Chapter V

Transfer Theorems and Their Applications to Logics

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This chapter is primarily concerned with the general problem of transferring results about one logic, say $\mathscr{L}(Q_1)$, to another logic, say $\mathscr{L}(Q_\alpha)$. A typical such property is $\aleph_0$-compactness. It is known from Chapter IV that $\mathscr{L}(Q_1)$ is $\aleph_0$-compact. Under certain set-theoretic assumptions on α discussed in this chapter, the logic $\mathscr{L}(Q_1)$ transfers to $\mathscr{L}(Q_\alpha)$. In such cases we can then conclude that $\mathscr{L}(Q_\alpha)$ is also $\aleph_0$-compact. The logics that we consider in this chapter are variants and generalizations of $\mathscr{L}(Q_1)$, and the properties of these logics which we are most concerned with are compactness and recursive enumerability for validity.

1. The Notions of Transfer and Reduction

After presenting the basic definitions that allow useful model-theoretic comparisons between logics, we present applications to compactness and recursive enumerability of logics and to two-cardinal questions.

1.1. Transfer

The substantive theme of this chapter is the notion of transfer and we will begin our explorations with

1.1.1 Definition. Suppose $\mathscr{L}_0$ and $\mathscr{L}_1$ are two logics which have exactly the same syntax but differ in their semantics. Then $\mathscr{L}_0$ *transfers to* $\mathscr{L}_1$ iff every sentence which is satisfiable relative to $\mathscr{L}_0$ is also satisfiable relative to $\mathscr{L}_1$. In symbols, we write $\mathscr{L}_0 \to \mathscr{L}_1$.

Transfer becomes quite fruitful when there is *mutual transfer*, when both $\mathscr{L}_0 \to \mathscr{L}_1$ and $\mathscr{L}_1 \to \mathscr{L}_0$ hold. For, in this situation $\mathscr{L}_0$ and $\mathscr{L}_1$ have exactly the same valid sentences, so that a syntactic property known to hold for $\mathscr{L}_0$ will also hold for $\mathscr{L}_1$. For example, if $\mathscr{L}_0$ has the Beth property, then so does $\mathscr{L}_1$. In this chapter we will generally be concerned with two properties which are especially amenable to verification using the methods of transfer. These properties are compactness and, to a lesser degree, recursive enumerability for validity. To be sure, if there is mutual transfer $\mathscr{L}_0 \to \mathscr{L}_1$ and $\mathscr{L}_1 \to \mathscr{L}_0$ and if either one of these logics is compact or recursively enumerable for validity, then so is the other. However, it often turns out that the proof of a specific transfer theorem yields a sort of self-transfer theorem of the form $\mathscr{L} \to \mathscr{L}$. And while the transfer $\mathscr{L} \to \mathscr{L}$ is evidently trivial, one nevertheless often obtains a stronger form having as a consequence the compactness and the recursive enumerability for validity of $\mathscr{L}$. This is the approach that Fuhrken and Vaught used in the original proofs of the compactness and the recursive enumerability for validity of $\mathscr{L}(Q_1)$.

To see how compactness typically obtains, we need a strengthening of the notion of transfer. For κ an infinite cardinal, we say that $\mathscr{L}_0 \to \mathscr{L}_1$ κ-compactly iff whenever a set of at most κ sentences is finitely satisfiable relative to $\mathscr{L}_0$, then it is satisfiable relative to $\mathscr{L}_1$. Fuhrken and Vaught observed that $\mathscr{L}(Q_{\alpha+1}) \to \mathscr{L}(Q_1)$ $\aleph_0$-compactly. In particular, $\mathscr{L}(Q_1) \to \mathscr{L}(Q_1)$ $\aleph_0$-compactly, which is just another way of saying that $\mathscr{L}(Q_1)$ is $\aleph_0$-compact.

For the sake of completeness, we will mention a further generalization of transfer at this point. For each $j \in J$, let $\mathscr{L}_j$ be a logic with the same syntax as $\mathscr{L}$. Then $\{\mathscr{L}_j : j \in J\} \to \mathscr{L}$ iff each sentence which is satisfiable relative to each $\mathscr{L}_j$ is also satisfiable relative to $\mathscr{L}$. Similarly, $\{\mathscr{L}_j : j \in J\} \to \mathscr{L}$ κ-compactly iff whenever a set of at most κ sentences is finitely satisfiable relative to each $\mathscr{L}_j$, then it is satisfiable relative to $\mathscr{L}$.

1.2. Reduction

Although it was noted at the outset of this Section that the notation of transfer provides the *substantive theme* of the present chapter, there is, nevertheless, a *methodological theme* appearing in this chapter: Reduction. This notion of reduction is of considerable importance in our exposition and the basic idea underlying it is to associate (usually effectively) with each sentence in some logic $\mathscr{L}$ a corresponding first-order sentence, and then reduce the study of the model theory of $\mathscr{L}$ to the study of those models of some first-order theory satisfying some additional property.

Much of what we do in this chapter will concern the logic $\mathscr{L}(Q)$ with various cardinality interpretations, which have already been discussed in Sections II.2.2 and III.2.4, and (for $\kappa = \aleph_1$), in Section IV.3. For any infinite cardinal κ, if we are defining the κ-interpretation of $\mathscr{L}(Q)$, then the key clause in the definition is that

$$\mathfrak{A} \models Qx\phi(x) \quad \text{iff} \quad |\{a \in A : \mathfrak{A} \models \phi(a)\}| \geq \kappa.$$

We will also adhere to the convention that if $\mathfrak{A}$ is a structure appropriate for $\mathscr{L}(Q)$ with the κ-interpretation, then $|A| \geq \kappa$; that is, $Qx(x = x)$ is a valid sentence. If $\kappa = \aleph_\alpha$, then $\mathscr{L}(Q_\alpha)$ simply denotes $\mathscr{L}(Q)$ with the κ-interpretation.

Fuhrken [1964] introduced the reduction of these logics to cardinal-like structures. We will consider the Fuhrken reduction in some detail, since it is quite typical of other reductions. Finally, typical applications will be given in Section 1.3.

1.2.1 Definition. A linearly ordered set $(A, <)$ is *κ-like* iff $(A, <) \models \forall x \neg Qy(y < x)$ under the κ-interpretation. A structure $\mathfrak{A} = (A, <, \ldots)$ is *κ-like* iff $(A, <)$ is *κ-like.* $\mathfrak{A}$ is *cardinal-like* iff it is κ-like for some κ. We let $K(\kappa)$ denote the class of κ-like structures.

Examples of κ-like linearly ordered sets are well-ordered sets with order type κ. If κ is uncountable, then there are linearly ordered sets which are κ-like but not well-ordered. On the other hand, $(\omega, <)$ is (up to order-isomorphism) the only $\aleph_0$-like linearly ordered set.

To begin the Fuhrken reduction, let us fix a vocabulary τ which includes neither the binary relation symbol $<$ nor the ternary relation symbol R. Consider the first-order sentence σ which is the conjunction of the universal closures of the following three formulas:

$$R(x_1, y, z) \wedge R(x_2, y, z) \rightarrow x_1 = x_2,$$

$$R(x, y_1, z) \wedge R(x, y_2, z) \rightarrow y_1 = y_2,$$

$$x_2 < x_1 \wedge R(x_1, y_1, z) \rightarrow \exists y_2 R(x_2, y_2, z).$$

The intention here is that σ should express the fact that as z varies, R encodes a set of bijections $x \mapsto y$ whose domains are (possibly improper) initial segments.

With each $\mathscr{L}(Q)(\tau)$-formula ϕ we will associate a first-order $(\tau \cup \{R, <\})$-formula ϕ^* having the same free variables as ϕ by the following inductive procedure:

$$\phi^* = \phi, \quad \text{if } \phi \text{ is atomic,}$$

$$(\neg\phi)^* = \neg\phi^*,$$

$$(\phi_1 \wedge \phi_2)^* = \phi_1^* \wedge \phi_2^*,$$

$$(\exists y\phi)^* = \exists y\phi^*,$$

$$(Qy\phi)^* = \exists z\, \forall x\, \exists y(R(x, y, z) \wedge \phi^*).$$

We will also associate with each $\mathscr{L}(Q)(\tau)$-formula ϕ a first-order $\tau \cup \{R, <\}$-sentence σ_ϕ by the following inductive procedure:

$$\sigma_\phi = \sigma, \quad \text{if } \phi \text{ is atomic,}$$

$$\sigma_{\neg\phi} = \sigma_\phi,$$

$$\sigma_{\phi_1 \wedge \phi_2} = \sigma_{\phi_1} \wedge \sigma_{\phi_2},$$

$$\sigma_{\exists y\phi} = \sigma_\phi,$$

$$\sigma_{Qy\phi} = \sigma_\phi \wedge \forall\bar{x}\, \exists z\, \forall y[\phi^*(\bar{x}, y) \leftrightarrow \exists x R(x, y, z)].$$

The following two lemmas give the essential properties of the Fuhrken reduction.

1.2.2 Lemma. *If $(\mathfrak{A}, R, <)$ is a κ-like $(\tau \cup \{R, <\})$-structure and $\phi(\bar{x})$ is an $\mathscr{L}(Q)(\tau)$-formula, then*

$$(\mathfrak{A}, R, <) \models \sigma_\phi \leftrightarrow \forall \bar{x}(\phi(\bar{x}) \leftrightarrow \phi^*(\bar{x}))$$

in the κ-interpretation. $\square$

1.2.3 Lemma. *If $\mathfrak{A}$ is a τ-structure with $|\tau| \leq \kappa = |A|$, then $\mathfrak{A}$ can be expanded to a κ-like structure $(\mathfrak{A}, R, <)$ such that for every $\mathscr{L}(Q)(\tau)$-formula ϕ, $(\mathfrak{A}, R, <) \models \sigma_\phi$.* $\square$

The proof of Lemma 1.2.2 can be obtained by a rather routine induction on formulas. In Lemma 1.2.3, the expansion of $\mathfrak{A}$ is done in the following manner. First, let $<$ be any well-ordering of A which has order type κ, and let d_ξ be the ξ-th element of A in this well-ordering. By the cardinality conditions imposed on σ and A, there are exactly κ subsets of A which are $\mathscr{L}(Q)$-definable. Let these be $\{D_\xi : \xi < \kappa\}$, and for each $\xi < \kappa$ let $f_\xi : D_\xi \to A$ be a one–one function onto an initial segment of A (which may, of course, be all of A). Now let $R \subseteq A^3$ be such that $R(a, b, c)$ holds iff there is $\xi < \kappa$ such that $c = d_\xi$, $b \in D_\xi$ and $a = f_\xi(b)$. It is now clear that $(\mathfrak{A}, R, <)$ is a κ-like model of σ. The problem of showing that $(\mathfrak{A}, R, <) \models \sigma_\phi$ involves merely another rather routine induction on formulas.

1.3. Applications of Reduction

In this subsection we will describe some applications of the specific reduction that was discussed in Subsection 1.2. We begin with the definition of transfer for cardinal-like models which is in complete analogy with the definitions of transfer given in Subsection 1.1.

1.3.1 Definition. Let λ, μ and κ_j, for $j \in J$, be infinite cardinals. Then $\{\kappa_j : j \in J\} \to \lambda$ *μ-compactly* iff every set of at most μ first-order sentences, each finite subset of which has a κ_j-like model for each $j \in J$, has a λ-like model.

We remark that by comparison with the corresponding definitions of transfer given in Subsection 1.1, the meaning of each of $\kappa \to \lambda$, $\kappa \to \lambda$ μ-compactly, and $\{\kappa_j : j \in J\} \to \lambda$ is obvious.

1.3.2 Proposition. *The following two statements are equivalent:*

(1) $\{\mathscr{L}(Q_{\alpha_j}) : j \in J\} \to \mathscr{L}(Q_\alpha)$ *μ-compactly;*
(2) $\{\aleph_{\alpha_j} : j \in J\} \to \aleph_\alpha$ *μ-compactly.*

Proof. We will first show that (1) implies (2). Actually, this is the trivial direction. Let T' be a set of at most μ first-order sentences each finite subset of which has an

$\aleph_{\alpha_j}$-like model, for each $j \in J$. Consider the set $T' \cup \{\forall x \neg Qy(y < x)\} \cup \{$"$<$ is a linear order"$\}$, and apply (1) to it.

We will now show that (2) implies (1). Clearly, if (2) holds, then we can assume $\mu < \aleph_\alpha$. Let T' be a set of at most μ $L(Q)$-sentences each finite subset of which has a model in each of the $\aleph_{\alpha_j}$-interpretations. By Lemma 1.2.3, for each finite $T'_0 \subseteq T'$ and each $j \in J$, we have that $T'_0 \cup \{\sigma_\phi : \phi \in T'_0\}$ has an $\aleph_{\alpha_j}$-like model; and by Lemma 1.2.2, this model is also a model of $\{\phi^* : \phi \in T'_0\}$. By (2), we thus have that $\{\phi^* : \phi \in T'\} \cup \{\sigma_\phi : \phi \in T'\}$ has an $\aleph_\alpha$-like model which, by Lemma 1.2.2, is also a model of T'. $\square$

The preceding proposition and its proof remain valid even when both references to the phrase "μ-compactly" are deleted.

1.3.3 Definition. Let K be a class of structures and μ an infinite cardinal. Then K is *μ-compact* iff any set of not more than μ first-order sentences which is finitely satisfiable in K is also satisfiable in K. Moreover, K is *recursively enumerable for validity* iff for any recursive vocabulary τ the set of all first-order sentences valid in every τ-structure in K is recursively enumerable.

1.3.4 Corollary. *The following are equivalent:*

(1) *$\mathscr{L}(Q_\alpha)$ is μ-compact;*
(2) *$K(\aleph_\alpha)$ is μ-compact.*

Proof. The proof for this result follows immediately from Proposition 1.3.2 upon noting the following obvious equivalences: $\mathscr{L}(Q_\alpha)$ is μ-compact iff $\mathscr{L}(Q_\alpha) \to \mathscr{L}(Q_\alpha)$ μ-compactly; $K(\aleph_\alpha)$ is μ-compact iff $\aleph_\alpha \to \aleph_\alpha$ μ-compactly. $\square$

It should be recognized that the Fuhrken reduction given in Subsection 1.2 is effective. That is to say, if τ is a recursive vocabulary, then both the functions $\phi \mapsto \phi^*$ and $\sigma \mapsto \sigma_\phi$ are recursive. This yields the following equivalence involving the recursive enumerability for validity of $\mathscr{L}(Q)$ under the cardinal interpretations.

1.3.5 Corollary. *The following are equivalent:*

(1) *$\mathscr{L}(Q_\alpha)$ is recursively enumerable for validity;*
(2) *$K(\aleph_\alpha)$ is recursively enumerable for validity.*

Proof. That (1) implies (2) is trivial. For the argument that (2) implies (1), we merely note that any $\mathscr{L}(Q)$-sentence ϕ has a model in the $\aleph_\alpha$-interpretation just in case $\sigma_\phi \wedge \phi^*$ has a κ-like model. $\square$

1.4. Two-Cardinal Models

In Subsection 1.2 we saw how to reduce $\mathscr{L}(Q)$ to cardinal-like structures. A further reduction to two-cardinal structures will be described in this subsection. The symbol U will always denote a unary relation symbol.

1.4.1 Definition. A structure $\mathfrak{A}$ is a (κ, λ)-structure if $|A| = \kappa$ and $|U| = \lambda$. Moreover, if $\kappa > \lambda$, then $\mathfrak{A}$ is a *two-cardinal* structure, and if $\kappa = \lambda^+ > \aleph_0$, then $\mathfrak{A}$ is a *gap*-1 two-cardinal structure. We will let $K(\kappa, \lambda)$ denote the class of (κ, λ)-structures.

1.4.2 Definition. $(\kappa_1, \lambda_1) \to (\kappa_2, \lambda_2)$ iff every first-order sentence which has a (κ_1, λ_1)-model also has a (κ_2, λ_2)-model.

We will leave the details of the remainder of this subsection as an easy, and yet instructive, exercise for the reader.

1.4.3 Proposition. *There is a first-order sentence σ in the vocabulary $\{<, U, S\}$, where S is a ternary relation symbol, such that*

(1) *if $\mathfrak{A} \models \sigma$ is κ-like, then for some λ, $\kappa = \lambda^+$ and $\mathfrak{A}$ is a (λ^+, λ)-structure;*
(2) *if $\mathfrak{A} \models \sigma$ is a two-cardinal (κ, λ)-structure, then $\kappa = \lambda^+$ and $\mathfrak{A}$ is κ-like;*
(3) *if τ is a vocabulary not including either $<$ or U or S, then*
   (i) *any λ^+-like $(\tau \cup \{<\})$-structure can be expanded to a model of σ, and*
   (ii) *any gap-1 two-cardinal $(\tau \cup \{U\})$-structure can be expanded to a model of σ.* $\square$

Obvious consequences of Proposition 1.4.3 equate transfer for gap-1 two-cardinal models with the corresponding transfer for successor cardinal-like models. This immediately yields that for cardinals κ and μ, $K(\kappa^+, \kappa)$ is μ-compact iff $K(\kappa^+)$ is μ-compact. Similarly, $K(\kappa^+, \kappa)$ is recursively enumerable for validity iff $K(\kappa^+)$ is recursively enumerable for validity.

2. The Classical Transfer Theorems

This section contains what might be referred to as the classical transfer theorems. Included under this rubric is the earliest of the two-cardinal theorems—the fundamental one of Vaught. Also included are those results which are directly inspired by Vaught's result, namely the transfer theorems of Keisler, Chang, Fuhrken and R. B. Jensen. The reduction of the previous section will yield information about the logics $\mathscr{L}(Q)$ under various cardinality interpretations. Some applications and counterexamples are also included in this section. We will conclude this section with a discussion of gap-n and multi-cardinal transfer theorems.

2.1. The Gap-1 Transfer Theorems

The earliest of the gap-1 transfer theorems is the following one. This result, first proven by Vaught in Morley-Vaught [1962], has already been discussed in Chapters II and IV.

2.1.1 Theorem. *For any cardinal $\kappa \geq \aleph_0$, $(\kappa^+, \kappa) \to (\aleph_1, \aleph_0)$ $\aleph_0$-compactly.* $\square$

A consequence of Theorem 2.1.1 is that $K(\aleph_1, \aleph_0)$ is $\aleph_0$-compact. Moreover, Vaught's proof of Theorem 2.1.1 shows that $K(\aleph_1, \aleph_0)$ is recursively enumerable for validity. Thus, the following corollary of Fuhrken [1964] and Vaught [1964] follows.

2.1.2 Corollary. *$\mathscr{L}(Q_1)$ is compact and recursively enumerable for validity.* $\square$

Keisler's proof of Theorem 2.1.1 in [1966b] also yields Corollary 2.1.2. In fact, his proof results in an elegant and comprehensible axiomatization for the class $K(\aleph_1)$. Corollary 1.4.6 suggests that there should also be an axiomatization for $K(\aleph_1, \aleph_0)$. Such an axiomatization, although less elegant than that for $K(\aleph_1)$, was indeed obtained by Keisler [1966a].

Theorem 2.1.1 is equivalent to $\kappa^+ \to \aleph_1$ $\aleph_0$-compactly. Fuhrken [1965] noticed that the proof of Theorem 2.1.1 can be used to prove the following generalization.

2.1.3 Theorem. *For any regular $\kappa \geq \aleph_0$, $\kappa \to \aleph_1$ $\aleph_0$-compactly.* $\square$

Yet another proof of Theorem 2.1.1 was given by Shelah [1978] using the method of identities. This method will be discussed in the next section.

The problem of the "converse" transfer of Theorem 2.1.1 was attacked by Chang [1965a] with notable partial success.

2.1.4 Theorem. *Assume* GCH. *For any regular cardinal κ, $(\aleph_1, \aleph_0) \to (\kappa^+, \kappa)$ κ-compactly.* $\square$

One of the byproducts of Theorem 2.1.4—or of any other instance of $(\aleph_1, \aleph_0) \to (\aleph_{\alpha+1}, \aleph_\alpha)$—is that there is then a completeness theorem for $\mathscr{L}(Q_{\alpha+1})$ which is, of course, the same completeness theorem as the one for $\mathscr{L}(Q_1)$ that is given in Section IV.3.

2.1.5 Corollary. *Assume* GCH. *If $\aleph_\alpha$ is regular, then $\mathscr{L}(Q_{\alpha+1})$ is $\aleph_\alpha$-compact and recursively enumerable for validity.* $\square$

In order to eliminate the requirement that κ be regular in the statement of Theorem 2.1.4, it is natural to replace the use of saturated models by special models. In fact, R. B. Jensen [1972] did just that, but only with an additional set-theoretic assumption which is a consequence of $V = L$.

2.1.6 Theorem. *Assume $V = L$. For any $\kappa \geq \aleph_0$, $(\aleph_1, \aleph_0) \to (\kappa^+, \kappa)$ κ-compactly.* $\square$

2.1.7 Corollary. *Assume $V = L$. For any α, $\mathscr{L}(Q_{\alpha+1})$ is $\aleph_\alpha$-compact and recursively enumerable for validity.* $\square$

We will end this subsection with an application to combinatorics. Shelah [1976a] proved the following result.

2.1.8 Theorem. *There is a linear order of power $\aleph_1$ whose square can be covered by countably many chains.* □

We present the following simple exercise for the reader. Write down a first-order sentence σ with the property that for any cardinals $\kappa \geq \lambda \geq \aleph_0$, σ has a (κ, λ)-model iff there is a linear order of power κ whose square can be covered by λ chains. This done, the following, for example, becomes an immediate consequence.

2.1.9 Corollary. *Assume $V = L$. For any κ, there is a linear order of power κ^+ whose square can be covered by κ^+ chains.* □

2.2. Trees: Some Applications

In this subsection an application and a counterexample, both of which are related to the previous subsection, will be presented. And both require special Aronszajn trees. Since trees will be useful at later points in this chapter, we will devote the first few paragraphs of the present discussion to the requisite definitions.

A *tree* is a partially ordered set $(A, <)$ such that the set of predecessors $\hat{a}$ of any element $a \in A$ is linearly ordered. Contrary to usual practice in set theory, we do not require that a tree be well-founded. A well-founded tree $(A, <)$ has associated with it a rank function rk, where $\mathrm{rk}(a)$ is the ordinal of the order type of $\hat{a}$. In the non-well-founded case there are no such intrinsic rank functions. However, we will overcome this deficiency by introducing *ranked trees* $(A, <, \preccurlyeq)$, where $\preccurlyeq$ is a quasi-order (that is, it is transitive, reflexive, and connected, although not necessarily anti-symmetric) on A such that $(A, <)$ is a tree and $(A, <, \preccurlyeq)$ satisfies the following two sentences:

$$x < y \rightarrow x \preccurlyeq y \wedge \neg y \preccurlyeq x,$$

$$x \preccurlyeq y \rightarrow \exists z(z \leq y \wedge x \preccurlyeq z \wedge z \preccurlyeq x).$$

A well-founded tree $(A, <)$ has a unique expansion to a ranked tree; and the rank order $\preccurlyeq$ is defined so that $a \preccurlyeq b$ iff $\mathrm{rk}(a) \leq \mathrm{rk}(b)$.

In order to make some definitions concerning ranked trees, we let $(A, <, \preccurlyeq)$ be an arbitrary ranked tree. For a regular cardinal κ, we say that $(A, <, \preccurlyeq)$ is a *κ-tree* if $|A| = \kappa$ and, for every $a \in A$. $|\{b \in A : b \preccurlyeq a\}| < \kappa$. A *branch* B of $(A, <, \preccurlyeq)$ is a maximal linearly ordered (by $<$) subset of A which has elements of arbitrarily high rank in the sense that for any $a \in A$ there is $b \in B$ such that $a \preccurlyeq b$. $(A, <, \preccurlyeq)$ is an *Aronszajn κ-tree* if it is a κ-tree which has no branches. At the other extreme, a κ-tree $(A, <, \preccurlyeq)$ is a *Kurepa κ-tree* if it has at least κ^+ branches. Suppose, now, that $(A, <, \preccurlyeq)$ is a λ^+-tree and that there is a function $f: A \rightarrow \lambda$ such that whenever

$x < y$ are elements of A, then $f(x) \neq f(y)$. Then $(A, <, \preccurlyeq)$ is an Aronszajn λ^+-tree. A λ^+-tree for which such a function exists is a *special* Aronszajn λ^+-tree.

The proof of the following result is left as an easy exercise for the reader.

2.2.1 Proposition. *There is a sentence σ of $\mathscr{L}(Q)$ such that for any regular cardinal κ the following are equivalent:*

(1) *there is a special Aronszajn κ-tree;*
(2) *there is a well-founded special Aronszajn κ-tree;*
(3) *there is a model for σ in the κ-interpretation.* □

The existence of an Aronszajn $\aleph_1$-tree was first established by Aronszajn. His construction actually produced a well-founded special Aronszajn $\aleph_1$-tree. The construction is well-known and can be found, for example, in Jech [1978].

2.2.2 Theorem. *There exists a special Aronszajn $\aleph_1$-tree.*

Later—although still prior to Chang's two-cardinal theorem—Specker [1949] proved the existence of special Aronszajn κ-trees, for some cardinals $\kappa > \aleph_1$. Assuming GCH, we can arrive at the same conclusion by use of Theorem 2.1.4.

2.2.3 Corollary. (1) *Assume GCH. If κ is regular, then there is a special Aronszajn κ^+-tree;*

(2) *Assume $V = L$. For any κ, there is a special Aronszajn κ^+-tree.* □

Special Aronszajn trees can be used to show the failure of two-cardinal transfer, or—to put it another way—the necessity of GCH in Chang's theorem (2.1.4). Mitchell [1972] proved the following consistency result concerning the non-existence of special Aronszajn trees. A different proof using iterated perfect set forcing, was developed by Baumgartner and Laver [1979].

2.2.4 Theorem. *If* Con(ZFC + "*there is a Mahlo cardinal*"), *then* Con(ZFC + "*there is no special Aronszajn $\aleph_2$-tree*"). □

2.2.5 Corollary. *If* Con(ZFC + "*there is a Mahlo cardinal*"), *then* Con(ZFC + "$\aleph_1 \not\rightarrow \aleph_2$"). □

Some further results along these lines, results which use generalizations of special Aronszajn trees, can be found in Schmerl [1974].

We will conclude this subsection with a result indicating that $V = L$ cannot be eliminated from the hypothesis of Jensen's theorem (2.1.6) unless there does not exist a certain kind of very large cardinal. This proof of Ben-David [1978a] and Shelah also makes use of trees.

2.2.6 Theorem. *If* Con(ZFC + GCH + " *there is a strongly compact cardinal*"), *then* Con(ZFC + GCH + "$\aleph_1 \nrightarrow \aleph_{\omega+1}$"). ∎

2.3. Gap-2 Transfer

The gap-1 transfer theorems of Section 2.1 suggest the possibility of "gap-2 transfer theorems", that is, theorems of the sort $(\kappa^{++}, \kappa) \to (\lambda^{++}, \lambda)$. Rather simple versions need not be true. For example, if the continuum hypothesis fails and yet $2^\kappa = \kappa^+$, then $(\aleph_2, \aleph_0) \nrightarrow (\kappa^{++}, \kappa)$. Even the GCH is not a sufficient hypothesis, as we shall now see.

From the previous subsection recall the notion of a Kurepa κ-tree. The following straightforward proposition relates Kurepa trees with gap-2 models.

2.3.1 Proposition. *There is a sentence σ such that, for any regular cardinal κ, the following are equivalent*:

(1) *there is a Kurepa κ-tree*;
(2) *there is a well-founded Kurepa κ-tree*;
(3) *there is a (κ^{++}, κ)-model of σ*. ∎

This result can be used to find examples of failure of gap-2 transfer. This is exactly what was done by Silver [1971b] where the following is proven.

2.3.2 Theorem. *If* Con(ZFC + "*there is an inaccessible cardinal*"), *then* Con(ZFC + GCH + "*there is a Kurepa $\aleph_2$-tree but no Kurepa $\aleph_1$-tree*"). ∎

2.3.3 Corollary. *If* Con(ZFC + "*there is an inaccessible cardinal*"), *then* Con(ZFC + GCH + "$(\aleph_3, \aleph_1) \nrightarrow (\aleph_2, \aleph_0)$"). ∎

In Theorem 2.3.2 it would not be sufficient to assume the consistency of just ZFC, for Solovay has shown that if there are no Kurepa κ-trees, then κ^+ is inaccessible in the constructible universe L. A proof of this result can be found in Devlin [1973a]. In particular, if $V = L$, then for every regular κ there exists a Kurepa κ-tree. This suggests the truth of the gap-2 transfer theorem assuming that $V = L$. Indeed this was proven by R. B. Jensen. A proof of this can also be found in Devlin [1973a].

2.3.4 Theorem. *If $V = L$, then $(\kappa^{++}, \kappa) \to (\lambda^{++}, \lambda)$ λ-compactly, for any infinite cardinals κ and λ.* ∎

The proof of Theorem 2.3.4 is quite difficult, using much of the intricate machinery of the fine structure of L. A simple proof by Burgess [1978a] yields just the consistency of gap-2 transfer relative to ZFC. A reduction of the type in Section 1 yields that $V = L$ implies, for example, that $\mathscr{L}(Q_1, Q_2)$ is $\aleph_0$-compact. The proof of Theorem 2.3.4 also shows that $V = L$ implies that $\mathscr{L}(Q_1, Q_2)$ is recursively enumerable for validity.

2.4. Gap-n and Multi-Cardinal Theorems

In order to generalize the gap-2 transfer of the previous subsection to gap-n, it will be useful to have the iterated successor function. For cardinal λ and ordinal $\alpha > 0$, let $\aleph_0(\lambda) = \lambda$ and $\aleph_\alpha(\lambda) = \sup\{(\aleph_\beta(\lambda))^+ : \beta < \alpha\}$. A gap-$n$ structure is an $(\aleph_n(\lambda), \lambda)$-structure, for some λ. We will use $U_1, U_2, U_3, \ldots$ to denote unary relations.

2.4.1 Definition. A structure $\mathfrak{A}$ is a $(\kappa_0, \kappa_1, \ldots, \kappa_n)$-structure if $|A| = \kappa_0, |U_i| = \kappa_i$ for $i = 1, 2, \ldots, n$ and (for the sake of orderliness), $\kappa_0 \geq \kappa_1 \geq \cdots \geq \kappa_n$.

$$(\kappa_0, \kappa_1, \ldots, \kappa_n) \to (\lambda_0, \lambda_1, \ldots, \lambda_n)$$

iff every sentence which has a $(\kappa_0, \kappa_1, \ldots, \kappa_n)$-model also has a $(\lambda_0, \lambda_1, \ldots, \lambda_n)$-model.

There are other notions of transfer for multi-cardinal models which are analogues of those in Definition 1.3.1.

Every gap-n theorem yields an ostensibly stronger multicardinal theorem. This is a consequence of the following observation which the reader should be able to prove.

2.4.2 Proposition. *For each $1 \leq n < \omega$ and each first-order sentence σ, there is a sentence σ' such that, for each infinite cardinal κ, the following are equivalent:*

(1) *σ has an $(\aleph_n(\kappa), \kappa)$-model;*
(2) *σ' has an $(\aleph_n(\kappa), \aleph_{n-1}(\kappa), \ldots, \kappa^+, \kappa)$-model.* □

The gap-2 theorem (2.3.4) has been extended by Jensen using techniques which are of such extreme difficulty that to date the proof remains unpublished, although it has been confirmed by rumor.

2.4.3 Theorem. *Assume $V = L$. For any $n < \omega$ and any infinite cardinals κ and λ, $(\aleph_n(\kappa), \kappa) \to (\aleph_n(\lambda), \lambda)$ λ-compactly.*

From Proposition 2.2.3 we can quite easily obtain the $\aleph_\alpha$-compactness of the logic $\mathscr{L}(Q_{\alpha+1}, Q_{\alpha+2}, \ldots, Q_{\alpha+n})$, assuming $V = L$. The proof of Theorem 2.4.3 also shows that $V = L$ implies that $\mathscr{L}(Q_1, Q_2, \ldots, Q_n)$ is recursively enumerable for validity.

At this point it is interesting to take note of the Lachlan multi-cardinal theorem for stable theories. The original proof is in Lachlan [1973] and a later, more simple proof can be found in Baldwin [1975].

2.4.4 Theorem. *Let T be a stable first-order theory which has a $(\kappa_0, \kappa_1, \ldots, \kappa_n)$-model, where $\kappa_0 > \kappa_1 > \cdots > \kappa_n$. Then T has a $(\lambda_0, \lambda_1, \ldots, \lambda_n)$-model whenever $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n$.*

Some multi-cardinal theorems have applications to the calculations of Hanf numbers. This will be discussed at the end of Section 3.3.

3. *Two-Cardinal Theorems and the Method of Identities*

This section will examine a powerful approach to analyzing two-cardinal transfer and two-cardinal compactness. These developments will, of course, have important implications for the logics $\mathscr{L}(Q)$ and for the various cardinal interpretations via Proposition 1.3.2.

In its simplest form, this method is the familiar one of employing indiscernibles as generators in such a way that throughout a very tight control is maintained over the generated model. For example, subsequent to the original proof of Vaught's gap-ω theorem in Vaught [1965a], a result that is here formulated as Corollary 3.3.7, Keisler and Morley used indiscernibles obtained via the Erdös-Rado theorem (see Example 3.1.2 below) to give an alternate and more simple proof of that result. Generators which are only partially indiscernible can be used with nearly the same resulting tight control. Moreover, there is an added flexibility that guarantees that the distinguished subset and the model itself have the desired cardinality. It will be seen that identities are used as a sort of local description of the partition of the set of all finite subsets of a set.

3.1. Identities

We will begin this subsection with the definition of an identity and some rather closely related notions.

3.1.1 Definition. An *identity* I is an equivalence relation on $[D]^{<\omega}$, where D is a finite set, such that if $X, Y \in [D]^{<\omega}$ and XIY, then $|X| = |Y|$. The set D is the *domain* of I, and $|D|$ is the *length* of I.

In general, we will not distinguish between equivalent identities. Two identities I_1 and I_2 are *equivalent* if there is a bijection $\alpha: D_1 \to D_2$, where D_1 and D_2 are domains of I_1 and I_2, respectively, such that whenever $X, Y \in [D_1]^{<\omega}$, then XI_1Y iff $\alpha[X]I_2\alpha[Y]$. Thus, for example, we will consider there to be only countably many distinct, that is, inequivalent, identities. An identity I_1 is called a *subidentity* of I_2 if there is an injection $\alpha: D_1 \to D_2$ from the domain of I_1 to the domain of I_2 such that whenever $X, Y \in [D_1]^{<\omega}$, then XI_1Y implies $\alpha[X]I_2\alpha[Y]$.

Suppose that $f: [A]^{<\omega} \to B$ is a partition of $[A]^{<\omega}$, and suppose also that $D \in [A]^{<\omega}$. Then f *induces* the identity I with domain D if, whenever $X, Y \in [D]^{<\omega}$, then XIY iff both $f(X) = f(Y)$ and $|X| = |Y|$. The set of identities which are subidentities of those induced by f is denoted by $\mathscr{I}(f)$. For infinite cardinals $\kappa \geq \lambda$, let $\mathscr{I}(\kappa, \lambda)$ be the set of all identities I which are in $\mathscr{I}(f)$ whenever $f: [\kappa]^{<\omega} \to \lambda$.

There is an immediate simple observation to be made regarding $\mathscr{I}(\kappa, \lambda)$: These sets are monotone in κ and λ. Specifically, if $\kappa_1 \geq \kappa_2 \geq \lambda_2 \geq \lambda_1$, then $\mathscr{I}(\kappa_2, \lambda_2) \subseteq \mathscr{I}(\kappa_1, \lambda_1)$. The further apart κ and λ happen to be the larger will be $\mathscr{I}(\kappa, \lambda)$, and the closer together they are, the smaller will be $\mathscr{I}(\kappa, \lambda)$. Thus, $\mathscr{I}(\kappa, \kappa)$ is minimal.

In fact, for every κ, the set $\mathscr{I}(\kappa, \kappa)$ consists merely of the *trivial* identities. Conversely, whenever $\kappa > \lambda$, then $\mathscr{I}(\kappa, \lambda)$ contains some nontrivial identity, the simplest one being the identity I with domain 2 in which $\{0\}$ and $\{1\}$ are equivalent.

The previous example will be generalized in Example 3.1.3 by using the iterated successor function. A more instructive example is one which uses the iterated exponential function defined in the following manner:

$$\beth_0(\lambda) = \lambda,$$

$$\beth_{\alpha+1}(\lambda) = 2^{\beth_\alpha(\lambda)},$$

$$\beth_\beta(\lambda) = \bigcup_{\alpha<\beta} \beth_\alpha(\lambda),$$

where α is any ordinal and β a limit ordinal. When $\lambda = \aleph_0$ reference to λ will be surpressed, resulting in the standard $\beth_\alpha$ for $\beth_\alpha(\aleph_0)$. This example indicates how identities are to be used in place of indiscernibles when complete indiscernibility is not possible.

3.1.2 Example. If $\kappa \geq \beth_\omega(\lambda)$, then the partition theorem of Erdös and Rado (see Chang–Keisler [1977]) implies that $\mathscr{I}(\kappa, \lambda)$ is the set of all identities. More specifically, if $\kappa > \beth_n(\lambda)$, then all identities of length at most $(n + 2)$ are in $\mathscr{I}(\kappa, \lambda)$. Conversely, by the Erdös–Hajnal–Rado [1965] converse to the Erdös–Rado theorem, if $\lambda \leq \kappa \leq \beth_n(\lambda)$, then there is an identity of length $(n + 2)$ which is not in $\mathscr{I}(\kappa, \lambda)$. The missing identity is the one in which all sets of the same size are equivalent.

Finally, we note that the reader should see Subsection 2.4 for the definition of $\aleph_\alpha(\lambda)$.

3.1.3 Example. Let I_n be the identity, having domain $D_n = \{a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}\}$, which is the equivalence relation that makes $X, Y \in [D_n]^{<\omega}$ equivalent iff either $X = Y$ or else, for each $i < n$, $|X \cap \{a_i, b_i\}| = |Y \cap \{a_i, b_i\}| \leq 1$. It is left as an interesting exercise to verify that $I_n \in \mathscr{I}(\kappa, \lambda)$ iff $\aleph_n(\lambda) \leq \kappa$.

Identities have a very close relationship with two-cardinal models. The proposition below indicates one direction of this relationship, the other direction being the deeper connection that is revealed in the next subsection by Theorem 3.2.1.

3.1.4 Proposition. *With each identity I one can effectively associate a first-order sentence σ_I such that whenever $\kappa \geq \lambda \geq \aleph_0$, then σ_I has a (κ, λ)-model iff $I \notin \mathscr{I}(\kappa, \lambda)$.*

Proof. Suppose that I is an identity of length n. The sentence σ_I will be in the vocabulary $\tau = \{U, f_1, f_2, \ldots, f_n\}$, where each f_i is an i-ary function symbol, and it will assert that each f_i is a function on the set of subsets of cardinality i, that the range of each f_i is included in U, and that the identity I is not a subidentity of one which is induced by the function $f_1 \cup \cdots \cup f_n$. It thus follows quite immediately from the definition of $\mathscr{I}(\kappa, \lambda)$ that the sentence σ_I has the required property. ☐

For instance, by applying Example 3.1.2 (or, respectively, Example 3.1.3) to the preceding proposition we obtain, for each $n < \omega$, an example of a sentence σ_n which has a (κ, λ)-model iff $\lambda \leq \kappa \leq \beth_n(\lambda)$ (or, respectively, $\lambda \leq \kappa \leq \aleph_n(\lambda)$).

3.2. The Two-Cardinal Compactness/Transfer Theorem

We now come to the fundamental two-cardinal compactness/transfer theorem, a result which was first enunciated by Shelah [1971d]. Some of its consequences will be given in the next subsection.

3.2.1 Theorem (The Two-Cardinal Compactness/Transfer Theorem). *Suppose that $\kappa \geq \lambda$ and that $\kappa_j \geq \lambda_j$ for each $j \in J$. Then each of the following is equivalent to each of the others:*

(1) $\{(\kappa_j, \lambda_j): j \in J\} \to (\kappa, \lambda)$ *$\aleph_0$-compactly;*
(2) $\{(\kappa_j, \lambda_j): j \in J\} \to (\kappa, \lambda)$ *λ-compactly;*
(3) *There exists a function $f: [\kappa]^{<\omega} \to \lambda$ such that $\mathscr{I}(f) \subseteq \bigcup \{\mathscr{I}(\kappa_j, \lambda_j): j \in J\}$.*

Proof. The implication (2) implies (1) is trivial. The implication (1) implies (3) is an easy consequence of Proposition 3.1.4. To see this, we let $\{I_i: i < \omega\}$ be the set of those identities not in each $\mathscr{I}(\kappa_j, \lambda_j)$. Let σ_{I_i} be the sentence from Proposition 3.1.4, so that each σ_{I_i} has a (κ_j, λ_j)-model, for each $j \in J$. Then, each finite subset of $\{\sigma_{I_i}: i < \omega\}$ has a (κ_j, λ_j)-model, for each $j \in J$. Thus, by (1) above, $\{\sigma_{I_i}: i < \omega\}$ has a (κ, λ)-model $(A, U, f_1, f_2, \ldots)$. Assuming that $A = \kappa$ and $U = \lambda$ both hold, we see that $f = \{f_i: i < \omega\}$ is the desired function.

The most interesting of the implications, and the one which demonstrates the real strength of identities, is the remaining one, (3) implies (2). Here, let T be a first-order theory in the vocabulary τ such that each finite subtheory $T_0 \subseteq T$ has a (κ_j, λ_j)-model for each $j \in J$. Because of the cardinality restrictions on T, it can be assumed that $|\tau| \leq \lambda$. The standard technique of adjoining Skolem functions can be used, so that we may as well assume that T is a Skolem theory. Thus, to every τ-formula $\phi(x_0, x_1, \ldots, x_{n-1}, y)$, there corresponds an n-ary term $\ell(x_0, \ldots, x_{n-1})$ in the vocabulary τ such that the sentence

$$\forall \bar{x}[\exists y \phi(\bar{x}, y) \to \phi(\bar{x}, \ell(\bar{x}))]$$

is a consequence of T.

The vocabulary τ will now be augmented by the adjunction of some constant symbols. For each $\xi < \lambda$, let b_ξ be a new individual constant; and, for each $\alpha < \kappa$, let c_α be a new individual constant, yielding the expanded vocabularies $\tau_1 = \tau \cup \{b_\xi: \xi < \lambda\}$ and $\tau_2 = \tau_1 \cup \{c_\alpha: \alpha < \kappa\}$. We will define a theory T_f in the expanded vocabulary τ_2 which depends only on the function $f: [\kappa]^{<\omega} \to \lambda$, whose existence is guaranteed by (3) and which consists of the following sentences:

(i) $b_\xi \neq b_\eta \quad (\xi < \eta < \lambda)$;
(ii) $U(b_\xi) \quad (\xi < \lambda)$;
(iii) $c_\alpha \neq c_\beta \quad (\alpha < \beta < \kappa)$;

(iv) $U(t(c_{\alpha_0}, \ldots, c_{\alpha_{m-1}})) \to t(c_{\alpha_0}, \ldots, c_{\alpha_{m-\tau}}) = t(c_{\beta_0}, \ldots, c_{\beta_{m-1}})$, where t is a τ_1-term, $\alpha_0 < \alpha_1 < \cdots < \alpha_{m-1} < \kappa$, $\beta_0 < \beta_1 < \cdots < \beta_{m-1} < \kappa$, and $f(\{\beta_0, \ldots, \beta_{m-1}\}) = f(\{\alpha_0, \ldots, \alpha_{m-1}\})$.

The key sentences are, of course, those occurring in (iv) above, and it should be noted that the terms t appearing there are τ_1-terms so that they may include some of the b_ξ.

There are two crucial facts about T_f that together will complete the proof of the theorem. The first is

Fact 1: Every minimal model of T_f is a (κ, λ)-model;

and the second is

Fact 2: $T \cup T_f$ is consistent.

By Fact 2, the theory $T \cup T_f$ has a model; and, by Fact 1, the minimal submodel of this model, which is also a model of T because T is a Skolem theory, is a (κ, λ)-model. It now remains to supply the proofs of these facts.

The proof of Fact 1 is very easy. Suppose that $\mathfrak{A} = (A, U, \ldots)$ is a minimal model of T_f. Then $|A| \geq \kappa$ holds, because of sentences (iii) above, and $|A| = \kappa$ since $\mathfrak{A}$ is a minimal model for a vocabulary τ_2, where $|\tau_2| \leq \kappa$. Thus, $|A| = \kappa$. Also, $|U| \geq \lambda$ holds, because of sentences (i) and (ii) above. Finally, to see that $|U| \leq \lambda$ holds, we observe that for each $b \in U$, there is some n-ary τ_1-term $t(x_0, \ldots, x_{n-1})$ and some $\xi < \lambda$ such that whenever $\alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \kappa$ and $f(\{\alpha_0, \ldots, \alpha_{n-1}\}) = \xi$, then $\mathfrak{A} \models t(c_{\alpha_0}, \ldots, c_{\alpha_{n-1}}) = b$. Therefore, $|U| \leq \lambda$ must hold since $|\tau_1| \leq \lambda$ holds, thus showing that $\mathfrak{A}$ is a (κ, λ)-model.

To demonstrate that Fact 2 holds, that is, that $T \cup T_f$ is consistent, we will show that every finite subtheory $T_0 \subseteq T \cup T_f$ is consistent. Thus, let $\{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}$ be the finite set consisting of those α for which c_α occurs in some sentence in T_0, where $\alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \kappa$. Then f induces an identity I with domain $\{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}$. Statement (3) of the theorem implies the existence of some $j \in J$ for which $I \in \mathscr{I}(\kappa_j, \lambda_j)$. Let $\mathfrak{A}$ be a (κ_j, λ_j)-model of $T_0 \cap T$; such a model exists by the assumption on T.

Let $\xi_0, \xi_1, \ldots, \xi_s < \lambda_j$ be such that if b_ξ occurs in T_0, then ξ is among $\xi_0, \xi_1, \ldots, \xi_s$. Expand $\mathfrak{A}$ to a structure $\mathfrak{A}_1 = (\mathfrak{A}, b_{\xi_0}, b_{\xi_1}, \ldots, b_{\xi_s})$, where each of the b_{ξ_i} denote distinct elements of U. By very simple cardinality considerations, there is a function $g\colon [A]^{<\omega} \to \lambda_j$ such that whenever $\{a_0, a_1, \ldots, a_{n-1}\}, \{a'_0, a'_1, \ldots, a'_{n-1}\} \in [A]^n$, then $g(\{a_0, a_1, \ldots, a_{n-1}\}) = g(\{a'_0, a'_1, \ldots, a'_{n-1}\})$ iff

$$\begin{aligned}\mathfrak{A}_1 \models [U(t(a_0, \ldots, a_{n-1})) \vee U(t(a'_0, \ldots, a'_{n-1}))] &\to t(a_0, \ldots, a_{n-1}) \\ &= t(a'_0, \ldots, a'_{n-1})\end{aligned}$$

for each τ_1-term t occurring in T_0.

Recall that $I \in \mathscr{I}(\kappa_j, \lambda_j)$. Hence, there exists $D \subseteq A$ such that the injection $h\colon \{\alpha_0, \ldots, \alpha_{n-1}\} \to D$ demonstrates that I is a subidentity of the identity with domain D induced by g. Expand $\mathfrak{A}_1$ to the structure $\mathfrak{A}_2 = (\mathfrak{A}_1, c_{\alpha_0}, \ldots, c_{\alpha_{n-1}})$, where $c_{\alpha_i} = h(\alpha_1)$. Then $\mathfrak{A}_2$ is a model of T_0, thus demonstrating the consistency of T_0. $\square$

3.3. Some Consequences

The two-cardinal compactness/transfer theorem of the previous subsection has many consequences. This subsection will be devoted to the most interesting and important of them. One immediate consequence is that in statements (1) through (3) it always suffices to consider just some countable subset $J_0 \subseteq J$.

3.3.1 Corollary. *If* $\{(\kappa_j, \lambda_j): j \in J\} \to (\kappa, \lambda)$ $\aleph_0$*-compactly, then for some countable* $J_0 \subseteq J$, $\{(\kappa_j, \lambda_j): j \in J_0\} \to (\kappa, \lambda)$ λ*-compactly.*

Proof. Consider statement (3) of Theorem 3.2.1. Since $\mathscr{I}(f)$ is countable, there is some countable $J_0 \subseteq J$ such that $\mathscr{I}(f) \subseteq \{\mathscr{I}(\kappa_j, \lambda_j): j \in J_0\}$. $\square$

A function such as the one whose existence is asserted by clause (3) of Theorem 3.2.1 is called a *fundamental function for the relation* $\{(\kappa_j, \lambda_j): j \in J\} \to (\kappa, \lambda)$. If $f: [\kappa]^{<\omega} \to \lambda$ is a fundamental function for $(\kappa, \lambda) \to (\kappa, \lambda)$, then we will say simply that $f: [\kappa]^{<\omega} \to \lambda$ is *fundamental.* Thus, as is very easy to see, $f: [\kappa]^{<\omega} \to \lambda$ is fundamental iff $\mathscr{I}(f) = \mathscr{I}(\kappa, \lambda)$.

The statement that $(\kappa, \lambda) \to (\kappa, \lambda)$ μ-compactly is evidently equivalent to $K(\kappa, \lambda)$ being μ-compact. Thus, Theorem 3.2.1 yields the following corollary.

3.3.2 Corollary. *If* $\kappa \geq \lambda \geq \aleph_0$, *then each of the following is equivalent to each of the others*:

(1) $K(\kappa, \lambda)$ *is* $\aleph_0$*-compact*;
(2) $K(\kappa, \lambda)$ *is* λ*-compact*;
(3) *There is a fundamental function* $f: [\kappa]^{<\omega} \to \lambda$. $\square$

The corollary thus characterizes compactness in terms of the purely combinatorial property of the existence of fundamental functions. In general, the question of the existence of fundamental functions remains unsolved. However, with some very mild restrictions imposed upon the cardinals, their existence can be easily demonstrated.

3.3.3 Lemma. *If* $\kappa > \lambda^{\aleph_0} = \lambda$, *then there is a fundamental function* $f: [\kappa]^{<\omega} \to \lambda$.

Proof. Let $\{I_n: n < \omega\}$ be the set of identities which are not in $\mathscr{I}(\kappa, \lambda)$. We pause at this point to observe that if this set is finite, or even empty, then things become even easier than is otherwise the case. For each n, let $f_n: [\kappa]^{<\omega} \to \lambda$ be such that $I_n \notin \mathscr{I}(f_n)$. Let $g: \lambda^\omega \to \lambda$ be a bijection. Define $f: [\kappa]^{<\omega} \to \lambda$ such that whenever $A \in [\kappa]^{<\omega}$ and $n < \omega$, then $f(A) = g(\langle f_n(A): n < \omega \rangle)$. We immediately see that $\mathscr{I}(f) = \mathscr{I}(\kappa, \lambda)$, so that f is fundamental. $\square$

The Shelah-Fuhrken two-cardinal compactness theorem, a result which was first proven in Fuhrken [1965] with stronger hypotheses using ultraproducts and which was later improved in Shelah [1971d], is an instantaneous consequence of Lemma 3.3.3 and Corollary 3.3.2.

3.3.4 Corollary. *If $\kappa > \lambda^{\aleph_0} = \lambda$, then $K(\kappa, \lambda)$ is λ-compact.* ☐

3.3.5 Corollary. *If $\aleph_\alpha^{\aleph_0} = \aleph_\alpha$, then $\mathscr{L}(Q_{\alpha+1})$ is $\aleph_\alpha$-compact.*

Proof. See Corollaries 1.3.4 and 3.3.4. ☐

Corollary 3.3.5 will be generalized later in Corollaries 4.2.1 and 5.1.3. There are instances of compactness of $\mathscr{L}(Q_{\alpha+1})$ not covered by Corollary 3.3.5, the most notable being $\mathscr{L}(Q_1)$ which is known to be $\aleph_0$-compact (see Chapter IV) even though $\aleph_0^{\aleph_0} > \aleph_0$. In fact, no example is known for even the consistency of the failure of $\aleph_0$-compactness of any $\mathscr{L}(Q_{\alpha+1})$. On the other hand, it is unknown whether it is a theorem of ZFC that $\mathscr{L}(Q_2)$ is $\aleph_0$-compact, although it does follow from ZFC + CH.

The two-cardinal compactness/transfer theorem has two transfer theorems as rather immediate corollaries. The first is the Chang–Keisler [1962] gap narrowing theorem, a result that was originally proven using ultrapowers, and the second is Vaught's gap-ω theorem, a result originally proven by Vaught [1965a] using self-extending models, a concept which will be discussed in Section 6.

3.3.6 Corollary. *If $\kappa \geq \mu \geq \lambda^{\aleph_0}$, then $(\kappa, \lambda) \to (\kappa, \mu)$ μ-compactly.*

Proof. Since $(\lambda^{\aleph_0})^{\aleph_0} = \lambda^{\aleph_0}$, we see from Lemma 3.3.3 that there is a fundamental function $f\colon [\kappa]^{<\omega} \to \lambda^{\aleph_0}$. Thus, we have that $\mathscr{I}(f) = \mathscr{I}(\kappa, \lambda^{\aleph_0}) \subseteq \mathscr{I}(\kappa, \lambda)$. Since $\lambda^{\aleph_0} \leq \mu$ holds, we can consider f to have range μ, so that $f\colon [\kappa]^{<\omega} \to \mu$ is fundamental for $(\kappa, \lambda) \to (\kappa, \mu)$. ☐

3.3.7 Theorem. *If $\kappa \geq \lambda \geq \aleph_0$ and if $\kappa_n \geq \beth_n(\lambda_n)$ for each $n < \omega$, then $\{(\kappa_n, \lambda_n)\colon n < \omega\} \to (\kappa, \lambda)$ λ-compactly.*

Proof. Because of Example 3.1.3 any $f\colon [\kappa]^{<\omega} \to \lambda$ is fundamental for $\{(\kappa_n, \lambda_n)\colon n < \omega\} \to (\kappa, \lambda)$. ☐

The following corollary can be extracted from the proof of Theorem 3.2.1.

3.3.8 Corollary. *If $K(\kappa, \lambda)$ is $\aleph_0$-compact, then $K(\kappa, \lambda)$ is recursively enumerable for validity iff $\mathscr{I}(\kappa, \lambda)$ is recursively enumerable.* ☐

If $\kappa \geq \beth_\omega(\lambda)$, then $\mathscr{I}(\kappa, \lambda)$ is the set of all identities, and is therefore evidently recursive. This observation yields the following corollary.

3.3.9 Corollary. *If $\kappa \geq \beth_\omega(\lambda)$, then $K(\kappa, \lambda)$ is recursively enumerable for validity.* ☐

The following three-cardinal theorem can be proven in a manner quite similar to the one that was used to prove Theorem 3.3.7.

3.3.10 Theorem. *If $\kappa \geq \aleph_1$ and if $\kappa_n \geq \beth_n(\lambda_n)$ and $\lambda_n > \mu_n$, for each $n < \omega$, then $\{(\kappa_n, \lambda_n, \mu_n)\colon n < \omega\} \to (\kappa, \aleph_1, \aleph_0)$ $\aleph_0$-compactly.* ☐

An immediate consequence of this theorem, a consequence that can be obtained by setting each $\lambda_n = \aleph_1$ and $\mu_n = \aleph_0$, is that the Hanf number of $\mathscr{L}(Q_1)$ is $\beth_\omega$. All that was needed concerning $\mathscr{L}(Q_1)$ was the $\aleph_0$-compactness of $\mathscr{L}(Q_1)$. Thus, the more general result on Hanf numbers $h_\lambda(\mathscr{L}(Q_\alpha))$ can be proven by the same technique.

3.3.11 Theorem. *If $\mathscr{L}(Q_\alpha)$ is λ-compact, then $h_\lambda(\mathscr{L}(Q_\alpha)) = \beth_\omega(\aleph_\alpha)$.* □

Consequently, Proposition II.5.2.4 yields the following characterization.

3.3.12 Corollary. *$h_{\aleph_0}(\mathscr{L}(Q_\alpha)) = \beth_\omega(\aleph_\alpha)$ iff $\mathscr{L}(Q_\alpha)$ is $\aleph_0$-compact.* □

In particular, Corollary 3.3.5 implies some specific Hanf numbers.

3.3.13 Corollary. *If $\aleph_\alpha^{\aleph_0} = \aleph_\alpha$, then $h_{\aleph_\alpha}(\mathscr{L}(Q_{\alpha+1})) = \beth_\omega(\aleph_\alpha)$.*

3.4. Employing the Methodology of Identities

The two-cardinal compactness/transfer theorem (3.2.1) suggests a method for proving specific two-cardinal transfer theorems. Suppose it is desired to prove the transfer $(\kappa_1, \lambda_1) \to (\kappa_2, \lambda_2)$ $\aleph_0$-compactly. Using the methodology of identities, we can employ the following three-step strategy:

(A) Define a set $\mathscr{I}_0$ of identities.
(B) Show that $\mathscr{I}_0 \subseteq \mathscr{I}(\kappa_1, \lambda_1)$.
(C) Show that there is a function $f: [\kappa_2]^{<\omega} \to \lambda_2$ such that $\mathscr{I}(f) \subseteq \mathscr{I}_0$.

This procedure has been used successfully by Shelah to prove several transfer theorems which will be discussed in this section. First, we will suggest an alternate proof of Vaught's theorem (2.1.1) that is due to Shelah [1978e]. In this proof we will only perform step (A), omitting steps (B) and (C) altogether. Second, we will discuss Shelah's transfer theorem $(\aleph_\omega, \aleph_0) \to (2^{\aleph_0}, \aleph_0)$, which was proven in Shelah [1977]. We will consider only steps (A) and (C).

Vaught's Theorem. Our first task will be to define a set $\mathscr{I}_{\mathrm{Vau}}$ of identities. To do this, a method for building a new identity from an old one will now be described. Let I be an identity with domain $n \in \omega$, and let $E \subseteq n$. The *identity J obtained from I by duplicating E* is constructed as follows: The domain of J is $(n + m)$, where $m = |E|$. Let $\alpha: n + m \to n$ be the function such that $\alpha | n$ is the identity function on n and $\alpha | \{n, n + 1, \ldots, n + m - 1\}$ is an order-preserving bijection onto E. Now J is defined so that if $X, Y \in [n + m]^{<\omega}$, then XJY iff either $X = Y$ or each of the following three conditions is satisfied:

(1) $X \subseteq n$ or $X \cap E = \phi$;
(2) $Y \subseteq n$ or $Y \cap E = \phi$;
(3) $\alpha[X]\, I \alpha[Y]$.

Define $\mathscr{I}^*$ to be the smallest set of identities containing the identity with domain 1 and such that whenever $I \in \mathscr{I}^*$ has domain n and $k < n$, then the ordered identity obtained from I by duplicating $\{k, k+1, \ldots, n-1\}$ is in $\mathscr{I}^*$.

Then $\mathscr{I}_{\mathrm{Vau}}$ can now be defined. It is the smallest set of identities which is closed under the taking of subidentities and which also contains all identities I which are in $\mathscr{I}^*$.

3.4.1 Theorem. $\mathscr{I}(\aleph_1, \aleph_0) = \mathscr{I}_{\mathrm{Vau}}$. □

This approach to Vaught's theorem is interesting since it yields a description of the set $\mathscr{I}(\aleph_1, \aleph_0)$. Now Theorem 2.1.2 and Corollary 3.3.8 predict that $\mathscr{I}(\aleph_1, \aleph_0)$ is merely recursively enumerable. However, since $\mathscr{I}(\aleph_1, \aleph_0) = \mathscr{I}_{\mathrm{Vau}}$, and this latter set is evidently recursive, the following corollary results.

3.4.2 Corollary. *The set* $\mathscr{I}(\aleph_1, \aleph_0)$ *is recursive.* □

Shelah's Theorem. The three-step strategy is the only known method for proving the theorem of Shelah [1977] that $(\aleph_\omega, \aleph_0) \to (2^{\aleph_0}, \aleph_0)$ $\aleph_0$-compactly. This theorem can be stated in a more general form for which a definition is required. For an infinite cardinal κ let ded*(κ) be the least cardinal λ such that every (well-founded) ranked tree (see Section 2.2) of cardinality κ has fewer than λ branches. Note that ded*$(\aleph_0) = (2^{\aleph_0})^+$ and that $\kappa^+ <$ ded*$(\kappa) \leq (2^\kappa)^+$. On the other hand, Mitchell [1972] has shown that ded*$(\aleph_1) \leq 2^{\aleph_1}$ is relatively consistent with ZFC.

3.4.3 Theorem. *If* ded*$(\lambda) > \kappa \geq \lambda$ *and if* $\kappa_n \geq \aleph_n(\lambda_n)$ *for each* $n < \omega$, *then* $\{(\kappa_n, \lambda_n) : n < \omega\} \to (\kappa, \lambda)$ λ*-compactly.*

In order to execute step (A), we will first define a set $\mathscr{I}^*$ of identities as the smallest set of identities containing the identity with domain 1, and such that whenever $I \in \mathscr{I}^*$ has domain n and $k < n$, then the identity obtained from I by duplicating $\{k\}$ is in $\mathscr{I}^*$. Then $\mathscr{I}_{\mathrm{She}}$ can now be defined as the smallest set of identities which is closed under the taking of subidentities and which contains all identities I which are in $\mathscr{I}^*$.

Having completed step (A), we will now proceed to develop a broad hint for Step (C). Let $(A, <)$ be a well-founded tree which has at least κ branches such that $|A| = \lambda$. Let B be a set of branches of $(A, <)$ of cardinality exactly κ. We will define a function f with domain $[B]^{<\omega}$. Suppose that $b_0, b_1, \ldots, b_n \in B$ are distinct branches. Then let α be the least ordinal such that the elements $a_0 \in b_0, a_1 \in b_1, \ldots, a_n \in b_n$ each of rank α are pairwise distinct. Finally, set $f(\{b_0, b_1, \ldots, b_n\}) = \{a_0, a\ , \ldots, a_n\}$. It is clear that the range of f has cardinality at most λ. A rather easy induction on n can be used to demonstrate that $\mathscr{I}(f) \subseteq \mathscr{I}_{\mathrm{She}}$.

The proof of step (B) can be found in Shelah [1977].

3.4.4 Corollary. *Suppose* $2^{\aleph_0} > \aleph_\omega$. *Then* $K(2^{\aleph_0}, \aleph_0)$ *is* $\aleph_0$*-compact and recursively enumerable for validity. In fact,* $\mathscr{I}(2^{\aleph_0}, \aleph_0)$ *is recursive.* □

It seems appropriate at this point to mention a closely related theorem that is due to Shelah [1975b], a result which is conveniently stated as a three-cardinal theorem.

3.4.5 Theorem. *For each $n < \omega$, let κ_n, p_n, q_n be cardinals such that $n^n \leq q_n^n \leq p_n < \aleph_0 \leq \kappa_n$. Also, let* $\mathrm{ded}^*(\lambda) > \kappa \geq \lambda$. *Then $\{(\kappa_n, p_n, q_n): n < \omega\} \to (\kappa, \kappa, \lambda)$ λ-compactly.* ☐

To prove this theorem, a modification of the aforementioned three-step procedure is used, steps (A) and (C) being almost exactly the same as in the proof of Theorem 3.4.3. A proof of a suitable version of step (B) can be given inside of Peano arithmetic, so the following corollary becomes a consequence of Theorem 3.4.5.

3.4.6 Corollary. *Let $\mathcal{M}$ be a model of Peano arithmetic and $I \subseteq M$ a proper initial segment closed under multiplication. Then, whenever* $\mathrm{ded}^*(\lambda) > \kappa \geq \lambda$ *there is a model $(\mathcal{N}, J) \equiv (\mathcal{M}, I)$ such that $|J| = \lambda$ yet every initial segment of $\mathcal{N}$ properly containing J has cardinality κ.* ☐

For the case in which $\kappa = 2^{\aleph_0}$ and $\lambda = \aleph_0$, this corollary was proven by Paris and Mills [1979]. Corollary 3.4.6 thus also follows from their result using Theorem 3.4.3 and some absoluteness considerations.

4. Singular Cardinal-like Structures

The topic of this section is the transfer theorem for singular cardinals which was obtained by Keisler [1968b]. This theorem and its proof have consequences concerning the compactness and recursive enumerability for validity of the language with cardinality quantifier Q_α with $\mathfrak{A}_\alpha$ a singular, strong limit cardinal.

4.1. Keisler's Transfer Theorem

In the following discussion Keisler's transfer theorem, which is the main result of this section, will be examined. To this purpose, we recall that a cardinal κ is a strong limit cardinal if $2^\lambda < \kappa$ whenever $\lambda < \kappa$. We will begin our development with a simple example limiting possible generalizations of the theorem.

4.1.1 Example. Let σ_1 be a first-order sentence in the vocabulary $\{<, R\}$, where R is a binary relation symbol, describing the fact that there is an injection of the universe in the power set of some proper initial segment. Then σ_1 has a κ-like model iff κ is not a strong limit cardinal.

4.1.2 Theorem (Keisler [1968b]). *Suppose that κ is a strong limit cardinal and that $\lambda > \mu \geq \aleph_0$, where λ is a singular cardinal. Then $\kappa \to \lambda$ μ-compactly.*

Only the initial portion of the proof will be presented here. Thus, suppose that τ is a vocabulary of cardinality at most μ which contains the n-ary Skolem function symbol f_ϕ for each $(n + 1)$-ary τ-formula ϕ. Let $\tau' = \tau \cup \{c_{i,j}: i, j < \omega\}$, where the $c_{i,j}$ are new, distinct, individual constants. Define a set Γ to consist of the following τ'-sentences:

(1) $\forall \bar{x}[\exists y \phi(\bar{x}, y) \to \phi(\bar{x}, f_\phi(\bar{x}))]$, for each τ-formula ϕ;
(2) $c_{i,j} < c_{i,k}$, whenever $i < \omega$ and $j < k < \omega$;
(3) $t < c_{i,j}$, where t is any τ'-term that does not involve any constant $c_{k,n}$ with $k \geq i$;
(4) $\forall x_0, \ldots, x_{n-1}[x_0 < c_{i,r} \wedge \cdots \wedge x_{n-1} < c_{i,r}$
$$\to (\phi(\bar{x}, c_{m,j_0}, c_{m,j_1}, \ldots, c_{m,j_s}) \leftrightarrow \phi(\bar{x}, c_{m,k_0}, c_{m_{k_1}}, \ldots, c_{m,k_s}))],$$
whenever $i < m$, $j_0 < j_1 < \cdots < j_s$, $k_0 < k_1 < \cdots < k_s$ and $\phi(\bar{x}, \bar{y})$ is a τ'-formula which does not involve any $c_{p,q}$ for $p \leq m$.

There are now two crucial properties that must be verified:

(I) Every set of τ-sentences consistent with Γ has a λ-like model;
(II) Any τ-sentence which has a κ-like model is consistent with Γ.

We end with a hint that in order to prove property (II) above, it is necessary to apply the Erdös–Rado theorem several times.

4.2. Some Consequences

Theorem 4.1.2 and its proof yield some immediate consequences.

4.2.1 Corollary. *If $\aleph_\alpha$ is a singular, strong limit cardinal and $\aleph_0 \leq \lambda < \aleph_\alpha$, then $\mathscr{L}(Q_\alpha)$ is λ-compact.* $\square$

By using a different approach to handle regular κ, we will see as a consequence of Theorem 5.1.3 that the requirement of singularity can be dropped in this corollary.

The upshot of (I) and (II) in the proof of Theorem 4.1.2 lies in the fact that if κ is a singular, strong limit cardinal and σ is a τ-sentence, then σ has a κ-like model iff σ is consistent with Γ. An inspection of the proof reveals that if τ is recursively enumerable, then so is Γ. Thus, the set of τ-sentences true in every κ-like model is recursively enumerable. This proves the following result.

4.2.2 Corollary. *If $\aleph_\alpha$ is a singular, strong limit cardinal, then $\mathscr{L}(Q_\alpha)$ is recursively enumerable for validity.* $\square$

As a consequence of Corollary 3.3.11, some more Hanf numbers can be computed.

4.2.3 Corollary. *If $\aleph_\alpha$ is a singular, strong limit cardinal and $\lambda < \aleph_\alpha$, then $h_\lambda(\mathscr{L}(Q_\alpha)) = \beth_\omega(\aleph_\alpha)$.* $\square$

Corollaries 4.2.1 and 4.2.2 have immediate consequences with respect to the logic $\mathscr{L}(Q^C)$ involving the Chang quantifier. (See Chapter VI.) Recall that the syntax of this logic is the same as the syntax of the logic $\mathscr{L}(Q)$ with the cardinality quantifier, and its interpretation in the structure $\mathfrak{A}$ is that of $\mathscr{L}(Q)$ using the $|A|$-interpretation, with the restriction that $\mathfrak{A}$ be infinite.

4.2.4 Corollary. *Assume* GCH. *$\mathscr{L}(Q^C)$ is recursively enumerable for validity and is compact.*

Proof. $\mathscr{L}(Q_1)$ is recursively enumerable for validity according to Theorem 2.1.2, and so is $\mathscr{L}(Q_\omega)$ by Corollary 4.2.2, since by GCH $\aleph_\omega$ is a strong limit cardinal. Now, by Theorems 2.1.3 and 4.1.2, σ is valid for $\mathscr{L}(Q^C)$ iff it is valid for both $\mathscr{L}(Q_1)$ and $\mathscr{L}(Q_\omega)$. Hence, $\mathscr{L}(Q^C)$ is recursively enumerable for validity.

Let Σ be a set of κ sentences of $\mathscr{L}(Q^C)$ which is finitely consistent. Then either every finite $\Sigma_0 \subseteq \Sigma$ is consistent for $\mathscr{L}(Q_1)$, or every finite $\Sigma_0 \subseteq \Sigma$ is consistent for $\mathscr{L}(Q_\omega)$. Using Theorem 2.1.4 in the first case and Theorem 4.1.2 in the second, there is a model $\mathfrak{A}$ of Σ in the $\aleph_\alpha$-interpretation for appropriate $\aleph_\alpha > \kappa$. Since the Löwenheim number $l_\kappa(\mathscr{L}(Q_\alpha)) = \aleph_\alpha$, we can require that $|A| = \aleph_\alpha$. Thus, we have that $\mathfrak{A}$ is also a $\mathscr{L}(Q^C)$-model of Σ. $\square$

5. *Regular Cardinal-like Structures*

By means of more elaborate forms of identities, κ-like anologues of some of the results given in Section 3 can be obtained. The main interest occurs when κ is inaccessible. Some of these results will be discussed in this section.

5.1. The Compactness/Transfer Theorem

We will begin this discussion with the basic compactness/transfer theorem.

5.1.1 Theorem (The Regular Cardinal-like Compactness/Transfer Theorem). *Suppose that $\kappa > \aleph_0$ and that κ_j is regular for each $j \in J$. Then the following are equivalent*:

(1) $\{\kappa_j : j \in J\} \to \kappa$ *$\aleph_0$-compactly;*
(2) $\{\kappa_j : j \in J\} \to \kappa$ *λ-compactly, for each $\lambda < \kappa$.* $\square$

5.1.2 Corollary. *If $\aleph_\alpha > \lambda \geq \aleph_0$ and $\aleph_\alpha$ is regular, then*

$$\mathscr{L}(Q_\alpha) \text{ is } \aleph_0\text{-compact} \quad \textit{iff} \quad \mathscr{L}(Q_\alpha) \text{ is } \lambda\text{-compact.}$$

A proof of Theorem 5.1.1 would yield the following instances of compactness as a consequence.

5.1.3 Corollary. *If $\aleph_\alpha > \lambda \geq \aleph_0$, for regular $\aleph_\alpha$, and $\aleph_\beta^{\aleph_0} < \aleph_\alpha$ for $\beta < \alpha$, then $\mathscr{L}(Q_\alpha)$ is λ-compact.* $\square$

Combining this result with Corollaries 2.1.7 and 4.2.1 yields the following general result.

5.1.4 Theorem. *Assume $V = L$. If $\aleph_\alpha > \lambda \geq \aleph_0$, then $\mathscr{L}(Q_\alpha)$ is λ-compact.* $\square$

This allows us to use Theorem 3.3.11 in computing Hanf numbers.

5.1.5 Theorem. *Assume $V = L$. If $\aleph_\alpha > \lambda \geq \aleph_0$, then $h_\lambda(\mathscr{L}(Q_\alpha)) = \beth_\omega(\aleph_\alpha)$.* $\square$

It is not known whether the $V = L$ hypothesis can be eliminated from Theorems 5.1.4 and 5.1.5.

A cardinal κ is 0-*Mahlo* iff it is inaccessible. For $\alpha > 0$, the cardinal κ is α-*Mahlo* if, whenever $\beta < \alpha$ and $C \subseteq \kappa$ is closed and unbounded, then there is a β-Mahlo cardinal in C. The cardinal κ is *strongly α-Mahlo* if it is strongly inaccessible in addition to being α-Mahlo. It is known that if κ is weakly compact, then κ is κ-Mahlo and also that there are many cardinals $\lambda < \kappa$ which are λ-Mahlo.

The following theorem was given a combinatorial proof in Schmerl [1972]. In this connection we point out that there is also the beautiful Silver–Kaufmann approach, which uses models of ZFC and which is detailed in Kaufmann [1983a].

5.1.6 Theorem. *For each $n < \omega$ there is an $\mathscr{L}(Q)$ sentence σ_n such that for each regular κ, σ_n is consistent in the κ interpretation iff κ is not strongly n-Mahlo.* $\square$

The following theorem of Schmerl and Shelah [1972] is a best possible result by Theorem 5.1.6.

5.1.7 Theorem. *For each $n < \omega$ let κ_n be strongly n-Mahlo, and let $\kappa > \lambda \geq \aleph_0$. Then $\{\kappa_n : n < \omega\} \to \kappa$ λ-compactly.* $\square$

One possible approach to proving this theorem uses generalizations of identities. For another approach, which uses self-extending models, see Theorem 6.1.3. Either approach enables us to obtain the following corollary.

5.1.8 Corollary. *If $\aleph_\alpha$ is strongly ω-Mahlo, then $\mathscr{L}(Q_\alpha)$ is recursively enumerable for validity.* $\square$

It is not known whether the hypothesis of the corollary can be weakened. For example, whether or not $\mathscr{L}(Q_\alpha)$ is recursively enumerable for validity when $\aleph_\alpha$ is the first strongly inaccessible remains open. Indeed, it is not even known whether it is even *consistent* with ZFC that there be *any* $\alpha > 0$ for which $\mathscr{L}(Q_\alpha)$ is not recursively enumerable for validity.

5.2. Strongly Cardinal-like Structures

Suppose we consider the vocabulary having only the binary relation symbol $<$ and the sentence of stationary logic which is the conjunction of a sentence asserting that $<$ is a linear order and the sentence

$$\text{aa}\ s\ \exists x\ \forall y(y \in s \leftrightarrow y < x).$$

Then $(A, <)$ is a model of this sentence iff it is $\aleph_1$-like and there is a closed, unbounded subset of A which has order type ω_1. A well-ordered subset $X \subseteq A$ is closed and unbounded iff whenever $a \in X$ is a limit point, then a is the least upper bound of the set $\{x \in X : x < a\}$ in A. The next definition generalizes this type of ordering.

5.2.1 Definition. A linearly ordered set $(A, <)$ is *strongly κ-like*, where κ is a regular, uncountable cardinal, if it is κ-like and contains a closed, unbounded subset. A structure $\mathfrak{A} = (A, <, \ldots)$ is *strongly κ-like* if $(A, <)$ is strongly κ-like.

There is a reduction of $\mathscr{L}(\text{aa})$ to strongly $\aleph_1$-like structures.

5.2.2 Theorem. *With each sentence σ of $\mathscr{L}(\text{aa})$ we can effectively associate a first-order sentence σ^* such that the following are equivalent:*

(1) *σ is consistent;*
(2) *σ has a model of cardinality $\aleph_1$;*
(3) *σ^* has a strongly $\aleph_1$-like model.* □

In order to get the κ-interpretation, where κ is regular and uncountable, we consider the set $P_\kappa(A)$ which is the set consisting of just those subsets of A having cardinality $<\kappa$. A subset $C \subseteq P_\kappa(A)$ is *closed* if it is closed under the union of chains of length $< \kappa$, and it is *unbounded* if, for every $s \in P_\kappa(A)$, there is $t \in C$ such that $s \subseteq t$. Let $D_\kappa(A)$ be the filter generated by the closed unbounded subsets of $P_\kappa(A)$. The new clause in the definition of satisfaction in the κ-interpretation is now clear:

$$\mathfrak{A} \models \text{aa}\ s\ \phi(s) \quad \text{iff} \quad \{s \in P_\kappa(A) : \mathfrak{A} \models \phi(s)\} \in D_\kappa(A).$$

Compare this definition with Definition IV.4.1.1. Stationary logic with the κ interpretation, where $\kappa = \aleph_\alpha$, will be denoted by $\mathscr{L}(\text{aa}_\alpha)$, so that $\mathscr{L}(\text{aa}_1) = \mathscr{L}(\text{aa})$.

The following transfer theorem becomes apparent upon checking that all the axioms for $\mathscr{L}(\text{aa})$ are valid in arbitrary $\mathscr{L}(\text{aa}_\alpha)$.

5.2.3 Theorem. *If $\aleph_\alpha > \aleph_0$ is regular, then $\mathscr{L}(\text{aa}_\alpha) \to \mathscr{L}(\text{aa}_1)$ $\aleph_0$-compactly.* □

Instead of proving transfer theorems of the form suggested by Theorem 5.2.3, we will concentrate on theorems concerning strongly κ-like structures. This is justified by the following two observations. The first is that in the κ-interpretation, the linearly ordered set $(A, <)$ is a model of the $\mathscr{L}$(aa) sentence displayed at the beginning of this subsection iff $(A, <)$ is strongly κ-like. In the second observation we state a theorem whose proof is identical to the proof of Theorem 5.2.2.

5.2.4 Theorem. *With each sentence σ of $\mathscr{L}$(aa) we can effectively associate a first-order sentence σ^* such that for each regular uncountable κ, the following are equivalent:*

(1) *in the κ-interpretation, σ has a model of cardinality κ;*
(2) *σ^* has a strongly κ-like model.* ☐

In light of the above, the next definition is natural.

5.2.5 Definition. For regular uncountable cardinals, κ and λ, $\kappa \underset{s}{\rightarrow} \lambda$ if whenever σ is a first-order sentence which has a strongly κ-like model, then σ has a strongly λ-like model.

The customary variations on the above definition will be in force. For example, for regular uncountable κ, Theorem 5.2.3 implies that $\kappa \underset{s}{\rightarrow} \aleph_1$ $\aleph_0$-compactly.

The following theorem is the compactness/transfer theorem for strongly cardinal-like models. Its proof resembles the proofs of Theorems 3.2.1 and 5.1.1, although it does use an even more elaborate notion of identity.

5.2.6 Theorem. *Suppose that κ and κ_j are regular, uncountable cardinals, for each $j \in J$, such that for each $n < \omega$ there is some $j \in J$ for which $\kappa_j \geq \aleph_n$. Then the following are equivalent*

(1) $\{\kappa_j : j \in J\} \underset{s}{\rightarrow} \kappa$ *$\aleph_0$-compactly;*
(2) $\{\kappa_j : j \in J\} \underset{s}{\rightarrow} \kappa$ *λ-compactly for each $\lambda < \kappa$.* ☐

Many corollaries of the same sort as those derived from Theorems 3.2.1 and 5.1.1 can be derived from this theorem. We will mention only one of them here.

5.2.7 Corollary. *If $\kappa > \lambda \geq \aleph_\omega$, κ is regular, and $\mu^{\aleph_0} < \kappa$ for each $\mu < \kappa$, then the class of strongly κ-like structures is λ-compact.* ☐

The subtle hierarchy of cardinals was defined in Baumgartner [1975] and in Schmerl [1976]. A cardinal κ is *subtle* iff whenever $\langle S_\alpha : \alpha < \kappa \rangle$ is such that each $S_\alpha \subseteq \alpha$ and whenever $C \subseteq \kappa$ is closed and unbounded, then there are $\alpha < \beta$, both in C, such that $S_\beta \cap \alpha = S_\alpha$. Subtle cardinals are large in the sense that they are all strongly inaccessible. And yet, the first one—if it exists—is far larger than the first strongly inaccessible. For each ordinal α, we will define α-subtle cardinals with 0-subtle cardinals being regular, uncountable cardinals and 1-subtle cardinals being the same as subtle cardinals. However, we will be even more general than this by defining what is meant by a subset $X \subseteq \kappa$ being α-subtle. To this end, let us assume that κ is a regular, uncountable cardinal and $X \subseteq \kappa$. Then X is 0-*subtle* iff X is stationary. Inductively, X is $(\alpha + 1)$-*subtle* iff whenever $\langle S_\nu: \nu < \kappa\rangle$ is such that each $S_\nu \subseteq \nu$, then

$$\{\mu \in X: \{\nu \in X \cap \mu: S_\nu = \nu \cap S_\mu\} \text{ is } \alpha\text{-subtle}\}$$

is stationary. If α is a limit ordinal, then X is α-*subtle* provided it is β-subtle for each $\beta < \alpha$. The cardinal κ is α-*subtle* if it is α-subtle when considered as a subset of itself.

The following theorem was proven in Schmerl [1976] using combinatorial techniques. However, for a much easier proof which uses models of set theory, see Kaufmann [1983a].

5.2.8 Theorem. *For each $n < \omega$, there is a first-order sentence σ_n such that for each regular, uncountable κ, σ_n has a strongly κ-like model iff κ is not n-subtle.* □

Can this theorem be extended, for example, by finding a sentence σ which has a strongly κ-like model iff κ itself is not ω-subtle? The answer is no because of the following theorem which is the analogue of Theorems 3.3.7 and 5.1.7. A proof of this result will be given in Section 6.

5.2.9 Theorem. *For each n let κ_n be an n-subtle cardinal and $\kappa > \lambda \geq \aleph_0$, where κ is regular. Then $\{\kappa_n: n < \omega\} \underset{s}{\rightarrow} \kappa$ λ-compactly.* □

6. Self-extending Models

Models which have canonical, internal proper elementary extensions of themselves will be considered in this section. By iterating these extensions many times, taking unions at limit stages, we can construct models with particular properties This method will be discussed in Section 6.1 where alternate proofs of Theorems 3.3.7, 5.1.7 and 5.2.9 will be indicated. This technique will be exploited in Subsection 6.2 to prove the MacDowell–Specker–Shelah theorem.

6.1. Self-Extending Theories

Consider the language $\mathscr{L}(Q)$, and consider a consistent theory T in this language which has the following two properties:

(1) *T is a Skolem theory*: For every formula $\phi(x_0, \ldots, x_{n-1}, y)$ there is a term $\ell(x_0, x_1, \ldots, x_{n-1})$ such that

$$\forall \bar{x}(\exists y \phi(\bar{x}, y) \rightarrow \phi(\bar{x}, \ell(\bar{x})))$$

is in T;

(2) *Q behaves as a nonprincipal ultrafilter*: All universal closures of formulas of the following form are in T:

$$Qx\phi(x) \leftrightarrow \neg Qx \neg \phi(x),$$

$$\forall y \neg Qx(x = y),$$

$$(\phi(x) \to \psi(x)) \to (Qx\phi(x) \to Qx\psi(x)),$$

$$Qx\phi(x) \wedge Qx\psi(x) \to Qx(\phi(x) \wedge \psi(x)).$$

A model of T has the form $(\mathfrak{A}, q)$, where q is a collection of subsets of A with the obvious additional clause needed in the definition of satisfaction:

$$(\mathfrak{A}, q) \models Qx\phi(x) \quad \text{iff} \quad \{a \in A : (\mathfrak{A}, q) \models \phi(a)\} \in q.$$

A model $(\mathfrak{A}, q)$ of T is *reduced* if every set in q is definable. Since replacing q by the subset of itself which consists only of definable sets does not alter the satisfaction relation, we can always assume that models of T are reduced.

There is a canonical elementary extension of $(\mathfrak{A}, q)$ which is obtained by a modified ultrapower construction. Let B be the set of definable functions $f: A \to A$ considered modulo q. That is, two definable functions $f, g: A \to A$ are to be considered as equal if $(\mathfrak{A}, q) \models Qx(f(x) = g(x))$. There is a unique reduced structure $(\mathfrak{B}, r)$ such that for any formula $\phi(x_0, \ldots, x_{n-1})$ and all functions $f_0, f_1, \ldots, f_{n-1} \in B$,

$$(\mathfrak{B}, r) \models \phi(f_0, \ldots, f_{n-1}) \quad \text{iff} \quad (\mathfrak{A}, q) \models Qx\phi(f_0(x), \ldots, f_{n-1}(x)).$$

The set r consists of all those sets of the form

$$\{g \in B : (\mathfrak{A}, q) \models Qx\phi(f_0(x), \ldots, f_{n-1}(x), g(x))\},$$

where $(\mathfrak{A}, q) \models QxQy\phi(f_0(x), \ldots, f_{n-1}(x), y)$. The structure $(\mathfrak{B}, r)$ is an elementary extension of $(\mathfrak{A}, q)$ if the elements of A are identified with the constant functions. Thus, the following definition is appropriate.

6.1.1 Definition. A consistent theory T satisfying (1) and (2) above is called a *self-extending* theory.

One important fact about the canonical extensions of models of a self-extending theory is that "large sets become larger." To make this precise, let $\imath: A \to A$ be the identity function so that if $(\mathfrak{B}, r)$ is the canonical extension of the model $(\mathfrak{A}, q)$ of a self-extending theory, then

$$(\mathfrak{B}, r) \models Qx\phi(x, \bar{a}) \to \phi(\imath, \bar{a}),$$

for any formula ϕ and $a_0, a_1, \ldots, a_{n-1} \in A$.

These self-extending theories can be applied to give alternate proofs of Theorems 3.3.7, 5.1.6 and 5.2.9. We state the relevant results in this regard.

6.1.2 Theorem. *Let T be a first-order theory such that, for each $n < \omega$, there are cardinals κ, λ, with $\kappa > \beth_n(\lambda)$, and a (κ, λ)-model of T. Then T can be extended to a self-extending theory which contains all universal closures of formulas of the form*

$$Qx\, \exists y(\phi(x, y) \wedge U(y)) \rightarrow \exists y Qx \phi(x, y). \quad \square$$

6.1.3 Theorem. *Let T be a first-order theory such that, for each $n < \omega$, there is a strongly n-Mahlo cardinal κ and a κ-like model of T. Then T can be extended to a self-extending theory which contains all universal closures of formulas of the form*

$$\forall z Qx\, \exists y(\phi(x, y) \wedge y < z) \rightarrow \exists y Qx \phi(x, y). \quad \square$$

Actually, a theorem which was first proven in Schmerl [1976] and which is slightly stronger than Theorem 5.2.9, will be considered here. In order to state it, we need the following

6.1.4 Definition. Let κ be a regular uncountable cardinal and $X \subseteq \kappa$. A linearly ordered set $(A, <)$ is *(κ, X)-like* if it is κ-like and there is an increasing function $e: X \rightarrow A$ such that whenever $\alpha \in X$ and $\alpha = \sup(\{v \in X: v < \alpha\}) \in X$, then $e(\alpha) = \sup(\{e(v): v \in X \cap \alpha\})$. A structure $\mathfrak{A} = (A, <, \ldots)$ is *(κ, X)-like* if $(A, <)$ is (κ, X)-like.

From this definition we see that $\mathfrak{A}$ is strongly κ-like iff it is (κ, κ)-like.

6.1.5 Theorem. *Suppose κ is a regular uncountable cardinal and T is a first-order theory such that $|T| < \kappa$. Also assume that, for each $n < \omega$, there is a cardinal κ_n and an n-subtle $X \subseteq \kappa_n$ such that that T has a (κ_n, X)-like model. Then T has a strongly κ-like model.*

In order to prove this theorem using self-extending models, we need

6.1.6 Theorem. *Let T be a first-order theory such that for each $n < \omega$ there is a cardinal κ, an n-subtle $X \subseteq \kappa$, and a (κ, X)-like model of T. Then T can be extended to a self-extending theory which contains the universal closures of all formulas of the form*

$$Qx\, \exists y(\phi(x, y) \wedge y < x) \rightarrow \exists y Qx \phi(x, y).$$

To see just how Theorems 6.1.2, 6.1.3 and 6.1.6 imply the corresponding transfer theorems, let us focus attention on Theorem 6.1.6 alone as a typical example. Suppose that T is a first-order theory satisfying the hypothesis of Theorem 6.1.6. Thus, according to that theorem, T can be extended to a self-extending theory T' containing the required sentences. Without loss of generality, we can require that

$|T'| = |T| + \aleph_0$. The sentences in T' imposed by Theorem 6.1.6 guarantee that the canonical extension of any model of T' is an end-extension. Furthermore, this extension has a least new element. Thus, in order to form a strongly λ-like model of T, where $\lambda > |T| + \aleph_0$ is regular, we begin with a model $(\mathfrak{A}_0, q_0)$ of T' with $|A_0| < \lambda$. We then form an increasing chain of models $\langle(\mathfrak{A}_\nu, q_\nu): \nu \leq \lambda\rangle$ by letting $(\mathfrak{A}_{\nu+1}, q_{\nu+1})$ be the canonical extension of $(\mathfrak{A}_\nu, q_\nu)$, and by letting $(\mathfrak{A}_\nu, q_\nu)$ be the union of the previously constructed structures if ν is a limit ordinal. Then $\mathfrak{A}_\lambda$ is a λ-like model of T. In order to see that it is strongly λ-like, we let a_ν be the least new element in the extension $(\mathfrak{A}_{\nu+1}, q_{\nu+1})$ of $(\mathfrak{A}_\nu, q_\nu)$. Thus, $A_\nu = \{x \in A_\lambda: x < a_\nu\}$. Then $\{a_\nu: \nu < \lambda\}$ is a closed subset of A_λ, demonstrating that $(A_\lambda, <)$ is strongly λ-like.

In order to see how to prove Theorems 6.1.2, 6.1.3 and 6.1.6, we will again consider Theorem 6.1.6 as a typical example. Our aim here is to show that T is consistent with some theory, call it T', so by compactness we can assume that T is countable, and then consider some finite $T_0 \subseteq T'$ and show the consistency of just $T \cup T_0$. To this end, we choose an $n < \omega$ which is sufficiently large (depending on T_0) and let $\mathfrak{A}_{n+1}$ be a (κ_{n+1}, X_{n+1})-like model of T, where X_{n+1} is an $(n+1)$-subtle subset of κ_{n+1}. Moreover, let $e: X_{n+1} \to A_{n+1}$ be the function which demonstrates that $(A_{n+1}, <)$ is (κ_{n+1}, X_{n+1})-like. Inductively, we will thus obtain structures $\mathfrak{A}_n, \mathfrak{A}_{n-1}, \ldots, \mathfrak{A}_0$ and $\mathfrak{B}_n, \mathfrak{B}_{n-1}, \ldots, \mathfrak{B}_0$. Each $\mathfrak{A}_i$ will be an expansion of $\mathfrak{B}_i$, and A_i will be an initial segment determined by an element $e(\kappa_i)$, where $\kappa_i \in X_{n+1}$; that is,

$$A_i = \{x \in A_{i+1}: x < e(\kappa_i)\}.$$

In order to get $\mathfrak{A}_n$ and $\mathfrak{B}_n$, let $\{\phi_\nu(v_0): \nu < \kappa_{n+1}\}$ be a nonrepeating list of all formulas with one free variable v_0 in the vocabulary of $\mathfrak{A}_{n+1}$ allowing parameters from A_{n+1}. There is a closed unbounded subset $C \subseteq \kappa_{n+1}$ such that whenever $\alpha \in C \cap X_{n+1}$ and $\phi_\nu(v_0)$ involves only parameters from the set $\{b \in A_{n+1}: b < e(\alpha)\}$, then $\nu < \alpha$. For each $\alpha \in C \cap X_{n+1}$, we let

$$S_\alpha = \{\nu < \alpha: \mathfrak{A}_{n+1} \models \phi_\nu(e(\alpha))\}.$$

We can also assume that if $\alpha \in C \cap X_{n+1}$, then $\mathfrak{A}_{n+1} | \{x \in A_{n+1}: x < e(\alpha)\} \prec \mathfrak{A}_{n+1}$. Using the definition of the subtle hierarchy, we find $\kappa_n \in C \cap X_{n+1}$ such that if

$$X_n = \{\nu \in X_{n+1} \cap \kappa_n: S_\nu = \nu \cap S_{\kappa_n}\},$$

then X_n is an n-subtle subset of κ_n. Let $A_n = \{b \in A_{n+1}: b < e(\kappa_n)\}$, and let $\mathfrak{B}_n = \mathfrak{A}_{n+1} | A_n$ so that $\mathfrak{B}_n \prec \mathfrak{A}_{n+1}$. The important fact to notice here is that, for any $\nu \in X_n$, both $e(\nu)$ and $e(\kappa_n)$ realize the same type over $\{b \in A_{n+1}: b < e(\nu)\}$.

Now let $\mathscr{D}$ be the collection of subsets D which are definable in $\mathfrak{A}_{n+1}$ using only parameters from A_n and for which $e(\kappa_n) \in D$. Now, expand $\mathfrak{B}_n$ to a structure $\mathfrak{B}'_n$ by adjoining a binary relation R_n so that

$$\{\{x \in A_n: \mathfrak{B}'_n \models R_n(b, x)\}: b \in A_n\} = \{D \cap A_n: D \in \mathscr{D}\}.$$

Let $\mathfrak{A}_n$ be the expansion of $\mathfrak{B}'_n$ obtained by adjoining all Skolem functions. The structure $\mathfrak{A}_n$ is (κ_n, X_n)-like for n-subtle $X_n \subseteq \kappa_n$.

The remainder of the $\mathfrak{A}_i$ and $\mathfrak{B}_i$ are constructed in exactly the same fashion. Having finally obtained $\mathfrak{A}_0$, we let $q = \{x \in A_0 : \mathfrak{A}_0 \models R_0(b, x),\ b \in A_0\}$. The structure $(\mathfrak{A}_0, q)$ is clearly a model of T and, without much difficulty, it can be shown to be a model of T_0 also. This demonstrates the consistency of $T \cup T_0$.

6.2. The MacDowell–Specker–Shelah Theorem

Our concern in this subsection is to use self-extending models to prove the following theorem.

6.2.1 Theorem. *If $\aleph_0 \leq \mu < \lambda$, then $\aleph_0 \to \lambda$ μ-compactly.* $\square$

Fuhrken [1965] observed that this theorem is a direct consequence of the well-known theorem of MacDowell and Specker [1961] which asserts that every model of Peano arithmetic has a proper, elementary end-extension. There are two features of Peano arithmetic that are used in the MacDowell–Specker theorem. One is that there is a definable pairing function which allows the coding of finite sequences. The other is that the induction scheme is true in Peano arithmetic, where by the *induction scheme* is meant the sentence

"$<$ is a linear order with a first but no last element"

together with all sentences which are universal closures of formulas of the form

$$[\exists x\phi(x) \wedge \forall x\, \exists y(\phi(x) \to \phi(y) \wedge x < y)] \to \forall x\, \exists y(\phi(y) \wedge x < y).$$

In words, this simply asserts that every nonempty definable set with no largest element is cofinal.

Shelah [1978b] showed that only the induction scheme is necessary. Notice that if we extend a theory which satisfies the induction scheme by adjoining all definable terms, then the extended theory is a Skolem theory. Thus, we will consider such theories to be already Skolem theories.

6.2.2 Theorem. *Let T be a consistent, countable first-order theory which satisfies the induction scheme. To each first-order formula $\phi(x_0, x_1, \ldots, x_{n-1}, y)$ there is associated another first-order formula $\sigma_\phi(x_0, x_1, \ldots, x_{n-1})$ such that T can be extended to a self-extending theory which contains the universal closures of all formulas of the form*

$$Qy\phi(\bar{x}, y) \leftrightarrow \sigma_\phi(\bar{x})$$

and of the form

$$\forall z Qx\, \exists y(\phi(x, y) \wedge y < z) \to \exists y Qx\phi(x, y).$$

Theorem 6.2.1 follows from this theorem. Furthermore, any model of the induction scheme in a countable vocabulary has a proper, elementary end-extension.

Proof. The first step in the proof is to observe that, for each $n < \omega$, there is a $2n$-ary formula $\psi_n(x_0, x_1, \ldots, x_{n-1}, y_0, y_1, \ldots, y_{n-1})$—which we will abbreviate by $\bar{x} <_n \bar{y}$—which defines a linear order on the set of n-tuples and which satisfies the induction scheme. These formulas can be obtained inductively by letting $<_1$ be $<$ and then allowing $\bar{x} <_{n+1} \bar{y}$ to be the formula

$$\begin{aligned}&(\max(x_0, \ldots, x_n) < \max(y_0, \ldots, y_n)) \vee [\max(x_0, \ldots, x_n)\\ &\quad = \max(y_0, \ldots, y_n) \wedge (x_n < y_n \vee (x_n = y_n \wedge ((x_0, \ldots, x_{n-1})\\ &\quad <_n (y_0, \ldots, y_{n-1})))))].\end{aligned}$$

Now consider a sequence $\langle \phi_n(\bar{x}, y) \colon n < \omega \rangle$ of all formulas, where ϕ_n has its free variable among $x_0, x_1, \ldots, x_n, y$. Our object is to find formulas $\sigma_n(\bar{x})$ and at the same time formulas $\theta_n(y)$ such that the following are all consequences of T:

$$\forall w\, \exists y > w\theta_n(y),$$

$$\theta_{n+1}(y) \rightarrow \theta_n(y),$$

$$\exists w\, \forall y > w(\theta_n(y) \rightarrow (\phi_n(\bar{x}_1 y) \leftrightarrow \sigma_n(\bar{x})).$$

We will proceed by induction on n. For convenience, we will let $\theta_{-1}(y)$ be $y = y$. Having $\theta_{n-1}(y)$ and $\sigma_n(\bar{x})$, we easily find an appropriate $\theta_n(y)$. For example, let $\theta_n(y)$ be

$$\begin{aligned}&\theta_{n-1}(y) \wedge \exists z < y[\forall \bar{x} <_{n+1} z^{n+1}(\sigma_n(\bar{x}) \leftrightarrow \phi_n(\bar{x}, y)\\ &\quad \wedge \forall w((\forall \bar{x} <_{n+1} z^{n+1}(\sigma_n(\bar{x}) \leftrightarrow \phi_n(\bar{x}, w) \wedge \theta_{n-1}(w))\\ &\quad \rightarrow w \le z \vee y \le z))],\end{aligned}$$

where by z^{n+1} is meant the $(n+1)$-tuple $(z, z, \ldots, z)$.

We have now reached the crux of the proof: To define $\sigma_n(\bar{x})$, knowing $\theta_n(y)$. Let $E(\bar{x}, y, z)$ be the formula

$$\forall \bar{w} <_{n+1} \bar{x}(\phi_n(\bar{w}, y) \leftrightarrow \phi_n(\bar{w}, z)).$$

For fixed $\bar{x}$, the formula $E(\bar{x}, y, z)$ defines an equivalence relation with only "boundedly" many equivalence classes. As $\bar{x}$ gets larger (in the sense of $<_{n+1}$), then the corresponding equivalence relation gets finer. Thus, the formula $E(\bar{x}, y, z)$ can be viewed as defining a tree, the nodes of rank $\bar{x}$ being the equivalence classes of the equivalence relation corresponding to $\bar{x}$. For each rank $\bar{x}$, there is an equivalence class containing an unbounded set of elements all of which satisfy θ_n. Call such an equivalence class *large*. Then, the following formula $L(\bar{x}, y, z)$ will assist us in selecting a canonical large equivalence class of each rank:

$$\exists \bar{w} <_{n+1} \bar{x}(\neg \phi_n(\bar{w}, y) \wedge \phi_n(\bar{w}, z) \wedge E(\bar{w}, y, z)).$$

The formula $L(\bar{x}, y, z)$ linearly orders the equivalence classes of rank $\bar{x}$. Thus, we let $S(\bar{x}, y)$ be a formula selecting the first large one. Thus, let $S(\bar{x}, y)$ be

$$\begin{aligned} &\forall w\, \exists z > w(\theta_n(z) \wedge E(\bar{x}, y, z)) \wedge \forall v(L(\bar{x}, v, y) \\ &\qquad \to \exists w\, \forall z > w(\theta_n(z) \to \neg E(\bar{x}, y', z))). \end{aligned}$$

The large classes selected in this way form a branch. That is, T implies $\bar{w} <_{n+1} \bar{x} \wedge S(\bar{x}, y) \to S(\bar{w}, y)$. It is now evident that $\sigma_n(\bar{x})$ should be $\forall \bar{w}\, \exists y(S(\bar{w}, y) \wedge \phi_n(\bar{x}, y))$. □

7. Final Remarks

The final section of this chapter mentions some results which would have been discussed in more detail had space allowed.

7.1. Other Logics

The logic of Magidor and Malitz [1977a] can be given cardinality interpretations other than the $\aleph_1$-interpretation discussed in Section IV.5. The logic $\mathscr{L}(Q, Q^2, Q^3, \ldots)$ which uses the $\aleph_\alpha$-interpretation is denoted by $\mathscr{L}(Q_\alpha, Q_\alpha^2, Q_\alpha^3, \ldots)$. The Magidor-Malitz completeness theorem (see Section IV.5.2) also proves the following transfer theorem.

7.1.1 Theorem. *Assume $\diamondsuit$. If $\kappa = \aleph_\alpha$ is regular, then $\mathscr{L}(Q_\alpha, Q_\alpha^2, Q_\alpha^3, \ldots) \to \mathscr{L}(Q_1, Q_1^2, Q_1^3, \ldots)$ $\aleph_0$-compactly.* □

A converse of the previous transfer theorem has been proven by Shelah [1980].

7.1.2 Theorem. *Assume $\diamondsuit_{\aleph_\alpha}$ and $\diamondsuit_{\aleph_{\alpha+1}}$. Then*

$$\mathscr{L}(Q_1, Q_1^2, Q_1^3, \ldots) \to \mathscr{L}(Q_{\alpha+1}, Q_{\alpha+1}^2, Q_{\alpha+1}^3, \ldots)$$

$\aleph_\alpha$-compactly. □

Theorems 7.1.1 and 7.1.2 together with the the Magidor-Malitz completeness theorem imply that $\mathscr{L}(Q_\alpha, Q_\alpha^2, Q_\alpha^3, \ldots)$ is recursively enumerable for validity under the appropriate hypothesis on $\aleph_\alpha$.

The cofinality quantifier (see Section II.2.4) yields a logic which is fully compact. We denote the quantifier by Q^{cf}, and for regular cardinal κ, its κ-interpretation is defined so that $Q^{\mathrm{cf}}xy\varphi(x, y)$ holds iff $\varphi(x, y)$ defines a linear order with cofinality κ. The logic with this quantifier with the $\aleph_\alpha$-interpretation is denoted by $\mathscr{L}(Q_\alpha^{\mathrm{cf}})$. A proof of the following transfer theorem can be found in Makowsky–Shelah [1981].

7.1.3 Theorem. *Let $\aleph_\alpha$ and $\aleph_\beta$ be regular cardinals. Then $\mathscr{L}(Q_\alpha^{\mathrm{cf}}) \to \mathscr{L}(Q_\beta^{\mathrm{cf}})$ λ-compactly for any cardinal λ.* □

Consequently, $\mathscr{L}(Q_0^{\mathrm{cf}})$ is fully compact. The proof also yields that $\mathscr{L}(Q_0^{\mathrm{cf}})$ is recursively enumerable for validity.

7.2. Infinitary Languages

Some of the transfer theorems we have discussed have extensions to infinitary languages. For example, the proof of Keisler [1966b] of Theorem 2.1.3 yields an $\mathscr{L}_{\omega_1, \omega}$ version.

7.2.1 Theorem. *If $\aleph_\alpha$ is regular, then $\mathscr{L}_{\omega_1,\omega}(Q_\alpha) \to \mathscr{L}_{\omega_1,\omega}(Q_1)$.* □

Some theorems of Section 5 also have infinitary versions which can be proven by the techniques of that section or those of Section 6. The reader should refer to Definition II.5.2.1 for the notion of the well-ordering number $w(\mathscr{L})$ of a logic and to Chapter VIII for $\mathscr{L}_A$, where A is an admissible set. If A is countable, then $w(\mathscr{L}_A) = A \cap \mathrm{Ord}$.

7.2.2 Theorem. *Let A be an admissible set and φ a sentence of $\mathscr{L}_A$.*

(1) *Suppose that for each $\alpha < w(\mathscr{L}_A)$, there is a strongly α-Mahlo cardinal κ and a κ-like model of φ. Then, for each $\lambda > |A|$, φ has a λ-like model.*
(2) *Suppose that, for each $\alpha < w(\mathscr{L}_A)$, there is an α-subtle cardinal κ and a strongly κ-like model of φ. Then, for each $\lambda > |A|$, φ has a strongly λ-like model.* □

Similarly, the Hanf numbers of admissible fragments can be computed.

7.2.3 Theorem. *Let A be admissible and $\omega < \alpha = w(\mathscr{L}_A)$. Then $h(\mathscr{L}_A) = \beth_\alpha$.* □