpretation of R is determined by $\mathfrak{A} \upharpoonright \tau$, then there is an $\mathscr{L}^t_{\omega\omega}[\tau]$-formula $\varphi(\vec{x})$ such that $T \models_t \forall\vec{x}(\varphi(\vec{x}) \leftrightarrow R(\vec{x}))$.*

We will now try to define the topology explicitly. Let $(\mathfrak{A}, \alpha)$ be a topological structure. A formula $\varphi(x, \vec{y})$ *defines* α, if

$$\{\{a \in A \mid \mathfrak{A} \models \varphi(a, \vec{b})\} \mid \vec{b} \in A\}$$

is a base of α. If, for example, α is the order topology of $(A, <^{\mathfrak{A}})$, then α is defined by $y_1 < x \wedge x < y_2$. In general, however, a topology is not definable. But we have:

3.3.2 Theorem (Flum–Ziegler [1980]). *Let T be an $\mathscr{L}^t_{\omega\omega}$-theory, then the following are equivalent:*

(a) *T defines the topology implicitly; that is, $(\mathfrak{A}, \alpha_i) \models T$ implies $\alpha_1 = \alpha_2$.*
(b) *There is an $\mathscr{L}_{\omega\omega}$-formula which defines the topology in all models of T.*

Proof. The reader should consult Flum–Ziegler [1980] for a more detailed proof of this result. The assertion that (b) implies (a) is clear. To prove the other implication, we assume that (a) is true. The interpolation theorem implies:

Claim 1. Every $\mathscr{L}^t_{\omega\omega}$-formula is equivalent (modulo T) to an $\mathscr{L}_{\omega\omega}$-formula.

Now we will further suppose that (b) does not hold and thus derive a contradiction. To this end, we assert

Claim 2. There is a countable model $(\mathfrak{A}, \alpha)$ of T, an element a_0 of A and an open neighborhood P of a_0 which contains no $\mathscr{L}_{\omega\omega}$-definable neighborhood of a_0.

Otherwise, there are $\mathscr{L}_{\omega\omega}$-formulas $\vartheta_1(x, y), \ldots, \vartheta_n(x, y)$ such that in every model $(\mathfrak{A}, \alpha)$ of T every $a_0 \in A$ has a base of neighborhoods of the form

$$\{a \mid \mathfrak{A} \models \vartheta_i(a, \vec{b})\}.$$

We can thus code the ϑ_i in one formula and so assume that $n = 1$. But then

$$\varphi'(x, \vec{y}) = I^1 x \vartheta_1(x, \vec{y})$$

defines the topology in all models of T. By Claim 1 φ' is equivalent to an $\mathscr{L}_{\omega\omega}$-formula $\varphi(x, \vec{y})$. Whence (b) must hold. Contradiction. We now make

Claim 3. There is a topological structure $(\mathfrak{A}^*, P^*, \alpha^*)$ such that $(\mathfrak{A}^*, \alpha^*) \models T$, $(\mathfrak{A}, P) \prec (\mathfrak{A}^*, P^*)$ and P^* is not a neighborhood of a_0.

This is a contradiction of Claim 1, because "S is a neighborhood of a" is an $\mathscr{L}^t_{\omega\omega}$-expression, and so the theorem will follow.

Proceeding with the argument we add a new sort C and a relation $E \subset A \times C$ such that (A, C, E) codes a countable base of α. We need, however,

Claim 4. Let $E_c = \{a: \mathfrak{A} \models a\mathbf{E}c\}$, for $c \in C$, be a neighborhood of a_0. Then there is an extension $(\mathfrak{A}', P', C', E')$ of $(\mathfrak{A}, P, C, E)$ such that $(\mathfrak{A}, P) \prec (\mathfrak{A}', P')$, $(\mathfrak{A}, C, E) \prec (\mathfrak{A}', C', E')$ and $E'c \not\subset P'$.

Otherwise,

$$\mathrm{Th}(\mathfrak{A}, P, a)_{a \in A} \cup \mathrm{Th}(\mathfrak{A}, C, E, a, d)_{a \in A, d \in C} \vdash \forall x(x\mathbf{E}c \to \mathbf{P}(x)).$$

By interpolation, there is an $\mathscr{L}_{\omega\omega}$-formula $\vartheta(x, \vec{a})$ ($\vec{a} \in A$) such that $(\mathfrak{A}, C, E) \models \forall x(x\mathbf{E}c \to \vartheta(x, \vec{a}))$ and $(\mathfrak{A}, P) \models \forall x(\vartheta(x, \vec{a}) \to P(x))$. But then $\vartheta(x, \vec{a})$ defines a neighborhood of a_0, which is contained in P. This contradicts Claim 2.

We can now continue the proof of Claim 3. Starting with $(\mathfrak{A}, P, C, E)$, we can iterate the construction of Claim 4 so as to construct an ascending sequence of countable structures with union $(\mathfrak{A}^*, P^*, C^*, E^*)$ such that $(\mathfrak{A}, P) \prec (\mathfrak{A}^*, P^*)$, $(\mathfrak{A}, C, E) \prec (\mathfrak{A}^*, C^*, E^*)$ and $E^*c \not\subset P^*$, whenever $c \in C^*$, $a_0 E^*c$. Let α^* be the topology generated by $\{E^*c \mid c \in C^*\}$.

3.3.3 Remark. Theorem 3.3.2 can be generalized to a Chang–Makkai type theorem: that is, for an $\mathscr{L}^t_{\omega\omega}$-theory T the following are equivalent:

(a) For all countable $\mathfrak{A}$, $\{\alpha \mid (\mathfrak{A}, \alpha) \models T\}$ is countable.
(b) For all countable models $(\mathfrak{A}, \alpha)$ of T,

$$|\{\beta \mid (\mathfrak{A}, \beta) \cong (\mathfrak{A}, \alpha)\}| < 2^{\aleph_0}.$$

(c) There is an $\mathscr{L}_{\omega\omega}$-formula $\vartheta(x, \vec{y}, \vec{z})$ such that in every model $(\mathfrak{A}, \alpha)$ of T there are $\vec{a} \in A$ for which $\vartheta(x, \vec{y}, \vec{a})$ defines a base of α.

3.3.4 Remark. In concluding this section, we point out two interesting facts about the notions we have discussed. First, we note that there is no Chang–Makkai version of Theorem 3.3.1; and, second, if T is an $\mathscr{L}^t_{\omega\omega}$-theory of structures with two topologies on it, and if we know that $(\mathfrak{A}, \alpha, \beta_i) \models T$ implies that $\beta_1 = \beta_2$, then in general we cannot conclude that B is definable in $(\mathfrak{A}, \alpha)$. The reader should consult Flum–Ziegler [1980] for a more detailed examination of this material.

4. *The Logic* $\mathscr{L}^t_{\omega_1\omega}$

Much of the theory of $\mathscr{L}_{\omega_1\omega}$ and $\mathscr{L}^t_{\omega\omega}$ can be transferred to $\mathscr{L}^t_{\omega_1\omega}$, the latter being equal to $\mathscr{L}^t_{\omega\omega}$ with countable conjunctions and disjunction. For example, the $\mathscr{L}^t_{\omega\omega}$-sentences are (up to equivalence) the basis-invariant $\mathscr{L}^t_{\mathrm{mon}\,\omega_1\omega}$-sentences, where $\mathscr{L}^t_{\text{mon}\,\omega_1\omega}$ is $\mathscr{L}^t_{\text{mon}}$ with countable disjunctions and conjunctions. Moreover, the interpolation theorem, the preservation theorems, and the definability theorem of Section 3, where α is defined by a sequence of formulas, are all true for $\mathscr{L}^t_{\omega_1\omega}$. In the present discussion, we will present the covering theorem (see Chapter X), a theorem which immediately implies the interpolation theorem.

4.1 Theorem. *Let $\tau \subset \tau^*$ be countable vocabularies, and let ψ be a sentence of $\mathscr{L}^t_{\omega_1\omega}[\tau^*]$. Then there is a sequence ϑ_α ($\alpha < \omega_1$) of $\mathscr{L}^t_{\omega_1\omega}[\tau]$-sentences such that*

(i) $\psi \models_t \vartheta_\alpha$;
(ii) *for all countable τ-structures $\mathfrak{A}$: if $\mathfrak{A} \models \bigwedge_{\alpha<\omega_1} \vartheta_\alpha$, then $\mathfrak{A}$ is the reduct of a model of ψ;*
(iii) *if $\tau^+ \cap \tau^* = \tau$, $\varphi \in \mathscr{L}^t_{\omega_1\omega}[\tau^+]$ and $\psi \models_t \varphi$, then $\vartheta_\alpha \models_t \varphi$, for some $\alpha < \omega_1$.*

Before undertaking the proof of the theorem, we will consider

4.2 Example. Let τ be empty, $\tau^* = \{P\}$, P a unary predicate, and ψ = "P is perfect." Then, for ϑ_α we can take the sentence which says that the α-th Cantor–Bendixson derivative is non-empty.

Proof. We will indicate the proof of the special case in which τ is one-sorted, $\tau^* = \tau \cup \{P\}$, and $\psi \in \mathscr{L}^t_{\omega\omega}$. It is easy to supply the details a proof of the general result (see Chapter VIII).

First, we observe that ψ can be put in the form

$$\forall x_1\, \forall X_1 \ni x_1\, \exists y_1\, \exists Y_1 \ni y_1\, \forall x_2, \ldots, \exists Y_n \ni y_n$$
$$\bigvee_{k<m} \left(\pi_k(x_1, X_1^+, \ldots, y_n, Y_n^-) \wedge \bigwedge_{j<r_k} P(t_j(\vec{x}, \vec{y})) \wedge \bigwedge_{j'<r'_k} \neg P(t'_{j'}(\vec{x}, \vec{y}))\right)$$

We now associate to ψ a game sentence. First, we choose a 1–1 enumeration $(s_i)_{i<\omega}$ of $\bigcup_{0<i\leq n} {}^i\omega$ such that $s_i \subset s_j$ implies $i \leq j$. Set

$$\Gamma = \forall u_0\, \forall U_0 \ni u_0\, \exists v_0\, \exists V_0 \ni v_0 \bigvee_{k_0<m} \forall u_1, \ldots, \bigwedge \Phi,$$

where Φ is the union of

$$\{\pi_{k_{i_n}}(u_{i_1}, U_{i_1}, \ldots, V_{i_n}) \mid s_{i_1} \subsetneq s_{i_2} \subsetneq \cdots \subsetneq s_{i_n}\}$$

and of

$$\{t_j(x_{i_1}, \ldots, y_{i_n}) \neq t'_{j'}(x_{i'_1}, \ldots, y_{i'_n}) \mid s_{i_1} \subsetneq \cdots \subsetneq s_{i_n},$$
$$s_{i'_1} \subsetneq \cdots \subsetneq s_{i'_n}, j < r_{k_{i_n}}, j' < r'_{k_{i'_n}}\}.$$

Now it is easy to see that $\psi \models_t \Gamma$ and that every countable model $(\mathfrak{A}, \alpha)$ of Γ can be expanded to a model of ψ. For the ϑ_α, we take the approximations of Γ:

$$\vartheta_0^{k_0, \ldots, k_{i-1}}(u_0, U_0, v_0, V_0, \ldots, V_{i-1}) \\ = \bigwedge \{\varphi \in \Phi \mid \varphi \text{ contains only } k_0, \ldots, k_{i-1}, u_0, \ldots, V_{i-1}\},$$

$$\vartheta_\alpha^{k_0, \ldots, k_{i-1}}(u_0, \ldots,) \\ = \forall u_i \, \forall U_i \ni u_i \, \exists v_i \, \exists V_i \ni v_i \bigvee_{k_i < m} \bigwedge_{\beta < \alpha} \vartheta_\beta^{k_0, \ldots, k_i}(u_0, \ldots, V_i).$$

Finally, one shows that $\Gamma \models_t \bigwedge_{\alpha < \omega_1} \vartheta_\alpha \models_t' \Gamma$ and that $\Gamma \models_t \varphi$ implies $\vartheta_\alpha \models_t \varphi$, for some $\alpha < \omega_1$, where $\models_t'$ means $\models_t$ for countable models. □

5. Some Applications

In the following discussions we will give four examples of the expressive power of $\mathscr{L}^t_{\omega\omega}$. In Section 5.1 we show that the theory of T_2-spaces is undecidable while the theory of T_3-spaces is decidable. We will also give invariants that determine the elementary type of T_3-spaces. In Section 5.2 we will show that the theory of torsion free locally pure abelian groups is decidable, although the theory of all topological groups is not. In Section 5.3 we present a complete axiomatization of the theory of the topological field of complex numbers. And finally, in Section 5.4, we show that all infinite dimensional, locally bounded real topological vector spaces are $\mathscr{L}^t$-equivalent: They are, in fact, models of an explicitly given complete theory. The results given in Section 5.1 are explored in Flum–Ziegler [1980].

5.1. Topological Spaces

Let T_2 be the theory of Hausdorff spaces; that is, the set

$$\forall x \, \forall y (x \neq y \to \exists X \ni x \, \exists Y \ni y \quad X \cap Y = \emptyset),$$

then we can consider

5.1.1 Theorem. *T_2 is hereditarily undecidable.*

Proof. Let $\varphi(x, y)$ be the formula $\neg(\exists X \ni x \, \exists Y \ni y \quad \bar{X} \cap \bar{Y} = \emptyset)$, then, for Hausdorff spaces $\mathfrak{A}$, we can make

$$(U, \{(a, b) \in U^2 \mid \mathfrak{A} \models \varphi(a, b)\}),$$

where $U = \{a \in A \mid \mathfrak{A} \models \exists y \neq a \varphi(a, y)\}$ is isomorphic to any graph without isolated points. But the theory of these graphs is known to be hereditarily undecidable. Thus, the assertion in the theorem is established. □

Remark. We recall that totally disconnected spaces are spaces in which any two points can be separated by a clopen set. Let T_ω be the theory of all totally disconnected spaces (that is, T_ω is the set of all $\mathscr{L}^t_{\omega\omega}$ sentences true in all these spaces), then every finite subtheory of T_ω is hereditarily undecidable (for example, the $T_{2,5}$ separation axiom). However, relative to T_ω, every formula is equivalent to a boolean combination of formulas $x = y$ and of formulas having only one free variable. Whether T_ω is decidable, remains an open question.

Let T_3 be the theory of regular Hausdorff spaces, then we have

5.1.2 Theorem. *T_3 is decidable.*

Proof. Every T_3-space is $\mathscr{L}^t_{\omega\omega}$-equivalent to a countable T_3-space. But the countable T_3-spaces are just the topological spaces which come from a countable linear order. Therefore, our result follows from the decidability of the elementary theory of linear orders.

Remark. T_ω is a subtheory of T_3, since in countable regular spaces disjoint closed sets can be separated by clopen sets.

In order to define elementary invariants of a T_3-space $\mathfrak{A}$ we divide A into sets A^s of all points of "type s", where s is an element of

$$S = \bigcup \{S^n \mid n \in \mathbb{N}\}, \quad \text{where} \quad S^0 = \{*\} \text{ and } S^{n+1} = \mathscr{P}(S^n).$$

We set $A^* = A$ and, for $s \in S^{n+1}$, we set

$$A^s = \{a \in A \mid a \text{ is an accumulation point of } A^r \text{ iff } r \in s, \text{ for all } r \in S^n\}.$$

5.1.3 Theorem. *Two T_3-spaces $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathscr{L}^t_{\omega\omega}$-equivalent iff $|A^s| = |B^s|$ (mod $\aleph_0$) for all $s \in S$.*

Example. All T_3-spaces without isolated points are $\mathscr{L}^t_{\omega\omega}$-equivalent. For then $A^s = A$, if s is of the form $*, \{*\}, \{\{*\}\}, \ldots$, and $A^s = \emptyset$ otherwise.

Proof. One direction follows from the observation that the A^s are $\mathscr{L}^t_{\omega\omega}$- definable in $\mathfrak{A}$. For the converse, we can assume that $\mathfrak{A}$ and $\mathfrak{B}$ have bases α and β of clopen sets such that (A, α) and (B, β) are $\aleph_0$-saturated. It is then easily proved that $(\mathfrak{A}, A^s)_{s\in S}$ and $(\mathfrak{B}, B^s)_{s\in S}$ are partially isomorphic via the system I which consists of all finite partial isomorphisms (π_0, π_1, π_2), where $\pi_1 = \pi_2 = \{(U_i, V_i) \mid i < n\}$, the $U_i \in \alpha$ (respectively the $V_i \in \beta$) form a clopen partition of A (respectively B), and $|U_i^s| = |V_i^s|$ (mod $\aleph_0$) for all $i < n$. ☐

5.2. Topological Abelian Groups

We will now consider Hausdorff topological abelian groups. Noting first that this is an $\mathscr{L}^t_{\omega\omega}$-elementary class, we proceed to examine several interesting results, the first of which is

5.2.1 Theorem (Cherlin–Schmitt [1981]). *The theory of Hausdorff abelian groups is undecidable.*

Proof. Let p be a prime and $q = p^9$. Baur [1976] has proven that the theory of all abelian groups (no topology) A of exponent q with a distinguished subgroup B is undecidable. But such a pair can be interpreted in a suitable topological group $\mathfrak{C}$ by letting

$$A = C/qC \quad \text{and} \quad B = \overline{qC}/qC. \quad \square$$

Call a group *locally pure*, if (partial) division by n is continous at 0. That is, a group is locally pure if the following $\mathscr{L}^t_{\omega\omega}$-sentence holds for every n

$$\forall X \ni 0\ \exists Y \ni 0\ \forall x(nx \in Y \to \exists y \in X \quad ny = nx).$$

Cherlin–Schmitt [1981] also proved that the theory of all locally pure groups is hereditarily undecidable. Furthermore, we have

5.2.2 Theorem (Cherlin–Schmitt [1980]). *The theory of all torsion free, locally pure groups is decidable.*

Proof. Since the theory of all (discrete) torsion free groups is decidable and the theory of all non-trivial ordered abelian groups is decidable (see Gurevic [1964]), the theorem follows from

5.2.3 Lemma. *A topological abelian group is torsion free, locally pure, and non-discrete iff it is $\mathscr{L}^t_{\omega\omega}$-equivalent to a non-trivial group with the order topology.*

Proof. One direction is easy to establish. For the converse suppose that $\bar{\mathfrak{A}}$ is torsion free, locally pure and non-discrete. We choose an $\mathscr{L}^t_{\omega\omega}$-equivalent group $(\mathfrak{A}_1, \alpha_1)$ where α_1 has a basis β_1 such that $(\mathfrak{A}_1, \beta_1)$ is $\aleph_1$-saturated. Then, as can be easily shown, α_1 is closed under countable intersections. Starting with an arbitrary U_0, we construct a sequence $(U_i)_{i<\omega}$ of open neighbourhoods of 0 such that for all $i = 0, 1, 2, \ldots$

$$U_{i+1} - U_{i+1} \subset U_i \quad \text{and} \quad nx \in U_{i+1} \to \exists y \in U_i \quad ny = nx.$$

Then the intersection of the U_i is an open pure subgroup of $\mathfrak{A}_1$. Thus, $\bar{\mathfrak{A}}_1$ has a base γ_1 of neighborhoods of 0 consisting of pure subgroups. Choose a countable $(\mathfrak{A}_2, \gamma_2)$ that is elementarily equivalent to $(\mathfrak{A}_1, \gamma_1)$. Then $\{a + U \mid a \in A_2, U \in \gamma_2\}$ is a base of a topology α_2 on A_2 such that $\bar{\mathfrak{A}}$ and $(\mathfrak{A}_2, \alpha_2)$ are $\mathscr{L}^t_{\omega\omega}$-equivalent.

From γ_2 we now choose a descending base for the neighborhood filter of 0, writing $U = U_0 \supset U_1 \supset \cdots$. We then fix an ordering $<_i$ of the torsion free group U_i/U_{i+1} ($i \in N$). If we define $x < y$ iff $x, y \in U_i$, $x + U_{i+1} <_i y + U_{i+1}$, for some i, we then obtain an ordering of $\mathfrak{A}_2$ which generates α_2.

5.3. Topological Fields

Theorem (Prestel–Ziegler [1978]). *The $\mathscr{L}^t_{\omega\omega}$-theory of the topological field of complex numbers is axiomatized by the sentences asserting*

(a) "*algebraically closed field of characteristic* 0";
(b) "*non-discrete Hausdorff topological ring*";
(c) "*V-topology*"; *that is, in symbols, we have*

$$\forall X \ni 0\ \exists Y \ni 0\ \forall x, y(xy \in Y \to x \in X \vee y \in X)$$

Proof. Let $\mathfrak{A}$ be a model of the axioms. Choose $(\mathfrak{B}, \beta)$ $\mathscr{L}^t_{\omega\omega}$-equivalent of $\mathfrak{A}$, where β is closed under countable intersections. Choose a sequence $(U_i)_{i<\omega}$ of neighborhoods of 0 such that $(i + 1) \notin U_i$, $U_{i+1}U_{i+1} \subset U_i$, $U_{i+1} - U_{i+1} \subset U_i$ and $x, y \in U_{i+1} \to x \in U_i$ or $y \in U_i$. Then the intersection U of the U_i is a neighborhood of 0 and has the following properties:

(1) $\mathbb{N} \cap U = \{0\}$.
(2) $UU \subset U$.
(3) $U - U \subset U$.
(4) $x, y \in U \Rightarrow x \in U$ or $y \in U$.

Set

(5) $R = \{b \in B \mid bU \subset U\}$.

Because of Property (3), R is a subring of B. In fact, we prove that R is a valuation ring of B. That is, we can prove that for all $b \in B$, either $b \in R$ or $b^{-1} \in R$. For, otherwise there are $u_i \in U$ such that $bu_1 \notin U$ and $b^{-1}u_2 \notin U$. But by (4) this implies that $u_1u_2 = bu_1b^{-1}u_2 \notin U$—a contradiction to (2).

By (3) U is an ideal of R and is is proper by (1) and prime by (4). But then (5) can hold only if U is the maximal ideal of the valuation ring R. Since $U \neq 0$, we must have that $R \neq B$. Furthermore, (1) implies that R/U has characteristic zero.

By Robinson [1956b], all $(\mathfrak{B}, R)$ are elementarily equivalent, where $\mathfrak{B}$ is algebraically closed and R is a proper valuation ring of $\mathfrak{B}$ with residue class of characteristic 0. Therefore, in order to show the completeness of our axioms, it remains to show that β is the valuation topology of $(\mathfrak{B}, R)$; that is, that $\{rU \mid r \in R\backslash\{0\}\}$ is a base for the neighborhoods of 0.

To that end, we now assume that V is a neighborhood of 0 and choose another neighborhood-W of 0 such that $x, y \notin V \cap U \Rightarrow xy \notin W$. Then $rU \subset V$, for any $r \in W$. For $u \in U$ implies $u^{-1} \notin U$ by (1) and (2). Therefore, $ru \notin V$ would imply that $r = ruu^{-1} \notin W$. □

The methods used in the above proof can be used to prove the following result, a theorem due to Stone [1969].

Approximation Theorem. *Let $\alpha_1, \ldots, \alpha_n$ be different V-topologies of the field K. Then the intersection of any sequence of non-empty open sets $U_i \in \alpha_i$ is non-empty.*

Proof. The theorem claims that $(K, \alpha_1, \ldots, \alpha_n)$ has a certain $\mathscr{L}^t_{\omega\omega}$-property. But we have seen that $(K, \alpha_1, \ldots)$ is $\mathscr{L}^t$-equivalent to a structure $(L, \beta_1, \ldots)$, where the β_i are defined by valuations. In this case, the theorem is well known from valuation theory. ☐

5.4. Topological Vector Spaces

We look at topological vector spaces as two sorted topological structures (R, V, α), where R is an ordered field, V is an R-vector space with a compatible non-discrete Hausdorff topology α. We let x, y range over V and ξ range over R.

Theorem (Sperschneider [1979]). *The $\mathscr{L}^t_{\omega\omega}$-theory of locally bounded real vector spaces of infinite dimension is complete and can be axiomatized by sentences asserting:*

"*infinite dimensional topological vector space over an ordered real closed field*";

"*locally bounded*": $\exists X \ni 0\ \forall Y \ni 0\ \exists \xi \quad X \subset \xi Y$;

"*the Riesz Lemma*": *For all n*, $\forall X \ni 0\ \exists Y \ni 0$ *such that for all subspaces F of dimension* $\leq n$ *and all* $x \notin F\ \exists y \quad y \in F + \langle x \rangle \wedge y \in X \wedge y \notin (F + Y)$.

Proof. It is easy to see that locally bounded real vector spaces satisfy our axioms. (If V is normed, the last axioms follow directly from the Riesz lemma.) Since all infinite dimensional vector spaces over a real closed field with a distinguished Euclidean bilinear form are elementarily equivalent, it is enough to show that every model (R, V, α) of our axioms is $\mathscr{L}^t_{\omega\omega}$-equivalent to a topological vector space whose topology is defined by an Euclidean norm.

We can suppose that α is closed under countable intersections. Then, taking the intersection of a suitable descending chain, we find a bounded neighborhood U of 0; (that is, $\{rU \mid r \in R\setminus\{0\}\}$ is a basis for the neighborhoods of 0) and an infinitesimal $r > 0$ such that $U - U \subset U$, $[-1, 1]U \subset U$ and for all finite dimensional F and $x \notin F$, there is $y \in F + \langle x \rangle$ such that $y \in U$ and $y \notin (F + rU)$. Finally, we choose a neighborhood V of 0 that is contained in all $r^n U$ $(n \leq N)$.

Now (proceed to an elementarily equivalent situation) we drop the assumption that α is closed under countable intersections, and instead assume that V is countable. We can then construct a basis $(x_i)_{i<\omega}$ of V such that $x_i \in U$ and $x_i \notin (\langle x_0, x_1, \ldots, x_{i-1} \rangle + rU)$. Define an Euclidean bilinear form on V such that $(x_i)_{i<\omega}$ becomes an orthonormal basis. Now set $B = \{x \in V \mid (x, x) \leq 1\}$. We will complete the proof by showing that $V \subset B \subset U$.

If $r_0 x_0 + r_1 x_1 + \cdots + r_n x_n \in V \subset r^{n+2} U$, we can conclude that $|r_n| < r^{n+1}$ and $r_0 x_0 + \cdots + r_{n-1} x_{n-1} \in r^{n+1} U$, etc. Whence, we have that $|r_i| < r^{i+1} \leq r$, for all $i = 0, 1, \ldots$. It now follows that $r_0 x_0 + \cdots + r_n x_n \in B$. This again implies that $|r_i| \leq 1$, for all i. Whence, $r_i x_i \in U$ and $r_0 x_0 + \cdots + r_n x_n \in U$. ☐

6. Other Structures

As a logic for topological structures, $\mathscr{L}^t_{\omega\omega}$ was constructed in the following three steps

(1) The second-order notion of a topology was replaced by the first-order notion of a base of topology.
(2) An appropriate logic ($\mathscr{L}^t_{\text{mon}}$) for the "weak structures" $(\mathfrak{A}, \beta)$ was chosen, where β is a base of a topology.
(3) That the $\mathscr{L}^t_{\omega\omega}$-sentences are (up to equivalence) just the base-invariant sentences of $\mathscr{L}^t_{\text{mon}}$ was shown.

There are many other cases in which this philosophy is successful. In the following examples, all of the general theorems given in Sections 1, 2, 3.3, and 4 hold true.

6.1. Quasitopologies

A set of subsets of A is a quasitopology on A, if it is closed under arbitrary unions. Every set β of subsets of A is the base of a quasi-topology α on A since it is possible to set $\alpha = \{\bigcup s \mid s \subset \beta\}$. Thus, a weak structure $(\mathfrak{A}, \beta)$ consists of a structure $\mathfrak{A}$ and a set of subsets of A. The appropriate logic for weak structures is $\mathscr{L}^t_{\text{mon}}$. The sentences of $\mathscr{L}^t_{\text{mon}}$ are basis-invariant are also, up to equivalence, the sentences of $\mathscr{L}^t_{\omega\omega}$. Thus, $\mathscr{L}^t_{\omega\omega}$ can also serve as a natural logic for quasi-topological structures. Topological structures form an elementary class of quasi-topological structures. It is now clear why φ_{bas} (see Corollary 1.2.4) was taken as an $\mathscr{L}^t_{\omega\omega}$-sentence.

6.2. Monotone Systems

Let n be a non-zero natural number. An n-monotone system on A is a system of subsets of A^n which is closed under supersets. A set β of subsets of A^n is the base of the n-monotone system

$$\{C \subset A^n \mid B \subset C \text{ for some } B \in \beta\}.$$

Thus, a weak structure $(\mathfrak{A}, \beta)$ is a structure $\mathfrak{A}$ with a set β of subset of A^n. The logic $\mathscr{L}$ for these weak structures adds set variables $X, Y, \ldots$ and atomic formulas $(t_1 \ldots t_n) \in X$ to $\mathscr{L}_{\omega\omega}$.

Now, up to equivalence, the base invariant $\mathscr{L}$-sentences are the sentences in which set quantification $\exists X\ \varphi$ (respectively $\forall X\ \varphi$) is allowed only if X occurs only negatively (respectively positively in φ).

We use these sentences as a logic $\mathscr{L}^*$ for n-monotone structures. We observe in passing that the same can be done for antitone systems.

Example. ν is a uniformity on A iff (A, ν) is a 2-monotone structure which satisfies the following $\mathscr{L}^*$-axioms:

$\exists X(\text{true})$, (that is, ν is non-empty;

$\forall X\ \forall x(x, x) \in X$;

$\forall X\ \forall Y\ \exists Z\ \forall x\ \forall y(x, y) \in Z \rightarrow ((x, y) \in X \wedge (x, y) \in Y)$;

$\forall X\ \exists Y\ \forall x\ \forall y\ \forall z((x, y) \in Y \wedge (x, z) \in Y) \rightarrow (y, z) \in X)$.

It is easy to prove that ν is an uniformity on A iff (A, ν) is $\mathscr{L}^*$-equivalent to a 2-monotone structure (B, μ) where μ is closed under finite intersections and has a base of equivalence relations.

6.3. Point Monotone Systems

A point monotone system μ on A assigns to every $a \in A$ an 1-monotone system $\mu(a)$ on A. The function $\beta: A \rightarrow \mathscr{P}(A)$ is a base of the point monotone system (monotone system with base $\beta(a) \mid a \in A$).

Precisely what constitutes a logic for these structures? Letting $\mathscr{L}$ denote the logic for such structures, we use sentences that are built-up like $\mathscr{L}_{\omega\omega}$-sentences along with set variables $X, Y, \ldots$, atomic formulas $t \in X$, and quantification $\exists X(t)\varphi$ and $\forall X(t)\varphi$ as the constituents of $\mathscr{L}$. The interpretation of these last two formulas is $X \in \beta(t)$ such that φ and for all $X \in \beta(t)$, φ. Now, the quantification $\exists X(t)\varphi$ (respectively, $\forall X(t)\varphi$) is only allowed in $\mathscr{L}^*$-sentences if X occurs only negatively (respectively, positively) in φ. These are, up to equivalence, the base invariant $\mathscr{L}$-sentences. Thus, we can use $\mathscr{L}^*$ as a logic for point monotone structures.

Example. We can interpret a topology on A as a point monotone structure (A, μ), where $\mu(a)$ is the neighborhood filter of a. Moreover, we can formulate Hausdorff's axioms in $\mathscr{L}^*$ as follows: A point monotone structure (A, μ) is a topological space iff the following $\mathscr{L}^*$-axioms are satisfied:

$\forall x\ \exists X(x)\ (\text{true})$;

$\forall x\ \forall X(x)\ x \in X$;

$\forall x\ \forall X(x)\ \forall Y(x)\ \exists Z(x)\ \forall y \quad y \in Z \rightarrow (y \in X \wedge y \in Y)$;

$\forall x\ \forall X(x)\ \exists Y(x)\ \forall y(y \in Y \rightarrow \exists Z(y)\ \forall z \quad z \in Z \rightarrow z \in X)$.

The resulting logic for topological structures is, of course, equivalent to $\mathscr{L}^t_{\omega\omega}$.

Remark. Call the point monotone structure $(\mathfrak{A}, \mu)$ an open substructure of the point monotone structure $(\mathfrak{B}, \nu)$, if $\mathfrak{A}$ is an substructure of $\mathfrak{B}$ and every $\mu(a)$ is a base of $\nu(a)$. Then, up to equivalence, the $\mathscr{L}^*$-sentences preserved under open substructures are the Π-sentences (which are similarly defined as in Theorem 3.1.2). This result generalizes both Theorem 3.1.2 and the Feferman–Kreisel theorem on end extensions.

6.4. Antitone Systems of Pairs of Sets

A set δ of pairs of subsets of A is *antitone*—and, for the sake of brevity, we write ASPS on A—if $(B_1, B_2) \in \delta$, $C_1 \subset B_1$, $C_2 \subset B_2$ implies $(C_1, C_2) \in \delta$. Every set of pairs of subsets of A is a base of an ASPS in the obvious way. This notion clear, we can arrive at the logic $\mathscr{L}^*$ for ASPS-structures $(\mathfrak{A}, \delta)$ as follows: We extend $\mathscr{L}_{\omega\omega}$ by set variables $X, Y, \ldots$ (for pairs of sets) and new atomic sentences $t \in_1 X$, $t \in_2 X$ whose meaning is that t is in the first (respectively, the second) component of X, and we allow quantification $\exists X\varphi$ (respectively, $\forall X\varphi$) only if X occurs only positively (respectively, negatively) in φ.

Example. A *proximity space* is an ASPS-structure (A, δ) with the following properties:

(a) if $B\,\delta\,C$, then $C\,\delta\,B$;
(b) if $B_1\,\delta\,C$ and $B_2\,\delta\,C$, then $B_2 \cup B_1\,\delta\,C$;
(c) for no $a \in A$ $\{a\}\,\delta\,\{a\}$;
(d) $\varnothing\,\delta\,A$;
(e) if $B\,\delta\,C$, then there are B', C' such that $B \subset B'$, $C \subset C'$, $B' \cap C' = \varnothing$, $B\,\delta\,(A \backslash B')$, and $(A \backslash C')\,\delta\,C$.

Each of the properties can be formulated in $\mathscr{L}^*$. Thus, for example, property (e) reads

$$\forall X\, \exists Y\, \exists Z(\forall x(x \in_2 Y \vee x \in_2 Z) \wedge \forall x(x \in_1 X \rightarrow x \in_1 Y) \wedge \forall x(x \in_2 X \rightarrow x \in_1 Z))$$

Finally, in concluding this discussion, we briefly note that we write $B\,\delta\,C$, for $(B, C) \in \delta$ to mean that Band C are *not proximate*.