Part C

Infinitary Languages

This part of the book is devoted to languages with infinitely long formulas and their applications. Again the structures are of the sort studied in first-order model theory. Languages with richer structures and infinitely long formulas are studied in Part E. The study of infinitely long formulas is more developed than some of the other parts of extended model theory. In particular, there are several books treating various aspects of the subject, notably Keisler [1971a] and Dickmann [1975]. This part of the present book was planned with the existence of these references in mind, containing chapters that give an introduction to the subject leading into these books as well as chapters that discuss more recent advances.

Chapter VIII presents a wealth of material on $\mathscr{L}_{\omega_1\omega}$ and some of its sublogics. Starting with the original motivations for studying languages with infinitely long formulas, the chapter provides both a basic introduction and an explanation of many of the developments that have taken place since Keisler's [1971a] publication. In addition, it discusses extensions of $\mathscr{L}_{\omega_1\omega}$ by new propositional connectives. The importance of these extensions is not for their intrinsic interest so much, as for the fact that they seem to have all the nice properties of $\mathscr{L}_{\omega_1\omega}$, and so make it difficult to find a characterization of $\mathscr{L}_{\omega_1\omega}$ by its model-theoretic properties.

Chapter IX presents an introduction to the stronger logics $\mathscr{L}_{\kappa\lambda}$, one that leads into Dickmann's book [1975] on this topic but also goes beyond it with the presentation of some more recent results. Special emphasis is given to partial isomorphisms and their applications, and to Hanf number computations.

One of the more recent developments in infinitary logic is that dealing with game quantification which has grown out of the work of Svenonius [1965], Moschovakis [1972] and Vaught [1973b]. The logic $\mathscr{L}_{\omega_1\omega}$ and $\mathscr{L}_{\infty\omega}$ allow only finite strings of quantifiers at any stage in the transfinite process of building formulas. $\mathscr{L}_{\omega_1\omega_1}$ and $\mathscr{L}_{\infty\omega_1}$ permit infinitely long strings of the forms

$$\forall x_1 \, \forall x_2 \ldots \phi(x_1, x_2, \ldots)$$

and

$$\exists x_1 \, \exists x_2 \ldots \phi(x_1, x_2, \ldots).$$

The logics $\mathscr{L}_{\infty G}$ and $\mathscr{L}_{\infty V}$ studied in this chapter are stronger than $\mathscr{L}_{\infty\omega}$ but are not comparable with $\mathscr{L}_{\infty\omega_1}$. They contain more powerful forms of infinite quantification, by allowing infinite strings with alternations.

$$\forall x_1\, \exists y_1\, \forall x_2\, \exists y_2 \ldots \phi(x_1, y_1, x_2, y_2, \ldots).$$

However, they are more restrictive in terms of the form of the matrix ϕ that can follow the quantifiers. As the name "game quantification" suggests, a basic motivation comes from game theory. We imagine a two-person game of perfect information played by "$\forall$" and "$\exists$". They are allowed to play in turns. The formula is true in some structure if "$\exists$" has a winning strategy. The restriction on the matrix ϕ represents a restriction on the complexity of the games they are allowed to play. Basically, the games should be "open" or "closed", so that one of the players has a winning strategy. As a consequence, one has

$$\neg(\forall x_1\, \exists y_1\, \forall x_2\, \exists y_2 \ldots \phi)$$

logically equivalent to

$$\exists x_1\, \forall y_1\, \exists x_2\, \forall y_2 \ldots \neg\phi$$

and equivalence which would fail without some such restriction. It is also exactly these open and closed games that arise in the analysis of inductive definitions, as Moschovakis showed. Vaught showed how these game formulas can be approximated by formulas of $\mathscr{L}_{\omega_1\omega}$, leading to interesting proofs of results about the latter logic. Svenonius' theorem relates the logics to the study of second-order logic on countable structures. All of these results are covered in Chapter X, as well as some of the connections with generalized recursion theory and descriptive set theory.

Chapter XI, the final one in Part C, presents several applications of infinitary logics to algebra. The chapter is organized by algebraic subject matter. The first two sections, on universal locally finite groups and on subdirectly irreducible algebras, respectively, contain "pure" applications, applications of infinitary logic to prove results that can be stated in standard algebraic terms without reference to concepts from logic. The remaining sections, on Lefschetz's principle, abelian groups, almost-free algebras, and concrete constructions, present the conceptually more interesting kind of application where concepts from logic are brought in to enrich some domain.

Chapter VIII

$\mathscr{L}_{\omega_1\omega}$ and Admissible Fragments

by M. Nadel

Of the many strengthenings of first-order logic that the reader will encounter in the course of this book, $\mathscr{L}_{\infty\omega}$ and its admissible fragments have attracted the most attention by a wide margin. Unlike many of the others, these logics are often studied by investigators who are not otherwise involved with questions of abstract model theory. A large body of "hard" model theory has already been developed, and it continues to grow. Such a wealth of material, when coupled with stringent space limitations, creates obvious difficulties for any researcher aiming to present an exposition of this fascinating and ever-growing theory. We have attempted to contend with these difficulties in as reasonable a way as possible while all the time fully recognizing that even the catalog of results that we do present here is indeed far from complete. In fact, entire areas are omitted. We have tried to compensate for this, at least to some extent, through an appendix. Moreover, of the topics we do cover, we try to mention at least the most basic results and then direct the reader to other sources for further information.

In keeping with the procedure sketched in the preceding paragraph, we have tried to strike a reasonable balance between "hard" and "soft" material, but have steered clear of results in the direction of stability theory. Sections 3 and 6 are concerned mainly with "softer" considerations, while Sections 4 and 7 deal mainly with those "harder" aspects that are particularly characteristic of infinitary logic. The distinction here is not absolute, of course, nor is it strictly observed. Sections 1 and 5 provide the necessary background material while Section 2 is concerned with elementary equivalence. Section 8 deals with propositional extensions, and is, perhaps, the "icing on the cake"—a part which some may like best, but which others may prefer to avoid. In any event, the methods used in that section make it a worthwhile discussion even for the reader whose interest in abstract logic is quite limited.

Again, we would like to emphasize that within the limitations imposed by strict space requirements and an already large (and rapidly growing) body of theory, it is hardly possible to completely eliminate one's own prejudices and preferences either with respect to the topics to be treated or to the treatment they are to receive. Fully aware of this, we have nevertheless tried to present a reasonably orthodox treatment of the subject. We hope we have succeeded.

Part I. Compactness Lost

1. Introduction to Infinitary Logics

1.1. Why We Need Infinitary Logic

In the practice of model theory, and in more general mathematics as well, it often becomes necessary to consider structures satisfying certain collections of sentences rather than just single sentences. This consideration leads to the familiar notion of a theory in a logic. For example, in ordinary finitary logic, $\mathscr{L}_{\omega\omega}$, if φ_n is a sentence which expresses that there are at least n elements, then the theory $\{\varphi_n: n \in \omega\}$ would express that there are infinitely many elements. Similarly, in the theory of groups, if ψ_n is the sentence $\forall x[x^n \neq 1]$, then $\{\psi_n: n \in \omega\}$ expresses that a group is torsion free.

Suppose we want to express the idea that a set is finite, or that a group is torsion. A simple compactness argument would immediately reveal that neither of these notions can be expressed by a theory in $\mathscr{L}_{\omega\omega}$. What we need to express in each case is that a certain theory is not satisfied, that is, that at least one of the sentences is false. While theories are able to simulate infinite conjunctions, there is no apparent way to simulate infinite disjunctions—which is just what is needed in this case.

A similar phenomenon occurs with respect to the description of the elements in a structure. In order to specify that there is some element satisfying a certain set of formulas—for instance, $x \neq \mathbf{0}$, $x \neq \mathbf{1}$, $x \neq \mathbf{2}$, and so on—we might simply introduce a new constant symbol, say c, and then consider the theory in the language augmented by c, containing $c \neq \mathbf{0}$. $\mathbf{c} \neq \mathbf{1}$, $\mathbf{c} \neq 2, \ldots$. Suppose, however, that we want to consider structures, say models of set theory, in which the set of natural numbers is standard. Here we must introduce the notion of a *type*; that is, a consistent set of formulas in some fixed finite set of variables. We say that a model $\mathfrak{M}$ *realizes* the type $\Phi(x) = \{\varphi_k(x): k \in \omega\}$ if there is some $m \in M$, such that for each $k \in \omega$, $\mathfrak{M} \models \varphi_k[m]$, or simply, $\mathfrak{M} \models \Phi(m)$. Otherwise, we say that $\mathfrak{M}$ *omits* Φ. In the example above, we want our structures to omit the type $\{x \in \omega, x \neq \mathbf{0}, x \neq \mathbf{1}, \ldots\}$. Of course, this is the same as requiring that each element satisfy at least one of the formulas $x \notin \omega$, $x = \mathbf{0}$, $x = 1, \ldots$. The original results on *omitting types* are due to Henkin [1954, 1957], Orey [1956], and Morley [1965].

The logics we will consider allow us to replace some or all types in the logic by formulas of the logic. Thus, the notion of omitting a type may be equivalent to satisfying a certain sentence. In fact, these logics may be viewed as being formed by closing under "omitting types" as well as the other standard logical operations. Somewhat earlier, model theorists considered *ω-logic* (See Keisler [1966]) in which there is a fixed unary relation symbol, say $\mathfrak{N}$, whose realization in all ω-models is taken to be the same, *viz.*, the set of standard natural numbers. However, as research developed, attention has moved from ω-logic to the more flexible setting which we will discuss in the remainder of this chapter.

1.2. Definition of the Infinitary Logics

We now formally define the formulas of the logic $\mathscr{L}_{\infty\omega}$ as the smallest class closed under the usual connectives and quantifiers of finitary logic and, in addition, under the conjunction of arbitrary sets of formulas. Thus, if Φ is a set of formulas of $\mathscr{L}_{\infty\omega}$, so is $\bigwedge \Phi$. The semantics for $\bigwedge \Phi$ is the obvious one, and the disjunction $\bigvee \Phi$ may be defined using de Morgan's law as $\neg \bigwedge \{\neg\varphi : \varphi \in \Phi\}$. We assume that the reader can supply correct definitions for such standard concepts as subformula, free variable, sentence, *etc.* In cases of doubt, the reader should consult Keisler [1971a] or Barwise [1975].

Formulas, as we have so far defined them, may have infinitely many free variables. However, from now on we will restrict our discussions to those formulas with only finitely many free variables. It should be noted that a subformula of such a formula—and specifically of a sentence—will again have only finitely many free variables.

For any infinite regular cardinal κ we define the sublogic $\mathscr{L}_{\kappa\omega}$ of $\mathscr{L}_{\infty\omega}$ by restricting the conjunctions to be of sets of cardinality less than κ. For κ singular, the definition is a bit different. This is so in order to prevent the conjunction of conjunctions from simulating a conjunction of cardinality κ, and we omit it here. Of special interest is $\mathscr{L}_{\omega_1\omega}$, in which only countable conjunctions and disjunctions occur. $\mathscr{L}_{\omega\omega}$ is simply the familiar finitary logic. For the sake of later comparison, we also introduce the stronger logic $\mathscr{L}_{\infty\infty}$, which, in addition to arbitrary conjunctions and disjunctions, allows either existential or universal quantification over an arbitrary set of variables; that is, if φ is a formula of $\mathscr{L}_{\infty\infty}$ and X is a set of variables, then $\exists X\varphi$ is a formula of $\mathscr{L}_{\infty\infty}$. Again, we leave the standard definitions to the reader. $\mathscr{L}_{\infty\lambda}$ is the sublogic of $\mathscr{L}_{\infty\infty}$ in which the quantifiers are over sets of variables of cardinality less than λ. By analogy to the situation for $\mathscr{L}_{\infty\omega}$, one only considers those formulas of $\mathscr{L}_{\infty\lambda}$ having fewer than λ free variables. The reader should consult Chapter IX for further details.

The structures for these logics are simply the structures of ordinary model theory, and we assume that the notions of satisfaction are self-explanatory. Structures will generally be denoted by $\mathfrak{M}$ or $\mathfrak{N}$ with their universes denoted by M and N, respectively. We save the letters A and B for other purposes. As is the custom in this book, when we wish to call attention to a particular vocabulary τ, we write $\mathscr{L}_{\infty\omega}(\tau)$ instead of $\mathscr{L}_{\infty\omega}$, *etc.*

1.3. Expressive Power

We next offer a few examples of the expressive power of the various logics that have been introduced. Some of these are quite simple; others take considerable ingenuity. It is easy to write a sentence of $\mathscr{L}_{\omega_1\omega}$ in the language with just equality that says that a structure is finite. Similarly, we can write a sentence of $\mathscr{L}_{\omega_1\omega}$ that says a group is torsion or finitely generated, or that a structure with distinguished unary predicate and constant symbols for the natural numbers is an ω-model. In fact, given any countable type in $\mathscr{L}_{\omega\omega}$ or $\mathscr{L}_{\omega_1\omega}$, it is easy to write a sentence in $\mathscr{L}_{\omega_1\omega}$ expressing that the type is omitted.

That an abelian group is $\aleph_1$-free, i.e. every countable subgroup is free, can be expressed by a sentence of $\mathscr{L}_{\omega_1\omega}$ (see Barwise [1973b]). On the other hand, whether or not there is a sentence of $\mathscr{L}_{\infty\infty}$ defining the class of free abelian groups depends upon the particular universe of set theory. See Chapter XI for more details. The Ulm invariants for a countable abelian torsion group can be "written" in $\mathscr{L}_{\omega_1\omega}$ (see Barwise [1973b]). One can do the same for uncountable groups, obtaining sentences of $\mathscr{L}_{\infty\omega}$ which, rather than characterize the group up to isomorphism, characterize its $\mathscr{L}_{\infty\omega}$ elementary class.

Turning now to the vocabulary of linear orderings, it is easy to characterize the well-orderings (at least when the axiom of choice is assumed) by a sentence of $\mathscr{L}_{\omega_1\omega_1}$. However, it can be shown (see Lopez-Escobar [1966a]) that no sentence of $\mathscr{L}_{\infty\omega}$ characterizes the well-orderings. In fact, this class is not even PC. As an exercise, the reader should show that for each ordinal α there is a sentence φ of $\mathscr{L}_{\infty\omega}$ characterizing it up to isomorphism. This can be accomplished by induction on α. While it is true (see Nadel [1974b]) that for any scattered linear order—that is, any linear order without a dense subordering—there is a sentence of $\mathscr{L}_{\infty\omega}$ characterizing it up to isomorphism, there is nevertheless no sentence in $\mathscr{L}_{\infty\omega}$ that characterizes the scattered linear orderings, though obviously there is one in $\mathscr{L}_{\omega_1\omega_1}$.

Finally, we mention that for each countable structure (and we will always assume the underlying vocabulary is countable as well) there is a sentence of $\mathscr{L}_{\omega_1\omega}$ which characterizes it, up to isomorphism, among countable structures. This very early and very fundamental result is due to Scott [1965] and will be considered in Section 4. We point out here that more generally, in the context of any logic $\mathscr{L}$, we may speak of a *Scott sentence* φ of a structure $\mathfrak{M}$ as a sentence of $\mathscr{L}$ which characterizes M up to elementary equivalence in $\mathscr{L}$. The reader should consult Chapter IX for a more complete discussion of the examples.

1.4. Reduction to Omitting Types

In this section we will give a paraphrase of a result which once again emphasizes the connection between $\mathscr{L}_{\omega_1\omega}$ and omitting types in $\mathscr{L}_{\omega\omega}$. See Chapter XI of this volume for details.

Let $\mathscr{L}_B(\tau)$ be a countable fragment of $\mathscr{L}_{\omega_1\omega}(\tau)$ (in a sense to be made precise later). Then, by adding countably many new symbols, τ can be expanded to a larger vocabulary τ' in which there is a set of types such that each τ-structure has a unique expansion to a τ'-structure omitting these types; and, on these τ'-structures, each formula of $\mathscr{L}_B(\tau)$ is equivalent to a formula of $\mathscr{L}_{\omega\omega}(\tau')$, and *vice versa*.

Remark. A similar result holds for arbitrary $\mathscr{L}_{\kappa\omega}$ and is discussed in Section 1.3 of Chapter IX.

1.5. $\mathscr{L}_{\omega_1\omega}$ *of an Abstract Logic*

Let $\mathscr{L}^*$ be some abstract logic. Beginning with $\mathscr{L}^*$, can be form an infinitary version of $\mathscr{L}^*$? For the sake of this discussion, let us consider a version which we will call $\mathscr{L}^*_{\omega_1\omega}$ and which allows closure under countable conjunctions and disjunctions, rather than the full $\mathscr{L}_{\infty\omega}$ analogue. A naive approach would be to close $\mathscr{L}^*$ under countable conjunctions and disjunctions, negation, and existential and universal quantifiers as well. However, this is really not what is wanted here. Suppose $\mathscr{L}^*$ is $\mathscr{L}(Q_1)$. Then in $\mathscr{L}^*_{\omega_1\omega}$ we would like to be able to have sentences of the form $Q_1x\varphi$, where φ is already a formula of $\mathscr{L}^*_{\omega_1\omega}$. In this situation, it is clear how to proceed. In addition to the above closure conditions, we also close $\mathscr{L}^*_{\omega_1\omega}$ under the "closure operations" of $\mathscr{L}^*$. The problem arises in the general context in which $\mathscr{L}^*$ may not be given in terms of "closure operations".

While the method we will use here and later in Section 6.6 is based on Barwise [1981], there are some difficulties involved in the treatment given there. First of all, the definition for $\mathscr{L}^*_{\omega_1\omega}$ used in that work does not seem to be adequate for the intended purposes; accordingly, we modify it slightly. Even more importantly, the discussion given there purports to include the case of logics involving second-, as well as, first-order variables, e.g. L(aa). As a matter of fact, however, the argument there does not really include this case. We will limit our attention to the first-order case, with the case of L(aa) being considered only briefly in Chapter IV.

In addition to requiring that $\mathscr{L}^*_{\omega_1\omega}$ include $\mathscr{L}^*$ and be closed under countable conjunction and disjunction in the obvious way, we impose a further condition in order to simulate "closing under $\mathscr{L}^*$ itself". This condition is as follows:

(∗) If $\varphi(\mathfrak{R}_1, \ldots, \mathfrak{R}_k)$ is an $\mathscr{L}^*$ sentence, and $\psi_i(c_{i_1}, \ldots, c_{i_{n_i}})$, are $\mathscr{L}^*_{\omega_1\omega}$ sentences, where $\mathfrak{R}_i$ is an n_i-ary relation symbol which does not occur in ψ_i, and $c_{i_1}, \ldots, c_{i_{n_i}}$ do not occur in φ, for $i = 1, \ldots, k$, then $\varphi(\psi_1/\mathfrak{R}_1, \ldots, \psi_k/\mathfrak{R}_k)$ is an $\mathscr{L}^*_{\omega_1\omega}$-sentence in which neither R_i nor $c_i, \ldots, c_{i_{n_i}}$ occur, for $i = 1, \ldots, k$.

The corresponding semantical clause is given by

(∗) $\mathfrak{M} \models \varphi(\psi_1/\mathfrak{R}_1, \ldots, \psi_k/\mathfrak{R}_k)$ iff $(\mathfrak{M}, R_1, \ldots, R_k) \models \varphi(\mathfrak{R}_1, \ldots, \mathfrak{R}_k)$, where $R_i = \{(a_{i_1}, \ldots, a_{i_{n_i}}): (\mathfrak{M}, a_{i_1}, \ldots, a_{i_{n_i}}) \models \psi_i\}$, for $i = 1, \ldots, k$.

Using the above definition we have now formally introduced $\mathscr{L}^*_{\omega_1\omega}$. However, yet another point remains to be considered. Suppose $\mathscr{L}^*$ itself were not closed under the analogue of (∗). Barwise [1981] refers to the closure condition as the *substitution axiom*. Then, even without adding any infinite conjunctions or disjunctions, new sentences may be added because of (∗) and this may ruin certain properties of $\mathscr{L}^*$, e.g. compactness. Thus, we will only consider $\mathscr{L}^*_{\omega_1\omega}$ for $\mathscr{L}^*$ satisfying the substitution axiom.

It is now easy to see that $\mathscr{L}^*_{\omega_1\omega}$ is closed, for example, under the conjunction of two sentences (for future use it is important to distinguish finite from infinite conjunctions and disjunctions), viz. the correct semantics for θ & ψ will apply to $\mathfrak{N}_1$ & $\mathfrak{N}_2(\theta/\mathfrak{N}_1, \psi/\mathfrak{N}_2)$. Since in (*) φ is required to be an $\mathscr{L}^*$-sentence, rather than an $\mathscr{L}^*_{\omega_1\omega}$-sentence, it is not clear, *a priori*, that $\mathscr{L}^*_{\omega_1\omega}$ will satisfy the substitution axiom. However, a simple argument by induction on the formation of φ shows that $\mathscr{L}^*_{\omega_1\omega}$ does.

Now, having obtained the definition of $\mathscr{L}^*_{\omega_1\omega}$ in working order, an entire new aspect of abstract model theory presents itself. Suppose P_1 and P_2 are properties of logics. We can then hope to prove theorems of the following form:

"Suppose that $\mathscr{L}^*$ satisfies P_1, then $\mathscr{L}^*_{\omega_1\omega}$ satisfies P_2."

We will mention some impressive results of this type in Section 6.6. In the meantime, let us note that the result we mentioned in Section 1.4 holds in the general context of $\mathscr{L}^*_{\omega_1\omega}$. It would be a worthwhile exercise for the reader to fill in the extra step in the proof for (*) and note where the substitution axiom is needed.

2. *Elementary Equivalence*

One reason that $\mathscr{L}_{\infty\omega}$ is such a fruitful logic is that its elementary equivalence relation $=_{\infty\omega}$ (we write this instead of $\equiv_{\mathscr{L}_{\infty\omega}}$) is very natural. Below we will give two useful characterizations of $\equiv_{\infty\omega}$. Lest the inexperienced reader jump to unfounded conclusions, we point out that there are logics other than $\mathscr{L}_{\infty\omega}$ with the same elementary equivalence relation (for example, see Keisler [1968a]).

2.1. *The Back-and-Forth Property*

A function f from a structure $\mathfrak{M}$ to a structure $\mathfrak{N}$ (for the same vocabulary) is said to be a *partial isomorphism* from $\mathfrak{M}$ to $\mathfrak{N}$ if f extends to an isomorphism of the substructure of $\mathfrak{M}$ generated by dom f onto the substructure of $\mathfrak{N}$ generated by range f.

Let κ be a cardinal. A set F of partial isomorphisms from $\mathfrak{M}$ to $\mathfrak{N}$ is said to be a *κ-back and forth set* if for any $f \in F$:

(i) $\forall X \subseteq M[|X| < \kappa \to \exists g \in F[f \subseteq g \;\&\; X \subseteq \text{dom } g]]$;
(ii) $\forall Y \subseteq N[|Y| < \kappa \to \exists h \in F[f \subseteq h \;\&\; Y \subseteq \text{ra } h]]$.

If such a set F exists, then we say that $\mathfrak{M}$ and $\mathfrak{N}$ have the *κ-back and forth property* or are *κ-partially isomorphic*, and write $\mathscr{L} \cong_{p,\kappa} \mathfrak{N}$.

It is easy to see that if we take $\kappa = \omega$, we get the same condition as by taking $\kappa = n$, for $2 \le n < \omega$. In this case we will simply omit κ from the notation. This property was first studied by Karp [1965], and for that reason a logic is said to have the *Karp property* if whenever $\mathfrak{M} \cong_p \mathfrak{N}$, $\mathfrak{M}$ and $\mathfrak{N}$ are elementarily equivalent in that logic. The uninitiated reader should become more familiar with these notions by convincing himself that if $\mathfrak{M}$ and $\mathfrak{N}$ are dense linear orderings without endpoints, then $\mathfrak{M} \cong_p \mathfrak{N}$. But if $\mathfrak{M}$ and $\mathfrak{N}$ are algebraically closed fields of transcendence rank distinct natural numbers, then $\mathfrak{M} \not\cong_p \mathfrak{N}$.

The first characterization of $\equiv_{\infty\omega}$, given below in Karp's theorem, is proved by a straightforward induction on the formation of formulas [see Chapter IX for a detailed discussion]. It should be mentioned that an earlier characterization of $\equiv_{\omega\omega}$ in a similar way has been given by Ehrenfeuct [1961] and Fraïssé [1954b]. The reader should consult Section IX.4 for a more detailed historical survey.

2.1.1 Theorem (Karp's Theorem). $\mathfrak{M} \equiv_{\infty\omega} \mathfrak{N}$ *iff* $\mathfrak{M} \cong_p \mathfrak{N}$.

If $\mathfrak{M}$ and $\mathfrak{N}$ are countable, then, in the process of going back-and-forth between them, we can use all the elements of each and obtain the following weak form of Scott's theorem.

2.1.2 Corollary. *If* $|M| = |N| = \aleph_0$ *and* $\mathfrak{M} \equiv_{\infty\omega} \mathfrak{N}$, *then* $\mathfrak{M} \cong \mathfrak{N}$.

2.1.3 Remarks. (1) The analogue of Karp's theorem for arbitrary infinite κ holds. However, the analogue of the corollary given in Corollary 2.1.2 does not—except for the case $\mathrm{cf}(k) = \omega$, a result which is due to Chang [1968c]. Quite early in the development of this area, Morley gave an example of two structures $\mathfrak{M}$ and $\mathfrak{N}$ of cardinality $\aleph_1$ such that $\mathfrak{M} \equiv_{\infty\omega_1} \mathfrak{N}$, but $\mathfrak{M} \not\cong \mathfrak{N}$. The reader may consult Nadel–Stavi [1978] for a fuller description of such examples. However, we note that contrary to the assertion there, the question of finding non-isomorphic structures $\mathfrak{M}$ and $\mathfrak{N}$ of power λ, for λ-singular, $\mathrm{cf}(\lambda) > \omega$, $\lambda^\omega = \lambda$, such that $\mathfrak{M} \equiv_{\infty\lambda} \mathfrak{N}$ has only recently been solved by S. Shelah. Given a structure $\mathfrak{M}$ of cardinality λ, let $n(M)$ be the number of non-isomorphic models $\mathfrak{N}$, such that $|N| = \lambda$ and $\mathfrak{M} \equiv_{\infty\lambda} \mathfrak{N}$. Under the assumption that $V = L$, Shelah [1981b] has shown that if λ is regular and not weakly compact, than $n(\mathfrak{M}) = 1$ or 2^λ. However, if λ is weakly compact, then $n(\mathfrak{M})$ can be any cardinal $\mu \leq \lambda$, as shown in Shelah [1982b].

(2) There are results analogous to Theorem 2.2.1, as well as for certain other results to follow, for the properties of a structure being embeddable in or a homomorphic image of another structure. These results can be found in Chang [1968c], or Nadel [1974b], or Chapter IX, and we will not discuss them further here.

2.2. *Potential Isomorphism*

The notion of partial isomorphism is of an algebraic nature. The characterization of $\equiv_{\infty\omega}$ we present in this section is metamathematical and involves the set-theoretic notions of forcing or boolean-valued models (see Jech [1978]). It is due independently to Barwise [1973b] and Nadel [1974b].

We say that structures $\mathfrak{M}$ and $\mathfrak{N}$ are *potentially isomorphic* iff they are isomorphic in some boolean extension of the universe, that is, iff for some complete boolean-algebra B, $[\check{\mathfrak{M}} \cong \check{\mathfrak{N}}]^{\mathbb{B}} = 1$. It is quite easy to show the equivalence given in

2.2.1 Theorem. $\mathfrak{M} \equiv_{\infty\omega} \mathfrak{N}$ *iff* $\mathfrak{M}$ *and* $\mathfrak{N}$ *are potentially isomorphic.*

To prove the equivalence one must first observe that $\equiv_{\infty\omega}$ is absolute. To see that $\mathfrak{M} \equiv_{\infty\omega} \mathfrak{N}$ is preserved in a boolean extension, we use Karp's theorem (2.1.1). To see that $\mathfrak{M} \not\equiv_{\infty\omega} \mathfrak{N}$ is preserved in a boolean extension, we merely use the absoluteness of satisfaction for sentences of $\mathscr{L}_{\infty\omega}$. Now, if $\mathfrak{M} \equiv_{\infty\omega} \mathfrak{N}$, to make $\mathfrak{M}$ and $\mathfrak{N}$ isomorphic, go to a boolean extension in which both $\mathfrak{M}$ and $\mathfrak{N}$ are countable and then use Corollary 2.1.2.

We have found the notion of potential isomorphism to be a very useful conceptual tool. As simple examples, note that it is now obvious that well-ordered structures of distinct order types are not $\equiv_{\infty\omega}$, while any two algebraically closed fields of infinite transcendence rank are $\equiv_{\infty\omega}$.

2.2.2 Remarks. It is natural to wonder if there are notions of potential isomorphism corresponding to $\equiv_{\infty\lambda}$ for $\lambda > \omega$. This question is investigated in some detail in Nadel–Stavi [1978] where it is shown that, for λ a successor cardinal, there is no such notion in quite a general sense. It is also suggested that one could begin with some very natural notion of potential isomorphism and then use it to fashion a logic with a corresponding notion of elementary equivalence. This idea was the motivation behind the paper by Nadel [1980a]. The investigation begun there was developed much further by D. Mundici and is described in Chapter V.

3. *General Model-Theoretic Properties*

In this section we will consider the most fundamental results in the model theory of $\mathscr{L}_{\infty\omega}$, or, more accurately, in $\mathscr{L}_{\omega_1\omega}$, since as we shall see, countability will make a very big difference. In fact, we will need to consider countable pieces of $\mathscr{L}_{\omega_1\omega}$. To this end, we now define our first—and quite weak—version of a "nice" piece of $\mathscr{L}_{\infty\omega}$. Later in Section 5, we will give a much stronger version.

3.1. The Model Existence Theorem

3.1.1 Definition. A *fragment* of $\mathscr{L}_{\infty\omega}(\tau)$ is a set $L_B(\tau)$ of formulas and variables of $\mathscr{L}_{\infty\omega}(\tau)$ such that:

(i) $\mathscr{L}_{\omega\omega}(\tau) \subseteq L_B(\tau)$;
(ii) if $\varphi \in L_B(\tau)$, then every subformula and variable of φ is in $L_B(\tau)$;
(iii) if $\varphi(v) \in L_B(\tau)$ and σ is a term of τ all of whose variables lie in $L_B(\tau)$, then $\varphi(\sigma/v) \in L_B(\tau)$; and

(iv) if φ, ψ and $v \in L_B(\tau)$, so are $\neg\varphi$, $\exists v\varphi$, $\forall v\varphi$, $\varphi \,\&\, \psi$, $\varphi \vee \psi$ and $\sim\varphi$, where $\sim\varphi$ is defined inductively as follows: $\sim\theta$ is $\neg\theta$ if θ is atomic, $\sim(\neg\theta)$ is θ, $\sim(\bigwedge\Theta)$ is $\bigvee\{\sim\theta\colon \theta \in \Theta\}$, $\sim(\bigvee\Theta)$ is $\bigwedge\{\sim\theta\colon \theta \in \Theta\}$, $\sim(\exists v\varphi)$ is $\forall v \neg\varphi$, $\sim(\forall v\varphi)$ is $\exists v \neg\varphi$.

Closure under $\sim$ is merely to guarantee that $L_B(\tau)$ is closed under taking equivalent formulas of a certain simple type. (A convention on terminology will be helpful here: We will use L_B rather than $L_B(\tau)$ to represent a fragment when the vocabulary τ does not come into play. In particular, we will speak of $L_{\omega_1\omega}$ and $L_{\omega\omega}$ as fragments, where the former corresponds to an arbitrary $L_{\omega_1\omega}(\tau)$, etc. Moreover, we may speak of L_B rather than τ, having certain symbols).

The following definition and the subsequent theorem due to Makkai [1969b] is the principal tool for building models. The precise formulation given here is from Barwise [1975].

3.1.2 Definition. Suppose that the fragment L_B contains a set of constant symbols $C = \{c_n : n \in \omega\}$. A *consistency property* for L_B is a set S such that each $s \in S$ is a set of sentences of L_B and such that the following hold for each $s \in S$:

(C0) $0 \in S$; if $s \subseteq s' \in S$, then $s \cup \{\varphi\} \in S$, for each $\varphi \in s'$;
(C1) If $\varphi \in s$, then $\neg\varphi \notin s$;
(C2) If $\neg\varphi \in s$, then $s \cup \{\sim\varphi\} \in S$;
(C3) If $\bigwedge\Phi \in s$, then for all $\varphi \in \Phi$, $s \cup \{\varphi\} \in S$;
(C4) If $(\forall v\varphi(v)) \in s$, then for every $c \in C$, $s \cup \{\varphi(c)\} \in C$;
(C5) If $\bigvee\Phi \in s$, then for some $\varphi \in \Phi$, $s \cup \{\varphi\} \in S$;
(C6) If $(\exists v\varphi(v)) \in s$, then for some $c \in C$, $s \cup \{\varphi(c)\} \in S$;
(C7) Let t be any term of the form $F(c_{i_1}, \ldots, c_{i_n})$,

F an n-any function symbol of L_B, and $c_{i_1}, \ldots, c_{i_n}, c, d, \in C$

(i) If $(c = d) \in s$, then $s \cup \{d = c\} \in S$;
(ii) If $\{\varphi(t), (c = t)\} \in s$ then $s \cup \{\varphi(c)\} \in S$;
(iii) For some $e \in C$, $s \cup \{e = t\} \in S$.

Condition (C0) is not essential at this stage, although it does come into play later when we are trying to obtain more refined results. The remaining conditions are just what is needed to build a canonical model in ω stages using the Henkin construction, where a canonical model is simply one in which each element interprets a constant. The point here is that unlike the case of $L_{\omega\omega}$ where compactness holds, one must actually have constructed the entire model after ω stages. It is usually not possible to iterate a construction beyond a limit stage.

3.1.3 Model Existence Theorem. (i) *Let L_B be a countable fragment, and let S be a consistency property for L_B. For each $s \in S$, there is a canonical model $\mathfrak{M} \models \bigwedge s$.*

(ii) **(Extended Version).** *If in addition, T is a set of sentences of L_B such that, for each $s \in S$ and $\varphi \in T$, $s \cup \{\varphi\} \in S$, then, for each $s \in S$, there is a canonical model of $T \cup \{s\}$.* □

We mention at this point, that the model existence theorem does not hold in the absence of the assumption of countability (allowing, of course, an uncountable set of constants in C). We will point out an example of this later.

3.2. Provability and Completeness

The first completeness result for $L_{\omega_1\omega}$ was given by Karp [1964]. To the usual Hilbert style proof system for $L_{\omega\omega}$ one adds for each sentence $\bigwedge \Phi$ and $\varphi \in \Phi$, the axiom

$$(\bigwedge \Phi) \to \varphi$$

and the rule of inference

$$\text{From } \psi \to \varphi, \quad \text{for all} \quad \varphi \in \Phi, \quad \text{infer } \psi \to \bigwedge \Phi.$$

Since an application of this rule involves infinitely many premises, proofs may be infinite in length. We now consider an extended form of completeness that is appropriate for countable fragments, and in Section 6 we will consider a more subtle version. We fix a fragment L_B and require that all formulas involved in proofs be in L_B as well as that the proofs be of countable length. We use the standard provability symbol $\vdash_{L_B}$ in the usual way to refer to this system.

3.2.1 Completeness Theorem. *Let L_B be a countable fragment of $\mathscr{L}_{\omega_1\omega}$. Then for any sentence φ of L_B and set of sentences T of L_B, $T \models \varphi$ iff $T \vdash_{L_B} \varphi$.* $\square$

Karp's original proof was boolean-algebraic. Alternatively, we can add to the vocabulary a countable set C of new constant symbols and show that the set $S = \{s\colon s$ is a finite set of sentences of L_B each containing only finitely many constants from C and not $T \vdash_{L_B} \neg \bigwedge s\}$ is a consistency property, and then appeal to the extended version of the model existence theorem.

3.2.2 Remarks. As a result of the completeness theorem, we see that the validity of a sentence of $\mathscr{L}_{\omega_1\omega}$ is absolute (for models of ZFC). On the other hand, it is easy to give examples showing that validity for sentences in $\mathscr{L}_{\infty\omega}$ is not generally absolute and thus no similar absolute notion of provability could give a completeness theorem. For uncountable fragments, being provable in the obvious generalization of the above sense is equivalent to validity in boolean-valued extensions of the universe rather than validity in V itself. That is, φ is provable iff "$\models \varphi$" has value 1 in every boolean-valued extension of V. It is easy to see that provable sentences are boolean valid. To see the other direction, one needs the absoluteness of provability which shall be obtained in Section 6.

Alternatively, (see Mansfield [1972]), there is a completeness theorem for $\mathscr{L}_{\infty\omega}$ where the models themselves (rather than the set-theoretical universe) are taken to be boolean-valued. Thus, provability as above is equivalent to boolean-validity in this second sense also.

3.3. Interpolation

The interpolation theorem for $\mathscr{L}_{\omega_1\omega}$ was first proved by Lopez-Escobar [1965b]. Since the idea involved in his proof can be used in other settings, we shall say a few words about it. The first step—which is the more difficult one—is to find a cut-free Gentzen system which is complete for $\mathscr{L}_{\omega_1\omega}$. This can be done either purely semantically as in Lopez-Escobar [1965b], where completeness is simply proven directly for the cut-free system or, more proof-theoretically, as in Feferman [1968a] where completeness is shown for the system with cut (another name for *modus ponens*), and then "cut elimination" is proven by examining proofs. This second method provides certain ordinal bounds as well.

The idea of the proof is to find the interpolant by induction on the derivation of the implication. For example, suppose the final step in a derivation uses the so-called ($\supset \bigwedge$-rule):

$$\frac{\varphi \supset \psi_i}{\varphi \supset \bigwedge \{\psi_i : i \in \omega\}} \quad \text{for all } i \in \omega.$$

Suppose, by induction, that for each $i \in \omega$ there is some interpolant θ_i such that $\varphi \supset \theta_i$ and $\theta_i \supset \psi_i$ are each derivable. Then, using the ($\supset \bigwedge$-rule) we may obtain $\varphi \supset \bigwedge \{\theta_i : i \in \omega\}$. By using the matching ($\bigwedge \supset$-rule), we may obtain, for each $i \in \omega$, $\bigwedge \{\theta_i : i \in \omega\} \supset \psi_i$. Using the ($\supset \bigwedge$-rule) again we obtain $\bigwedge \{\theta_i : i \in \omega\} \supset \bigwedge \{\psi_i : i \in \omega\}$. It is now easy to check that $\bigwedge \{\theta_i : i \in \omega\}$ is an interpolant. The problem with the cut-rule is that this sort of induction step simply does not work, and that is why cut must be eliminated.

An alternate proof for a countable fragment L_B using the model existence theorem is given in Keisler [1971a]. We describe it very briefly. Suppose $\models \varphi \to \psi$. First, we add an infinite set of new constant symbols $C = \{c_1, c_2, \ldots\}$ to the alphabet. We define S_φ to be the set of all sentences φ' of L_B such that every symbol of the original alphabet that occurs in φ' also occurs in φ; and, in addition, finitely many of the c_n's may occur. S_ψ is defined analogously. We let S be the set of all finite sets of sentences which can be written as $s_1 \cup s_2$, where $s_1 \subseteq S_\varphi$, $s_2 \subseteq S_\psi$; and, if $\theta_1, \theta_2 \in S_\varphi \cap S_\psi$ and $\models \bigwedge s_1 \to \theta_1$, $\models \bigwedge s_2 \to \theta_2$, then $\theta_1 \,\&\, \theta_2$ is consistent. We then show that S is a consistency property and apply the model existence theorem. Since $\models \varphi \to \psi$, we have that $\{\varphi, \neg\psi\} \notin S$. But this means there must be $\theta_1, \theta_2 \in S_\varphi \cap S_\psi$ such that $\models \varphi \to \theta_1$, $\models \neg\psi \to \theta_2$ and $\theta_1 \,\&\, \theta_2$ is inconsistent. Thus, $\models \theta_1 \to \neg\theta_2$. Now, since $\models \neg\theta_2 \to \psi$, we have $\models \theta_1 \to \psi$. Now, by quantifying out the new constants in θ_1 we get the desired interpolant.

There are other more refined interpolation results of Lopez-Escobar [1965b] and Malitz [1969]. A good reference is Keisler [1971a].

The automatic consequences of interpolation, such as the Beth property, naturally hold. Robinson joint consistency fails, but a weaker version of it, a version in which the joint theory T is complete for $L_{\omega_1\omega}$ rather than just for L_B, does hold.

3.3.1 Remarks. The reader should consult Chapter IX for a full discussion of interpolation and definability results for infinitary logics. In particular, it is worth emphasizing in this context that interpolation fails for $\mathscr{L}_{\infty\omega}$.

3.3.2 Remarks. One of the main uses for interpolation results is in obtaining preservation theorems. As in the case of $\mathscr{L}_{\omega\omega}$, the more refined interpolation theorems alluded to above give rise to preservation theorems. For example, Malitz's interpolation theorem shows that a sentence φ of $L_{\omega_1\omega}$ is preserved under submodels relative to some other sentence ψ of $L_{\omega_1\omega}$ (that is, if $\mathfrak{M}$, $\mathfrak{N} \models \psi$, $\mathfrak{M} \subseteq \mathfrak{N}$ and $\mathfrak{N} \models \varphi$, then $\mathfrak{M} \models \varphi$) iff there is some universal sentence θ such that $\psi \models \varphi \leftrightarrow \theta$. By a universal sentence we mean a sentence which is formed from atomic and negated atomic formulas using only $\bigwedge$, $\bigvee$ and $\forall$. For a fuller discussion of preservation results the reader should consult Chapters 6 and 7 of Keisler [1971a].

3.4. *Kueker's Filter*

The reader will have noticed by now that many fundamental facts about $\mathscr{L}_{\omega_1\omega}$ fail to extend to $\mathscr{L}_{\infty\omega}$. Some outstanding examples of this are the corollary to Karp's theorem; completeness, and interpolation. D. Kueker [1972, 1977, 1978] (see also Barwise [1974b]) found a way of reformulating these and other results so that they do extend to $\mathscr{L}_{\infty\omega}$. Kueker's reformulation involves countable approximations to structures and formulas as well as a notion of "almost everywhere" corresponding to the closed unbounded filter on $\mathscr{P}_{<\omega_1}(X)$. A description of this very interesting approach can be found in Chapter XVII.

3.5. *Omitting Types*

Given a fragment L_B, we speak of types over L_B just as we do for $L_{\omega\omega}$, that is, sets of formulas in L_B in some fixed finite set of free variables. Then, using the model existence theorem (see Keisler [1971a] for details), we see that an omitting types theorem can be proved in much the same way as the original Henkin–Orey result for $L_{\omega\omega}$. Since the infinite disjunction is now officially available, it is customary to use it in the statement.

3.5.1 Theorem (Omitting Types Theorem). *Let L_B be a countable fragment of $L_{\omega_1\omega}$ and let T be a set of sentences of L_B which has a model. For each $n \in \omega$, let Φ_n be a set of formulas of L_B in the free variables $v_1, \ldots, v_{k_n}$. Assume that for each $n \in \omega$ and formula $\psi(v_1, \ldots, v_{k_n})$ of L_B, if $T \cup \{\exists v_1 \ldots v_{k_n}\psi\}$ has a model, so does $T \cup \{\exists v_1 \ldots v_{k_n}(\psi \;\&\; \varphi)\}$, for some $\varphi \in \Phi_n$. Then there is a model of*

$$T \cup \left\{\bigwedge_{n\in\omega} \forall v_1 \ldots v_{k_n} \bigvee_{\varphi\in\Phi_n} \varphi\right\}. \quad \square$$

The omitting types theorem is, of course, closely related to the ω-completeness theorem. The latter—especially the ω-rule, *viz.*, from $\varphi(n)$, for each $n \in \omega$, infer $\forall x(N(x) \to \varphi(x))$—is an important precursor of the study of infinitary logic in its present form.

3.5.2 Remarks. Shelah [1978a] has shown that a stronger version of omitting types is true. In that version there are fewer than continuum many Φ's over the fixed countable fragment L_B. The proof of this may be gleaned from the proof of Lemma 8.2.2 and, hence, we will omit it here.

It should be mentioned that because of the omitting types theorem, we are able to obtain the equivalence of prime models with countable atomic models, just as can be done for $\mathscr{L}_{\omega\omega}$. We shall have more to say about omitting types in Section 6.6.

3.6. Löwenheim–Skolem Results

Since the model existence theorem produces a countable model, we have, in effect, already shown that $\mathscr{L}_{\omega_1\omega}$ has Löwenheim number $\aleph_0$. That is to say, if a sentence of $\mathscr{L}_{\omega_1\omega}$ has a model, it has a countable model. The upward Löwenheim–Skolem result is more complicated. Unlike $\mathscr{L}_{\omega\omega}$, the Hanf number of $\mathscr{L}_{\omega_1\omega}$ is not $\aleph_0$. Examples showing this are easy to find. The proof for $\mathscr{L}_{\omega\omega}$ is simple enough using compactness, but that is not available. It is not surprising that the results for $\mathscr{L}_{\omega_1\omega}$ resemble rather the Hanf number results for omitting types over $\mathscr{L}_{\omega\omega}$, results which were proven slightly earlier by Morley [1965b]. The next result first appeared in Lopez-Escobar [1966a] who credits it to Helling.

3.6.1 Theorem (Upward Löwenheim–Skolem Theorem). *The Hanf number of $\mathscr{L}_{\omega_1\omega}$ is $\beth_{\omega_1}$. This means,*

(i) *if φ is a sentence of $\mathscr{L}_{\omega_1\omega}$ with models of all cardinalities $\beth_\alpha$, $\alpha < \omega_1$, then φ has models of all infinite cardinalities;*

(ii) *for each $\kappa < \beth_{\omega_1}$, there is a sentence φ with a model of cardinality at least κ with no model of cardinality $\beth_{\omega_1}$.* $\square$

3.6.2 Remarks. There is also an upward Löwenheim–Skolem theorem for arbitrary $\mathscr{L}_{\kappa\omega}$ given in Lopez-Escobar [1966a]. This result is discussed in Chapter IX.

Part (i) of Theorem 3.6.1, the difficult part of the result, is proven by using the hypothesis, together with a combinatorial property known as the Erdös–Rado theorem (Erdös and Rado [1956]), to produce a model generated by indiscernibles. The reader should consult Kunen [1977] for a nice treatment of the Erdös–Rado result.

To obtain (ii) for each $\alpha < \omega_1$, Morley gave a sentence φ_α that had models in all cardinalities up to $\beth_\alpha$. In essence, φ_α says that the model is a subset of V_α, the set of all sets of rank $\leq \alpha$. Morley also shows how to get φ_α for $\aleph_\alpha$ instead of $\beth_\alpha$. To do this, one "says" of a linear ordering that it is $\aleph_\alpha$-like.

3.6.3 Remarks. We can ask a similar question about sentences that are complete for $\mathscr{L}_{\omega_1\omega}$. Trivially, the Hanf number is at most $\beth_{\omega_1}$. Malitz [1968] using GCH showed that it is $\beth_{\omega_1}$ and found a sentence φ_α for each $\beth_\alpha$ as above. Later, Baumgartner [1974] was able to accomplish this without the GCH. Shelah [1974a], in a related result, showed that the Hanf number for omitting complete types over $\mathscr{L}_{\omega\omega}$ is $\beth_{\omega_1}$ and obtained a complete type for each $\beth_\alpha$. Can a complete sentence of $\mathscr{L}_{\omega_1\omega}$ be obtained for $\aleph_\alpha$? At this time the only result in this direction is due to Knight [1977] who has found a complete sentence for $\aleph_1$.

3.6.4 Remarks. There is an attractive result of Landraitis [1980] on linear orderings that is worth mentioning at this point, and this we do in

3.6.5 Theorem. *Let $\mathfrak{M}$ be a denumerable linear ordering and let φ be a Scott sentence of $\mathfrak{M}$ in $\mathscr{L}_{\omega_1\omega}$. The spectrum of φ, $S(\varphi) = \{\kappa: \kappa = |\mathfrak{N}|$ for some $\mathfrak{N} \models \varphi\}$ is either*

(i) *$\aleph_0$ iff each (isomorphism) orbit of $\mathfrak{M}$ is scattered;*
(ii) *all infinite cardinals iff $\mathfrak{M}$ has a self-additive interval or*
(iii) *$\{\kappa: \aleph_0 \leq \kappa \leq 2^{\aleph_0}\}$, otherwise;*

and each case occurs. $\square$

4. "Harder" Model Theory

4.1. Scott Sentences

Certainly the most striking of the early results in infinitary logic was Scott's theorem which is stated without proof in Scott [1965].

4.1.1 Theorem (Scott's Theorem). *For each countable structure $\mathfrak{M}$ for a countable vocabulary τ there is a sentence φ of $\mathscr{L}_{\omega_1\omega}(\tau)$ such that for any countable τ-structure $\mathfrak{N}$, $\mathfrak{M} \cong \mathfrak{N}$ iff $\mathfrak{N} \models \varphi$.*

We will now proceed to sketch a proof of Scott's theorem. We will assume that the reader can supply the obvious inductive definition of the quantifier rank of a formula of $\mathscr{L}_{\infty\omega}$. We write $\mathfrak{M} \equiv_\alpha \mathfrak{N}$ to mean that $\mathfrak{M}$ and $\mathfrak{N}$ agree on all sentences of $\mathscr{L}_{\infty\omega}$ of quantifier rank at most α. Karp [1965] gave an algebraic characterization of $\equiv_\alpha$.

4.1.2 Lemma. *For any structures $\mathfrak{M}$ and $\mathfrak{N}$ for the same vocabulary, and any ordinal α the following are equivalent:*

(i) $\mathfrak{M} \equiv_\alpha \mathfrak{N}$.
(ii) *There is a sequence $I_0 \supseteq I_1 \supseteq \cdots \supseteq I_\alpha$ of partial isomorphisms from $\mathfrak{M}$ to $\mathfrak{N}$ such that if $\beta + 1 \leq \alpha$ and $f \in I_{\beta+1}$, then for each $m \in M$* (resp. $n \in N$), *there is some $g \in I_\beta$, $g \supseteq f$ with $m \in$* dom g (resp. $n \in$ ra g).

The proof of Lemma 4.1.2 is by induction on α and is very similar to that of Karp's theorem (2.1.1).

Now, for each structure $\mathfrak{M}$, $m_1, \ldots, m_k \in M$, and ordinal α, we define a formula $\sigma^{\alpha}_{\mathfrak{M}, m_1, \ldots, m_k}(x_1, \ldots, x_k)$ of $\mathscr{L}_{\infty\omega}$ by induction on α.

4.1.3 Definition. (i) For $\alpha = 0$, $\sigma^{\alpha}_{\mathfrak{M}, m_1, \ldots, m_k}(x_1, \ldots, x_k) = \bigwedge \{\theta(x_1, \ldots, x_k)$: θ is atomic or the negation of an atomic formula and $\mathfrak{M} \models \theta(m_1, \ldots, m_k)\}$.

(ii) For $\alpha = \beta + 1$,

$$\begin{aligned}\sigma^{\alpha}_{\mathfrak{M}, m_1, \ldots, m_k}(x_1, \ldots, x_k) &= \sigma^{\beta}_{\mathfrak{M}, m_1, \ldots, m_k}(x_1, \ldots, x_k)\\ &\quad \& \ \forall x_{k+1} \bigvee_{m \in M} \sigma^{\beta}_{\mathfrak{M}, m_1, \ldots, m_k, m}(x_1, \ldots, x_k, x_{k+1})\\ &\quad \& \ \bigwedge_{m \in M} \exists x_{k+1} \sigma^{\beta}_{\mathfrak{M}, m_1, \ldots, m_k, m}(x_1, \ldots, x_k).\end{aligned}$$

(iii) For α a limit,

$$\sigma^{\alpha}_{\mathfrak{M}, m_1, \ldots, m_k}(x_1, \ldots, x_k) = \bigwedge_{\beta < \alpha} \sigma^{\beta}_{\mathfrak{M}, m_1, \ldots, m_k}(x_1, \ldots, x_k).$$

It is obvious from inspection that $\sigma^{\alpha}_{\mathfrak{M}, m_1, \ldots, m_k}$ has quantifier rank α, and that $\mathfrak{M} \models \sigma^{\alpha}_{\mathfrak{M}, m_1, \ldots, m_k}(m_1, \ldots, m_k)$. More importantly, this formula is complete for formulas of quantifier rank of at most α.

4.1.4 Lemma. *For any structures $\mathfrak{M}$, $\mathfrak{N}$, for the same vocabulary, elements $m_1, \ldots, m_k \in M$, $n_1, \ldots, n_k \in N$ and ordinal α, the following are equivalent*:

(i) $(\mathfrak{M}, m_1, \ldots, m_k) \equiv_{\alpha} (\mathfrak{N}, n_1, \ldots, n_k)$;
(ii) $\mathfrak{N} \models \sigma^{\alpha}_{\mathfrak{M}, m_1, \ldots, m_k}(n_1, \ldots, n_k)$;
(iii) $\sigma^{\alpha}_{\mathfrak{M}, m_1, \ldots, m_k} = \sigma^{\alpha}_{\mathfrak{N}, n_1, \ldots, n_k}$.

The only non-trivial step in the proof is that of showing that (ii) implies (i). This fact follows from Lemma 4.1.2 if we define for each $\beta \leq \alpha$,

$$\begin{aligned}I_{\beta} = \{f\colon \operatorname{dom} f = \{m_1, \ldots, m_i\}, f(m_j) = n_j, \text{ for } j \leq i \text{ and}\\ \mathfrak{N} \models \sigma^{\beta}_{\mathfrak{M}, m_1, \ldots, m_i}(n_1, \ldots, n_i)\}.\end{aligned}$$

Two observations are now needed to find the sentence which will characterize a structure up to $\equiv_{\infty\omega}$. First, if it happens that $I_{\alpha} = I_{\alpha+1}$ for some α, then I_{α} is easily seen to be a back-and-forth set. Second, for any $\mathfrak{M}$, there is an ordinal α, such that for any $k \in \omega$, $m_1, \ldots, m_k, m'_1, \ldots, m'_k \in M$,

$$(\mathfrak{M}, m_1, \ldots, m_k) \equiv_{\alpha} (\mathfrak{M}, m'_1, \ldots, m'_k)$$

implies

$$(\mathfrak{M}, m_1, \ldots, m_k) \equiv_{\infty\omega} (\mathfrak{M}, m'_1, \ldots, m'_k).$$

The least ordinal for which this happens is called the *Scott height* of $\mathfrak{M}$ and is denoted SH($\mathfrak{M}$). Using the first observation, we see that the Scott height of $\mathfrak{M}$ is the first ordinal α such that, for all $k \in \omega$, $m_1, \ldots, m_k, m'_1, \ldots, m'_k \in M$,

$$(\mathfrak{M}, m_1, \ldots, m_k) \equiv_\alpha (\mathfrak{M}, m'_1, \ldots, m'_k)$$

implies

$$(\mathfrak{M}, m_1, \ldots, m_k) \equiv_{\alpha+1} (\mathfrak{M}, m'_1, \ldots, m'_k).$$

Thus, it is easy to see that the Scott height of $\mathfrak{M}$ is below $|\mathfrak{M}|^+$. In Section 7 we will obtain a better bound.

4.1.5 Definition. We now define the sentence $\sigma(\mathfrak{M})$ to be

$$\sigma^\alpha_{\mathfrak{M}} \;\&\; \bigwedge_{\substack{k \in \omega \\ m_1, \ldots, m_k \in M}} \forall x_1 \ldots x_k[\sigma^\alpha_{\mathfrak{M}, m_1, \ldots, m_k}(x_1, \ldots, x_k) \leftrightarrow \sigma^{\alpha+1}_{\mathfrak{M}, m_1, \ldots, m_k}(x_1, \ldots, x_k)]$$

where $\alpha = $ SH($\mathfrak{M}$).

This sentence appears first in Chang [1968c] and is called the *canonical Scott sentence* of $\mathfrak{M}$ in view of the next theorem.

4.1.6 Theorem. *For any structures $\mathfrak{M}$ and $\mathfrak{N}$ for the same vocabulary, the following are equivalent:*

(i) $\mathfrak{M} \equiv_{\infty\omega} \mathfrak{N}$;
(ii) $\mathfrak{N} \models \sigma(\mathfrak{M})$;
(iii) $\sigma(\mathfrak{M}) = \sigma(\mathfrak{N})$. ☐

The non-trivial implication from (ii) to (i) is established much as in Lemma 4.1.4.

We see from Theorem 4.1.6 that $\sigma(\mathfrak{M})$ characterizes $\mathfrak{M}$ up to $\equiv_{\infty\omega}$ and depends only on the $\mathscr{L}_{\infty\omega}$-theory of $\mathfrak{M}$. If $\mathfrak{M}$ is countable, then, by Corollary 2.1.2, $\sigma(\mathfrak{M})$ is the sentence required in Scott's theorem (4.1.1). The quantifier rank of $\sigma(\mathfrak{M})$ is SH($\mathfrak{M}$) + ω and there are often Scott sentences for $\mathfrak{M}$ of lower quantifier rank. However, it will be observed in Section 7 that at least for countable $\mathfrak{M}$, $\sigma(\mathfrak{M})$ cannot have quantifier rank too much above any other Scott sentence for $\mathfrak{M}$.

4.2. *Automorphisms and Local Definability in Countable Models*

It was observed by Scott [1965] and follows quite readily from the preceding discussion that a countable model $\mathfrak{M}$ is *rigid* (that is to say, has no non-trivial automorphisms) iff each element of $\mathfrak{M}$ is definable in $\mathfrak{M}$ by a formula of $\mathscr{L}_{\omega_1\omega}$.

A similar result holds for countable models having fewer than continuum many automorphisms. This result has been shown by Kueker [1968].

4.2.1 Theorem. *Let $\mathfrak{M}$ be a countable structure. The following are equivalent:*

(i) *$\mathfrak{M}$ has countably many automorphisms.*
(ii) *$\mathfrak{M}$ has fewer than continuum many automorphisms.*
(iii) *There is some tuple of elements $n_1, \ldots, n_j \in M$ such that $(\mathfrak{M}, n_1, \ldots, n_j)$ is rigid.*
(iv) *There is some tuple of elements $n_1, \ldots, n_j \in M$ such that for each $m \in M$ there is a formula $\varphi(x_1, \ldots, x_j, y)$ of $\mathscr{L}_{\omega_1\omega}$ such that*

$$M \models \exists!\, y\varphi(n_1, \ldots, n_j, y) \;\&\; \varphi(n_1, \ldots, n_j, m),$$

that is, m is definable from $n_1, \ldots, n_j$ in $\mathfrak{M}$ by a formula of $\mathscr{L}_{\omega_1\omega}$.

The main step in the proof comes in showing that (ii) implies (iii). This can be accomplished by using the negation of (iii) to construct a full binary tree all of whose branches give rise to distinct automorphisms of $\mathfrak{M}$. It should be observed that the equivalence of (i) and (ii) can be obtained via general descriptive set-theoretic considerations, since the set of automorphisms of $\mathfrak{M}$ forms a $\mathbf{\Sigma}_1^1$ set. In Section 7 we will also get a better bound on the defining formulas in (iv).

It follows easily from Theorem 4.2.1 that if $\mathfrak{M}$ is countable, $\mathfrak{N}$ uncountable and $\mathfrak{M} \equiv_{\infty\omega} \mathfrak{N}$, then $\mathfrak{M}$ will have $2^{\aleph_0}$ automorphisms.

Another result that was already noted in Scott [1965] is that if $\mathfrak{M}$ is countable and R is a relation on $\mathfrak{M}$, then R is definable by a formula of $\mathscr{L}_{\omega_1\omega}$ iff every automorphism of $\mathfrak{M}$ is an automorphism of $(\mathfrak{M}, R)$. This is a local version of Beth definability and follows from Beth definability for $\mathscr{L}_{\omega_1\omega}$ together with Scott's theorem, if one assumes the vocabulary is countable. However, there is an even more elementary proof. For the non-trivial direction, if each automorphism of $\mathfrak{M}$ is an automorphism of $(\mathfrak{M}, R)$, then for each $\bar{m} = (m_1, \ldots, m_k) \in R$ and $\bar{m}' = (m_1', \ldots, m_k') \notin R$ there is some $\varphi_{\bar{m},\bar{m}}$ in $\mathscr{L}_{\omega_1\omega}$ such that $\mathfrak{M} \models \varphi_{\bar{m},\bar{m}'}(m_1, \ldots, m_k)$ but $\mathfrak{M} \models \neg\varphi_{\bar{m},\bar{m}'}(m_1', \ldots, m_k')$. Now R is definable by $\bigvee_{\bar{m}\in R} \bigwedge_{\bar{m}'\in R} \varphi_{\bar{m},\bar{m}'}$.

By analogy to the case for rigid models that was considered above, Kueker [1968] and Reyes [1967] have shown the result given in

4.2.2 Theorem. *Let $\mathfrak{M}$ be a countable structure. Let R be a k-ary relation on M and define $S = \{Q\colon (\mathfrak{M}, R) \cong (\mathfrak{M}, Q)\}$. The following are equivalent:*

(i) $|S| = \aleph_0$.
(ii) $|S| < 2^{\aleph_0}$.
(iii) *There is some formula $\varphi(x_1, \ldots, x_j, y_1 \ldots y_k)$ in $\mathscr{L}_{\omega_1\omega}$ and $n_1, \ldots, n_j \in M$ such that*

$$R = \{(m_1, \ldots, m_k)\colon \mathfrak{M} \models \varphi(n_1, \ldots, n_j, m_1, \ldots, m_k)\}. \quad \square$$

In Section 7 we will give a better bound for this result also.