Part B

Finitary Languages with Additional Quantifiers

Part B of the book is devoted to the study of logics with added quantifiers and the applications of such. The logics considered, for the most part, express properties of ordinary structures. Logics with additional quantifiers based on richer structures are studied in Part E.

Chapter IV begins the discussion by investigating the logic $\mathscr{L}(Q_1)$ with the quantifier "there exist uncountably many." It also discusses various extensions of $\mathscr{L}(Q_1)$ including stationary logic $\mathscr{L}(\mathrm{aa})$ and the Magidor–Malitz logic $\mathscr{L}^{<\omega}$. The primary emphasis of the chapter is on the method of constructing models of size $\aleph_1$ used by Keisler [1970] to prove his completeness theorem for $\mathscr{L}(Q_1)$, a method that has become one of the standard tools of the subject. Each of these logics comes with its own intended concepts of "small" set and "large" set. The basic idea of Keisler-type proofs is to use an elementary chain $\langle A_\alpha : \alpha < \omega_1 \rangle$ of countable non-standard or "weak" models to build a standard model, one where the quantifier has its intended interpretation. The key step is always from A_α to $A_{\alpha+1}$, constructing $A_{\alpha+1}$ so that all small definable subsets of A_α stay fixed, but where a fixed definable subset of A_α that is supposed to be large receives a new element.

Chapter V discusses the general problem of transferring results known about $\mathscr{L}(Q_\alpha)$ to some other $\mathscr{L}(Q_\beta)$, especially the problem of taking results known about $\mathscr{L}(Q_1)$, where we have powerful techniques for building models, to $\mathscr{L}(Q_{\beta+1})$ for larger β. For example, if we assume the Generalized Continuum Hypothesis, it follows that the axioms and rules that are complete for $\mathscr{L}(Q_1)$ are also complete for any logic of the form $\mathscr{L}(Q_{\beta+1})$, as long as $\aleph_\beta$ is regular. In general, this chapter depends heavily on various set-theoretical assumptions which are independent of the usual axioms of set theory, however.

Chapter VI surveys and compares the strength of a host of other logics with additional quantifiers. One of these is the class of partially ordered quantifiers like Q^{H} whose meaning is given by: $Q^{\mathrm{H}}x, y; z, w\phi(x, y, z, w)$ is true just in case for every x there is a y, and for every z there is a w, such that y depends only on x, w only on z, such that $\phi(x, y, z, w)$. Quantifiers of this kind are called partially ordered because they are often written:

$$\begin{matrix} \forall x\, \exists y \\ \forall z\, \exists w \end{matrix}\ \phi(x, y, z, w).$$

Some other quantifiers discussed in Chapter VI include:

- the Hartig quantifier I, defined so that $Ix, y[\phi(x), \psi(y)]$ means that the number of ϕ's is the same as the number of ψ's;
- the similarity quantifier S, defined so that $Sx, x'[\phi(x), \psi(x')]$ means that the substructures defined by ϕ and ψ are isomorphic; and
- the well-ordering quantifier W, defined so that $Wx, y\phi(x, y)$ means that ϕ defines a well-ordering.

The relative strengths of these logics, and their Δ-closures are discussed. For example, it is shown that $\Delta(\mathscr{L}(Q^{\mathrm{H}})) = \Delta(\mathscr{L}(S)) = \Delta(\mathscr{L}^{\mathrm{mII}})$, where $\mathscr{L}^{\mathrm{mII}}$ is monadic second-order logic. Under the assumption of the axiom of constructibility, it is also shown that $\Delta(\mathscr{L}(I)) = \Delta(\mathscr{L}^{\mathrm{mII}})$.

Chapter VII is devoted to identifying decidable and undecidable theories in logics with generalized quantifiers, especially $\mathscr{L}(Q_1)$, the Magidor–Malitz logic $\mathscr{L}^{<\omega}$, logic $\mathscr{L}(I)$ with the Hartig quantifier, and stationary logic $\mathscr{L}(\mathrm{aa})$. The chapter is organized around three main methods of proof, quantifier elimination, the method of interpretations, and the use of "dense systems." These are all well-known methods from first-order logic which have interesting extensions to stronger logics. The mathematical theories discussed include abelian groups and modules, orderings, and boolean algebras. This chapter leads into a rich literature on the decidability of theories with extra quantifiers.

Chapter IV

The Quantifier "There Exist Uncountably Many" and Some of Its Relatives

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The idea of adding quantifiers to first-order logic goes back at least to Mostowski [1957]. Fuhrken [1964] and Vaught [1964] were the first investigators to prove compactness and (abstract) completeness theorems for such a logic, namely the logic $\mathscr{L}(Q_1)$ obtained by adjoining the quantifier Q_1 (there exist uncountably many) to first-order logic. The first systematic study of $\mathscr{L}(Q_1)$ and, in fact, of any well-behaved logic obtained by adding a quantifier to first-order logic, appeared in Keisler's 1970 paper. By giving the completeness of a simple *explicit* set of axioms for $\mathscr{L}(Q_1)$, along with other nice features of a logic such as an omitting types theorem (with applications), Keisler's work encouraged the further study of $\mathscr{L}(Q_1)$ as well as the search for extensions of $\mathscr{L}(Q_1)$ that retain some of the nice properties of first-order logic. In this chapter we will present some of the progress in this study.

A main focus of this chapter is on the development of methods of proving completeness theorems for logics extending $\mathscr{L}(Q_1)$. (Such an approach allows compactness theorems to be derived as corollaries.) In Section 3, the proof of Keisler's concrete completeness theorem for $\mathscr{L}(Q_1)$ leads to new methods of constructing models and to a version of the omitting types theorem which differs a bit from the first-order version, and which leads to a completeness theorem for the corresponding infinitary version of $\mathscr{L}(Q_1)$. These methods, and the resulting intuition developed for $\mathscr{L}(Q_1)$, make possible the completeness proofs for the other logics that are examined in Sections 4 and 5. Although concrete completeness is a desirable feature of a logic, our main purpose here is to present the *methods* that go into the proofs of such theorems.

The basic plan for proving each of these completeness theorems is to reduce the given logic to first-order logic in some manner so that familiar tools from first-order model theory may then be applied. One such reduction is used in Section II.3 to prove that the set of validities for $\mathscr{L}(Q_1)$ is r.e. in the vocabulary; another reduction—one that is due to Fuhrken—is given in Section 1.1 below. However, in order to prove a *concrete* completeness theorem, we need a reduction that is somehow more closely tied to the logic. The notion of *weak model* is thus developed for this purpose in Section 2 although some of the details are relegated to the appendix. The general approach adopted in Section 2 enables us to give a reasonably unified treatment of the completeness theorems in Sections 3, 4, and 5.

In Section 6 we conclude our study with an investigation of interpolation and definability questions for various extensions of $\mathscr{L}(Q_1)$. The interest in these questions is largely due to the use of a variety of back-and-forth arguments for proving $\mathscr{L}$-equivalence (for various logics $\mathscr{L}$), although the original motivation was largely due to the search for well-behaved extensions of first-order logic. Several of the proofs given in Section 6 elaborate the basic model-theoretic practice of showing that certain partial isomorphisms preserve elementary equivalence.

This chapter is essentially self-contained, its only prerequisite being a reasonable familiarity with first-order model theory.

1. *Introduction to* $\mathscr{L}(Q_\alpha)$

Probably the simplest quantifiers which are stronger than $\exists$ and $\forall$ are the cardinality quantifiers Q_α, "there exist at least $\aleph_\alpha$" defined in Section II.2.2. When $\alpha = 1$, the subscript on Q will be omitted. In this case, Q asserts that "there exist uncountably many." The notation $\mathscr{L}(Q_\alpha)(\tau)$ denotes the set of $\mathscr{L}(Q_\alpha)$-formulas of the vocabulary τ. In the present chapter, however, we will rarely consider the case $\alpha > 1$, since it comprises part of Chapter V.

Of course, $\mathscr{L}(Q_\alpha)$ is strictly stronger than first-order logic. For example, the sentence $Q_\alpha x(x = x) \wedge \forall x \neg Q_\alpha y(y < x)$ holds in a linear order if and only if that order is $\aleph_\alpha$-like. Examples of the expressive power of $\mathscr{L}(Q_\alpha)$ tend to be rather obvious. In order to express more interesting notions in the logic, we must extend $\mathscr{L}(Q_\alpha)$. This is done in Sections 4 and 5.

As is shown in Section II.3, $\mathscr{L}(Q_1)$ is countably compact (compact for countable theories), a fact which we will again prove in this chapter, in Section 3. However, our method and emphasis are somewhat different from the one in Section II.3, as was explained in the introduction above. For now, we will begin our work by discussing the incompactness of $\mathscr{L}(Q_0)$ in subsection 1.1 and then examine some Löwenheim–Skolem properties of $\mathscr{L}(Q_\alpha)$ in Section 1.3, giving also a brief outline (with comments) of Fuhrken's original compactness proof for $\mathscr{L}(Q_1)$ in Section 1.2.

1.1. *Incompactness of* $\mathscr{L}(Q_0)$

The following finite theory T has only one model (up to isomorphism), namely $(\omega, <)$: $T = \{\forall x \neg Q_0 y(y < x)$, "$<$ is a linear order without last element"$\}$. It follows then that $\mathscr{L}(Q_0)$ is not countably compact. Moreover, the set of valid sentences of $\mathscr{L}(Q_0)(\tau)$ is not recursively enumerable (it is actually complete Π_1^1) if τ contains a binary relation symbol. In fact, Barwise [1974] has shown that the Δ-closure of $\mathscr{L}(Q_0)$ is equivalent to $\mathscr{L}_{\infty\omega} \cap \mathscr{L}_{\omega_1^{CK}}$ (see Section II.7.2), the latter being the hyperarithmetic fragment of $\mathscr{L}_{\infty\omega}$ (see also Theorems VI.2.3.3 and XVII.3.2.2).

Since most of the emphasis in this chapter is on logics that are countably compact, we will now turn to $\mathscr{L}(Q_i)$.

1.2. On Completeness and Compactness of $\mathscr{L}(Q_1)$

Mostowski [1957] asked whether $\mathscr{L}(Q_1)$ has a recursively enumerable set of validities. The chief result in this direction was Vaught's two-cardinal theorem (see Morley–Vaught [1962]), or, perhaps more accurately, the proof of the theorem. To be precise, Fuhrken discerned that $\mathscr{L}(Q_1)$ is countably compact by abstracting the following lemma from the proof of Vaught's theorem.

1.2.1 Lemma (Fuhrken [1964; 1.7]). *Suppose that T is a set of (first-order) sentences in a countable vocabulary τ which contains a unary relation symbol U. Let W be a new unary predicate symbol, and let Δ be the set of all sentences*

$$\forall v_0 \ldots \forall v_{n-1}[W(v_0) \wedge \cdots \wedge W(v_{n-1}) \rightarrow [\phi \leftrightarrow \phi^W]],$$

where ϕ is any τ-formula having only $v_0, \ldots, v_{n-1}$ as free variables, and ϕ^W is obtained from ϕ by relativizing all quantifiers to W. That is, W defines an elementary submodel of the universe. Then the following are equivalent:

(i) *$T \cup \Delta \cup \{\forall x(U(x) \rightarrow W(x)), \exists x \neg W(x)\}$ is consistent;*
(ii) *T has a model $\mathfrak{A}$ for which $|U^{\mathfrak{A}}| < |A| = \aleph_1$;*
(iii) *T has a model $\mathfrak{A}$ for which $|U^{\mathfrak{A}}| < |A|$.* □

A proof of this result is carefully worked out in Chang–Keisler [1973; §3.2, especially 3.2.12]. We will now examine the two relevant corollaries of this lemma, discussing their proofs in 1.2.4.

1.2.2 Corollary (Fuhrken [1964; Theorem 3.4]). *$\mathscr{L}(Q)$ is countably compact.* □

1.2.3 Corollary (Vaught [1964]). *For countable τ, the set of valid sentences of $\mathscr{L}(Q_1)(\tau)$ is recursively enumerable in τ. In fact, $\mathscr{L}(Q_1)$ is recursively enumerable for consequence (in the sense of Definition* II.1.2.4). □

1.2.4 Idea of Proofs of Corollaries 1.2.2 and 1.2.3. These corollaries both follow from Fuhrken [1964, Theorem 2.2]. The idea is that one can replace $\neg Qx\phi(x, \mathbf{y})$ by a statement asserting that there is a function mapping $\{x: \phi(x, \mathbf{y})\}$ one–one into U; and that one can replace $Qx\phi(x, \mathbf{y})$ by a statement asserting that there is a one–one function from the universe of the model into $\{x: \phi(x, \mathbf{y})\}$. The details of how this may be accomplished can be found in Fuhrken [1964]. However, the result is that questions about satisfiability of an $\mathscr{L}(Q_{\beta+1})$ theory Σ may be reduced to the satisfiability of a corresponding $\mathscr{L}_{\omega\omega}$ theory Σ^* in a model $\mathfrak{A}$ with $U^{\mathfrak{A}} \leq \aleph_\beta < |A|$. Setting $\beta = 0$ gives the corollaries. These ideas were expanded in Keisler [1966a] in giving an axiomatization of 2-cardinal models. The reader should also see Section V.1 for more about the method of reduction.

Comparison of Completeness Proofs and the Related Literature. As we have pointed out, Fuhrken's Lemma (1.2.1) is based largely on the proof of Vaught's

2-cardinal theorem. That is generally proved by using homogeneous models to build an appropriate elementary chain. However, the proof of Keisler's completeness theorem (see Section 3.2, also Section II.3.2) is based on the proof of Keisler's 2-cardinal theorem. That is, homogeneous models are replaced by an omitting types argument. The latter technique is what really enables Keisler to give an *explicit* set of axioms for $\mathscr{L}(Q_1)$, and to prove an omitting types theorem for $\mathscr{L}(Q_1)$. The reader should see Section 3 for more on this.

It is also interesting to compare the method of Section II.3.2 (and also of Section 3.2) to that used for the MacDowell–Specker theorem for models of arithmetic. The latter asserts that every model of Peano arithmetic (even if it is uncountable) has an elementary end extension. (See Section V.7 for a related result.) The former is more closely related to the methods used to prove an analogous theorem for models of set theory, Theorem 3.2.5 below (Keisler–Morley [1968]). The Keisler–Morley theorem does not hold for all uncountable models. However, the fact that it requires the collection schema, rather than the (stronger) induction schema does speak in its favor. The connection between the Keisler–Morley theorem and Keisler's $\mathscr{L}(Q)$ completeness theorem is made somewhat more explicit in the proof of Theorem 3.2.5 given below (the Keisler–Morley theorem), which uses the Main Lemma (3.2.1) from the proof of completeness of $\mathscr{L}(Q_1)$.

1.3. Observations on $\mathscr{L}(Q_\alpha)$

We will close this introduction by making some easy observations about $\mathscr{L}(Q_\alpha)$. The first was noticed by Mostowski, and it generalizes easily to the $\aleph_\alpha$-interpretation of $\mathscr{L}^{<\omega}$ (see Definition 5.1.3).

Before we examine the argument for this result, we should make a comment on the notation and notions involved. By $\mathfrak{B} \prec_{\mathscr{L}(Q_\alpha)} \mathfrak{A}$ we mean that $\mathfrak{B} \prec \mathfrak{A}$ and that both $\mathfrak{B}$ and $\mathfrak{A}$ satisfy the same $\mathscr{L}(Q_\alpha)$ formulas at any assignment of $\mathfrak{B}$. These ideas clear, we now turn to

1.3.1 Proposition. *If $\mathfrak{A}$ is any model, then there exists $\mathfrak{B} \prec_{\mathscr{L}(Q_\alpha)} \mathfrak{A}$ such that $|B| \leq \aleph_\alpha$.*

Sketch of Proof. For $\alpha = \omega_1$, the result follows from Fuhrken's normal form (see subsection 1.2.4) together with Lemma 1.2.1, if we only require $\mathfrak{B} \equiv_{\mathscr{L}(Q_\alpha)} \mathfrak{A}$. However, the more general statement has an even easier direct proof. Assuming that $|A| > \aleph_\alpha$ (for otherwise, the argument is done), the usual proof of the downward Löwenheim–Skolem theorem can be easily modified to provide $\aleph_\alpha$ witnesses to each $Qx\phi$ instead of only one. □

On the other hand, as we will now show, the upward Löwenheim–Skolem property clearly fails. (The reader should consult Theorem II.6.1.6 and V.4.2.3 for theorems on Hanf numbers.)

1.3.2. Proposition. *For each of the conditions* (i) *through* (iv) *below, there is a sentence ϕ of $\mathscr{L}(Q)$ such that for all α and β: ϕ has a model of power $\aleph_\beta$ in the α-interpretation* (*that is, considering ϕ as a sentence of $\mathscr{L}(Q_\alpha)$*) *iff that condition holds.*

(i) $\beta < \alpha$.
(ii) $\beta = \alpha$.
(iii) $\beta \leq \alpha + n$, *for any* $n < \omega$.
(iv) $\aleph_\beta \leq \beth_n(\aleph_\alpha)$, *for any* $n < \omega$, *where* $\beth_0(\alpha) = \alpha$ *and* $\beth_{n+1}(\alpha) = 2^{\beth_n(\alpha)}$).

Hence, full compactness fails for all $\mathscr{L}(Q_\alpha)$.

Proof. (i) ϕ is, of course, simply $\neg Q_\alpha x(x = x)$. Thus, it follows that compactness fails for $\mathscr{L}(Q_\alpha)$: Consider the set $\{\neg Q_\alpha x(x = x)\} \cup \{c_\beta \neq c_\gamma : \beta < \gamma < \aleph_\alpha\}$.

(ii) ϕ says that $\leq$ is a (reflexive) $\aleph_\alpha$-like linear order: "$\leq$ is a linear order" $\bigwedge \forall x \neg Q_\alpha y(y \leq x) \wedge Q_\alpha x(x = x)$.

(iii) Here, such a sentence ϕ_n can be constructed by induction on n. Thus, ϕ_0 is "$\leq$ is a linear order" $\bigwedge \forall x \neg Q_\alpha y(y \leq x)$, while ϕ_{n+1} says "$<$ is a linear order and every proper initial segment can be expanded to a model of ϕ_n."

(iv) We assume that $n \geq 1$ (for, in the absence of this assumption, (iii) clearly applies). Thus, the language of ϕ includes $<$, $P_0, P_1, \ldots, P_n$, and ε. And, that much being so, we assert that each P_{i+1} is contained in the power set of P_i. (See also Theorem II.6.1.6.) ∎

This contrasts with Theorem 8 of Yasuhara [1966], which gives full compactness when one removes $=$, $\exists$, and $\forall$ from $\mathscr{L}(Q_\alpha)$, $\alpha \geq 1$.

2. *A Framework for Reducing to First-Order Logic*

Our goal in this section is to provide some means of reducing a given logic to first-order logic in order that we may develop some model theory for $\mathscr{L}(Q)$ and some of its extensions in Sections 3, 4, and 5. As we will see, when we transform a given logic into first-order logic in some manner—say, by enlarging the vocabulary—we may apply methods of first-order model theory to obtain results about the given logic. The reduction given here works for any logic that possesses some basic syntactic properties, "concrete syntax". Our notion of "concrete syntax" is neither memorable nor worthy of study in its own right. Indeed, every reasonable logic probably has this property in some sense. However, it is a notion which will enable us to prove theorems about so-called weak models, and these, in turn, will enable us to carry out the more interesting model constructions later on. In fact, we will omit the precise definition of "concrete syntax" here as well as most proofs. These are, however, included in Section 7 (the appendix) where they may be safely ignored. The reader might want to read this section with $\mathscr{L}(Q)$ in mind.

Keisler's notion of weak model is presented in Section 2.3, where it is related to the notion given here in Definition 2.1.3. That done we will then briefly touch on the logic of monotone structures.

2.1. Logics With Concrete Syntax and Weak Models

A precise definition of *concrete syntax* can be found in Definition 7.1.1. For present purposes, it suffices to say that the properties include:

- closure under $\neg$, $\vee$, $\exists$;
- possession of a notion $\vdash_{\mathscr{L}}$ of finitary proof, with a deduction theorem;
- existence of a rank function $r(\phi)$ which measures the complexity of ϕ in a reasonable way;
- existence of a function $\mathrm{frvar}(\phi)$ which gives the set of free variables of each formula ϕ, as well as a notion of substitution $\phi(f)$ for any function $f: \mathrm{frvar}(\phi) \to C$, for some set C of constants.

These properties are sufficient (when stated precisely) to prove the deduction theorem in the usual way, as in Enderton [1972].

2.1.1 Theorem (Deduction Theorem). $\Gamma \cup \{\phi\} \vdash_{\mathscr{L}(\tau)} \psi$ *iff* $\Gamma \vdash_{\mathscr{L}(\tau)} \phi \to \psi$. ☐

Any logic with concrete syntax can be transformed into first-order logic by using extra relation symbols and "weak models" as follows in

2.1.2 Definition. Let $\mathscr{L}$ be a logic with concrete syntax. We define a map $\phi \mapsto \phi^*$ which sends $\mathscr{L}(\tau)$-formulas to $\mathscr{L}_{\omega\omega}(\tau^+)$-formulas, where $\tau^+ = \tau \cup \{R_\phi : \phi$ is an $\mathscr{L}(\tau)$-formula, neither atomic nor of the form $\neg\psi$, $\psi_1 \vee \psi_2$, or $\exists x\psi\}$. The arity of R_ϕ is $|\mathrm{frvar}(\phi)|$. The definition is by recursion on rank $r(\phi)$. If ϕ is atomic, set $\phi^* = \phi$. Also, set $(\neg\psi)^* = \neg(\psi^*)$, $(\psi_1 \vee \psi_2)^* = \psi_1^* \vee \psi_2^*$, and $(\exists x\psi)^* = \exists x(\psi^*)$. If ϕ is neither atomic nor of the form $\neg\psi$, $\psi_1 \vee \psi_2$, nor $\exists x\psi$, and if $\mathrm{frvar}(\phi) = \{v_{i_1}, \ldots, v_{i_n}\}$ with $i_1 < \cdots < i_n$, then set $\phi^* = R_\phi(v_{i_1}, \ldots, v_{i_n})$.

2.1.3 Definition (Weak Models). *A weak model for* a logic $\mathscr{L}$ with concrete syntax is a τ^+-structure $\mathfrak{A}^* = \langle \mathfrak{A}, R_\phi^{\mathfrak{A}^*} \rangle_{\phi \in \mathscr{L}(\tau)}$, for some τ, which satisfies every instance of ϕ^* for every $\vdash_{\mathscr{L}}$-axiom ϕ in $\mathscr{L}(\tau)$. For ϕ any formula of $\mathscr{L}(\tau)$, we write $\mathfrak{A}^* \models \phi[s]$ to denote $\mathfrak{A}^* \models \phi^*[s]$. Since "$*$" commutes with $\neg$, $\vee$, and $\exists$, "$\models$" obeys the usual inductive clauses for first-order satisfaction.

For weak models $\mathfrak{A}^*$ and $\mathfrak{B}^*$ of vocabulary τ^+, we write $\mathfrak{A}^* \prec^w \mathfrak{B}^*$ if $A \subseteq B$ and for all assignments s into A and all $\phi \in \mathscr{L}(\tau)$, $\mathfrak{A}^* \models \phi[s]$ iff $\mathfrak{B}^* \models \phi[s]$. Notice that this is weaker than $\mathfrak{A}^* \prec \mathfrak{B}^*$, since we restrict ourselves to formulas of the form ϕ^*.

2.2. Some Weak Model Theory

In this discussion we will present completeness (and related) theorems for weak models. The proofs, although routine, are given in the appendix. Throughout this section we assume that $\mathscr{L}$ has concrete syntax.

2.2.1 Proposition (Soundness). *Let $\mathfrak{A}^*$ be a weak model for $\mathscr{L}(\tau)$, and suppose that ϕ is an $\mathscr{L}(\tau)$-formula and $f\colon X \to C$, for some one-one function f, some $X \subseteq \mathrm{frvar}(\phi)$, and some set C of constants which is disjoint from τ. If $\vdash_{\mathscr{L}(\tau \cup C)} \phi(f)$ then for all $s\colon \mathrm{frvar}(\phi) \to A$, $\mathfrak{A}^* \models \phi[s]$.* $\square$

2.2.2 Proposition (Elementary Chain Theorem). *Let $\mathfrak{A}^*_\alpha$ be a τ^+_α-structure for all $\alpha < \gamma$, where $\alpha < \beta$ implies that $\tau_\alpha \subseteq \tau_\beta$ and $\mathfrak{A}^*_\alpha \prec^w \mathfrak{A}^*_\beta \restriction \tau^+_\alpha$. Let $\mathfrak{A}^*$ be the union of $\{\mathfrak{A}^*_\alpha\colon \alpha < \gamma\}$, that is, $\mathfrak{A}^*$ is a $(\bigcup_{\alpha<\gamma} \tau^+_\alpha)$-structure and for all $\alpha < \gamma$, $\mathfrak{A}^* \restriction \tau^+_\alpha = \bigcup_{\beta \in \gamma - \alpha} \mathfrak{A}^*_\beta$. Then for all $\alpha < \gamma$, $\mathfrak{A}^*_\alpha \prec^w \mathfrak{A}^*_\gamma \restriction \tau^+_\alpha$.* $\square$

2.2.3 Theorem (Weak Completeness). *Let T be an $\mathscr{L}(\tau)$-consistent set of $\mathscr{L}(\tau)$ sentences, where τ is countable. Then T has a countable weak model, that is, there is a countable weak model $\mathfrak{A}^*$ for $L(\tau)$ such that $\mathfrak{A}^* \models \phi$ for all $\phi \in T$.*

The following extension of the weak completeness theorem will also be useful. First, however, we need a related definition which, in applications, will be equivalent to a more familiar condition.

2.2.4 Definition. Let T be an $\mathscr{L}(\tau)$-consistent set of $\mathscr{L}(\tau)$-sentences. Also let Σ be a set of $\mathscr{L}(\tau)$-formulas such that $\mathrm{frvar}(\sigma) \subseteq \mathbf{x}$ for all $\sigma \in \Sigma$; then we write $\mathrm{frvar}(\Sigma) \subseteq \mathbf{x}$. T is said to *$\mathscr{L}(\tau)$-locally omit* Σ, if for every finite set C of constant symbols, every $\mathscr{L}(\tau \cup C)$-sentence ϕ which is $\mathscr{L}(\tau \cup C)$-consistent with T, and every function f mapping $\mathbf{x}$ into the set C, there exists $\sigma \in \Sigma$ such that $\phi \wedge [\neg\sigma(f)]$ is $\mathscr{L}(\tau \cup C)$-consistent with T. Notice that range (f) may include constants of ϕ.

2.2.5 Weak Omitting Types Theorem. *Let T be an $\mathscr{L}(\tau)$-consistent set of $\mathscr{L}(\tau)$-sentences, where τ is countable. Also let $\{\Sigma_n\colon n < \omega\}$ be a family of countable sets of $\mathscr{L}(\tau)$-formulas with $\mathrm{frvar}(\Sigma_n) \subseteq \mathbf{x}_n$. If T $\mathscr{L}(\tau)$-locally omits Σ_n for all $n < \omega$, then T has a countable weak model omitting each Σ_n, that is, which satisfies $\bigwedge_{n<\omega} \forall \mathbf{x}_n \bigvee \{\neg\sigma\colon \sigma \in \Sigma_n\}$.* $\square$

The following technical lemma is used in Sections 3, 4, and 5, to extend weak models while omitting types. The exact statement can be found as Lemma 7.2.3; for the present, we will use this slightly imprecise but considerably more readable statement of it.

2.2.6 Lemma (Extension Lemma). *Suppose $\mathfrak{A}^*$ is a countable weak model for $\mathscr{L}(\tau)$, where τ is countable. Also let T be any consistent countable extension of the elementary diagram of $\mathfrak{A}^*$ which $\mathscr{L}(\tau)$-locally omits sets $\Sigma_n(x_n)$, each $n < \omega$. Then there exists a weak model $\mathfrak{B}^*$ of T which omits each set Σ_n, such that $\mathfrak{A}^* \prec \mathfrak{B}^* \restriction \tau^+$.*

$\square$

2.3. Connections With Monotone Structures

We will conclude this section by relating the notion of weak model as given by Keisler [1970] (and studied later by others: see Definition 2.3.3) to the notion given above. Keisler considered structures $(\mathfrak{A}, q)$, where $q \subseteq \mathscr{P}(A)$, and inductively defined satisfaction for $\mathscr{L}(Q)$ formulas in such models with the new clause

$$(\mathfrak{A}, q) \models Qx\phi[s] \quad \text{iff} \quad \{a \in A: (\mathfrak{A}, q) \models \phi[s(x, a)]\} \in q.$$

Here, $s(x, a)$ denotes $[s \restriction (\mathrm{dom}(s) - \{x\})] \cup \{\langle x, a\rangle\}$.

2.3.1 Definition. $\mathscr{L}^0(Q)$ is the logic with concrete syntax with the usual notions of substitution, frvar(ϕ), and $r(\phi)$(= complexity of ϕ). The axioms are simply the schemas of first-order logic together with the universal closure of each formula $\forall x(\phi \leftrightarrow \psi) \rightarrow (Qx\phi \leftrightarrow Qx\psi)$, as well as of each formula $Qx\phi \leftrightarrow Qy(\phi^x_y)$ whenever y does not occur in ϕ.

Strictly speaking, $\mathscr{L}^0(Q)$ is a logic only if we give a "standard semantics", that is, a global interpretation of Q. But this is not a problem, since in this discussion we are only concerned with weak models. For a fuller explanation of this point see Remark 7.1.2.

2.3.2 Proposition. *Suppose $\mathfrak{A}^*$ is a weak model for $\mathscr{L}^0(Q)$. Let q consist of all sets of the form $\{a \in A: \mathfrak{A}^* \models \phi[s(x, a)]\}$ such that $\mathfrak{A}^* \models Qx\phi[s]$. Then for all $\phi \in \mathscr{L}^0(Q)$ and s, $\mathfrak{A}^* \models \phi[s]$ iff $(\mathfrak{A}, q) \models \phi[s]$.*

Proof. The proof is a straightforward induction on complexity. The only interesting step is that of assuming that $(\mathfrak{A}, q) \models Qv\phi[s]$ holds and showing that $\mathfrak{A}^* \models Qv\phi[s]$ must hold also. By definition, there exist $Qu\psi$ and t such that $\mathfrak{A}^* \models Qu\psi[t]$ and for all $a \in A$,

(1) $$\mathfrak{A}^* \models \psi[t(u, a)] \Leftrightarrow \mathfrak{A}^* \models \phi[s(v, a)].$$

The following two facts are easy to establish.

(2) Suppose $\mathbf{x}$ and $\mathbf{y}$ are disjoint. For every $\mathscr{L}^0(Q)$ formula $\theta(\mathbf{x})$ there is an $\mathscr{L}^0(Q)$ formula $\theta'(\mathbf{x})$ of the same vocabulary, such that no y_i from $\mathbf{y}$ occurs in θ', and $\vdash_{\mathscr{L}^0(Q)} \forall\mathbf{x}(\theta \leftrightarrow \theta')$.

(3) For any formulas θ and θ' and sequences $\mathbf{x}$ and $\mathbf{y}$ as in (2), if f maps $\mathbf{y}$ to $\mathbf{x}$, that is, $f(y_i) = x_i$, all i, then for all s, $\mathfrak{A}^* \models \theta[s]$ iff $\mathfrak{A}^* \models \theta'[s]$ iff $\mathfrak{A}^* \models \theta'^{\mathbf{x}}_{\mathbf{y}}[s \circ f]$.

For, (2) follows by induction on θ, using the axioms $Qy\alpha \leftrightarrow Qz(\alpha^y_z)$ and the theorems $\exists y\alpha \leftrightarrow \exists z(\alpha^y_z)$, while the second "iff" in (3) follows from the equality axiom $\mathbf{x} = \mathbf{y} \wedge \theta'(\mathbf{x}) \rightarrow \theta'(\mathbf{y})$. Thus, we may assume that ϕ and ψ have disjoint sets of free variables; and by changing ψ again, we may assume that u and v are the same variable. Accordingly, (1) then yields

(4) $$\mathfrak{A}^* \models (\psi \leftrightarrow \phi)[(s \cup t)(v, a)] \quad \text{for all } a \in A;$$

and hence,

(5) $$\mathfrak{A}^* \models \forall v(\psi \leftrightarrow \phi)[s \cup t].$$

By the axioms, we have

$$\mathfrak{A}^* \models (Qv\psi \leftrightarrow Qv\phi)[s \cup t].$$

Since $\mathfrak{A}^* \models Qv\psi[t]$, $\mathfrak{A}^* \models Qv\phi[s]$ and the argument is complete. □

We can also define the class of *monotone structures* as in

2.3.3 Definition. A structure $(\mathfrak{A}, q)$, where $q \subseteq \mathscr{P}(A)$, is said to be a *monotone structure* if for all X and Y, $X \supseteq Y \in q$ implies that $X \in q$.

For more on monotone structures, the reader should consult Makowsky–Tulipani [1977] or Ziegler [1978]. In the present volume, Chapter III, Section 4, Chapter XV, Section 6 and Section 6.4 of this chapter offer some further material along these lines.

The logic with concrete syntax $\mathscr{L}^{\mathrm{m}}(Q)$ where the "m" stands for monotone, is obtained from $\mathscr{L}^0(Q)$ by strengthening the axioms $\forall x(\phi \leftrightarrow \psi) \to (Qx\phi \leftrightarrow Qx\psi)$ to $\forall x(\phi \to \psi) \to (Qx\phi \to Qx\psi)$.

2.3.4 Proposition. *Suppose $\mathfrak{A}^*$ is a weak model for $\mathscr{L}^{\mathrm{m}}(Q)$. Let q consist of all subsets of A which contain $\{a \in A: \mathfrak{A}^* \models \phi[s(x, a)]\}$ for some ϕ and s such that $\mathfrak{A}^* \models Qx\phi[s]$. Then $(\mathfrak{A}, q)$ is a monotone structure and for all $\phi \in \mathscr{L}^{\mathrm{m}}(Q)$ and s, $\mathfrak{A}^* \models \phi[s]$ iff $(\mathfrak{A}, q) \models \phi[s]$.*

Proof. Of course, $(\mathfrak{A}, q)$ is a monotone structure. The remainder of the proof is obtained from the proof of Proposition 2.3.2 by changing "$\leftrightarrow$" to "$\to$" in (1), (4), and (5). □

Although our main purpose in this section has been to pave the way for completeness proofs in Sections 3, 4, and 5, we should notice that our digression here in Section 2.3 has brought us to the well-known weak completeness theorem given in

2.3.5 Corollary (Folklore Weak Completeness). *Let T be a consistent set of sentences in $\mathscr{L}^0(Q)$. Then, for all $\kappa \geq \omega$, there exists $(\mathfrak{A}, q) \models T$ such that $|A| = \kappa + |T|$. If T is in fact $\mathscr{L}^{\mathrm{m}}(Q)$-consistent, we may take $(\mathfrak{A}, q)$ to be a monotone structure. The converses (soundness) also hold, regardless of cardinalities.*

Proof. For countable T, this is immediate from the weak completeness theorem (2.2.3) together with Propositions 2.3.2 and 2.3.4. In general, we can obtain a weak model of each countable subset of T, apply first-order compactness and Löwenheim–Skolem arguments to get a weak model $\mathfrak{A}^*$ of T of the desired cardinality, and then apply Propositions 2.3.2 or 2.3.4. The argument for soundness is clear. ☐

2.3.6 Corollary (Compactness for Weak Models). *Let T be a set of sentences of $\mathscr{L}^0(Q)$ such that every finite subset of T has a weak model. Then T has a weak model.* The term "weak model" may have either of the two meanings from Proposition 2.3.2.

If, in fact, every finite subset of T has a weak model which is a monotone structure, then T has a weak model which is a monotone structure. The reader can find an ultraproduct proof for this in Makowsky–Tulipani [1977, §7].) ☐

3. $\mathscr{L}(Q_1)$ and $\mathscr{L}_{\omega_1\omega}(Q_1)$: Completeness and Omitting Types Theorems

This section consists primarily of the main results from Keisler's paper [Ke][1] on $\mathscr{L}(Q)$, where Q = "there exist uncountably many." Although we will base the proofs on the notion of weak model as presented in Section 2, the reader may prefer to use Keisler's notion (see Section 2.3) or any other notion having reasonable properties. Further applications of the completeness theorem for $\mathscr{L}(Q)$ can be found in [Ke].

3.1. The Axioms, Basic Notions, and Properties

3.1.1 Definition ([Ke]). The axioms of $\mathscr{L}(Q)$ include the universal closures of all first-order axiom schemas as well as the following axioms, all of which may have free variables other than those displayed.

(1) $\neg Qx(x = y \vee x = z)$;

(2) $\forall x(\phi \to \psi) \to (Qx\phi \to Qx\psi)$;

(3) $Qx\phi(x) \leftrightarrow Qy\phi(y)$, where $\phi(x, \ldots)$ is a formula of $\mathscr{L}(Q)$ in which y does not occur, and $\phi(y, \ldots)$ is obtained by replacing each free occurrence of x by y;

(4) $Qy\,\exists x\phi \to \exists xQy\phi \vee Qx\,\exists y\phi$.

[1] Henceforth, [Ke] will refer to Keisler [1970]. Except as otherwise noted all results in Section 3 are proved in [Ke].

The rule of inference is *modus ponens* (but universal generalization may be derived as in Enderton [1972]). Notice that the axioms are all valid (where again, $Q = Q_1$). To see that Axiom 4 is valid, consider its contrapositive $\neg Qx\,\exists y\phi \wedge \neg\exists xQy\phi \to \neg Qy\,\exists x\phi$, which asserts that a countable union of countable sets is countable. Throughout the following discussion we will assume that the axiom of choice holds. Keisler [Ke] also credits Craig and Fuhrken with the conjecture that these axioms are complete.

In order to apply the results of Section 2 (on weak models) to the problems at hand, we need the following lemma. The proof, though routine, is omitted since it lacks interest. Nevertheless, we note that the proof of (i) is similar to the $\mathscr{L}_{\omega\omega}$ case as treated in Enderton [1972].

3.1.2 Lemma. (i) *With the notion of proof as defined above, $\mathscr{L}(Q)$ has a concrete syntax (in the sense of Section 2).*

(ii) *The notion "$\mathscr{L}(Q)$-locally omits" as given in Definition 2.2.4 is equivalent to the usual notion. That is, for a fixed vocabulary τ, T $\mathscr{L}(Q)$-locally omits $\Sigma(\mathbf{x})$ iff whenever $\exists\mathbf{x}\phi$ is consistent with T, then so is $\exists\mathbf{x}(\phi \wedge \neg\sigma)$ for some $\sigma \in \Sigma$.* □

For the remainder of this section, we fix a countable vocabulary τ. The proof of the completeness theorem is composed of three steps. First, the weak completeness theorem (2.2.3) is applied to obtain a countable weak model of a consistent theory T. That done, we then prove a "main lemma" which will, in effect, show how to expand "uncountable" sets while keeping "countable" sets unexpanded. Extending the given countable weak model and iterating ω_1 times using this process, we will find that the union of the structures gives the desired model of T. First, however, let us formally state the kind of extension we need.

3.1.3 Definition. Let $\mathfrak{A}^*$ and $\mathfrak{B}^*$ be countable weak models for $\mathscr{L}(Q)$. We say that $\mathfrak{B}^*$ is a *precise extension of* $\mathfrak{A}^*$ *relative to* ϕ, if $\phi(x)$ is a formula of $\mathscr{L}(Q)$ with parameters in A and

(i) $\mathfrak{A}^* \prec^w \mathfrak{B}^*$.

(ii) If $\mathfrak{A}^* \models Qx\phi$, then $\mathfrak{B}^* \models \phi(b)$ for some $b \in B - A$.

(iii) Whenever $\mathfrak{A}^* \models \neg Qx\psi$ for $Qx\psi$ a sentence with parameters in A, then $\mathfrak{B}^* \models \neg\psi(b)$ for all $b \in B - A$.

3.1.4 Remarks on Notation. Notice that the notation has become more informal than that used in Section 2. A precise definition would consider precise extensions relative to $\langle\phi, s\rangle$, where ϕ is a formula of $\mathscr{L}(Q)$, and s is an assignment into A with domain including all but at most one free variable x of ϕ. Then, for example, (ii) would be worded thus: "if $\mathfrak{A}^* \models Qx\phi[s]$ then $\mathfrak{B}^* \models \phi[s(x, b)]$ for some $b \in B - A$." The more informal notation will generally be used in the sequel.

The symbol Q^*x is an abbreviation for $\neg Qx\neg$, "for all but countably many x." Before moving to the "main lemma", we should summarize some easy consequences of the axioms. Accordingly, we have

3.1.5 Lemma. *Every formula in the following schema is a theorem of $\mathscr{L}(Q)$ and is therefore valid in every weak model for $\mathscr{L}(Q)$.*

(i) $\neg Qx\psi \leftrightarrow Q^*x \neg \psi$.

(ii) $Qx(x = x) \to Q_1x_1 \ldots Q_nx_n(\phi \wedge Q_{n+1}y_1 \cdots Q_{n+m}y_m\psi)$ $\leftrightarrow Q_1x_1 \ldots Q_nx_nQ_{n+1}y_1 \cdots Q_{n+m}\, y_m(\phi \wedge \psi)$, *whenever $y_1, \ldots, y_m$ are not free in ϕ, and each $Q_i \in \{\exists, \forall, Q, Q^*\}$.*

(iii) (Monotonicity) $\forall \mathbf{x}(\phi \to \psi) \to (q\mathbf{x}\phi \to q\mathbf{x}\psi)$, *where $q\mathbf{x}$ is any string of $\exists$, $\forall$, Q, Q^* quantifiers on $\mathbf{x}$.*

Moreover, we also have the following "Intersection principles":

(iv) $\bigwedge_{i \in I} Q^*x\psi \to Q^*x \bigwedge_{i \in I} \psi$ *(I finite).*

(v) $Qx\phi \wedge Q^*x\psi \to Qx(\phi \wedge \psi)$.

(vi) $\forall x\phi \wedge qx\psi \to qx(\phi \wedge \psi)$ (for $q = Q$ or Q^*). $\square$

3.2. Towards a Proof of Keisler's Completeness Theorem

3.2.1 Main Lemma. *Suppose $\mathfrak{A}^*$ is a countable weak model for $\mathscr{L}(Q)$, and suppose $\phi(x, \mathbf{p})$ is a formula of $\mathscr{L}(Q)$ with parameters $\mathbf{p}$ in A. Then there is a precise extension of $\mathfrak{A}^*$ relative to ϕ.*

Proof. If $\mathfrak{A}^* \models \neg Qx\phi(x, \mathbf{p})$, then we set $\mathfrak{B}^* = \mathfrak{A}^*$. So, assume that $\mathfrak{A}^* \models Qx\phi(x, \mathbf{p})$, and let $C_A = \{c_a : a \in A\}$ be a set of new constant symbols. Also let $D = C_A \cup \{c\}$ for yet another constant symbol c, and form the following set $T_\phi(\mathfrak{A}^*)$ of $\tau \cup D$-sentences of $\mathscr{L}(Q)$. The notation $\mathbf{c_a}$ denotes $\langle c_{a_1}, \ldots, c_{a_n}\rangle$, when $\mathbf{a} = \langle a_1, \ldots, a_n\rangle$ is any sequence of elements of A.

$$T_\phi(\mathfrak{A}^*) = \{\theta(\mathbf{c_a}) : \mathfrak{A}^* \models \theta(\mathbf{a})\} \cup \{\phi(c, \mathbf{c_p})\}$$
$$\cup \{\neg\psi(c, \mathbf{c_a}) : \mathfrak{A}^* \models \neg Qx\psi(x, \mathbf{a})\}.$$

For each $\psi(x, \mathbf{a})$, we define a set Σ_ψ such that

$$\Sigma_\psi = \{\psi(x, \mathbf{c_a})\} \cup \{x \neq c_b : \mathfrak{A}^* \models \psi(b, \mathbf{a})\}. \quad \square$$

Claim A. $T_\phi(\mathfrak{A}^*)$ is an $\mathscr{L}(Q)$-consistent theory which $\mathscr{L}(Q)$-locally omits Σ_ψ, for each $\psi(x, \mathbf{a})$, such that $\mathfrak{A}^* \models \neg Qx\psi(x, \mathbf{a})$.

Deferring the proof of Claim A for the moment, we will see how the theorem follows. Let $\mathfrak{B}^* \succ^w \mathfrak{A}^*$ be the countable weak model guaranteed by the extension lemma (2.2.6) or by Lemma 7.2.3. That is, $\mathfrak{B}^*$ omits each Σ_ψ, and there exists $e \in B$ (corresponding to c) such that for all $\theta(c, \mathbf{c_a}) \in T_\phi(\mathfrak{A}^*)$, $\mathfrak{B}^* \models \theta(e, \mathbf{a})$. Since

$\phi(c, \mathbf{c_p}) \in T_\phi(\mathfrak{A}^*)$, it follows that $\mathfrak{B}^* \models \phi(e, \mathbf{p})$. Moreover, $\neg(c = c_a) \in T_\phi(\mathfrak{A}^*)$ for all $a \in A$, since $\mathfrak{A}^* \models \neg Qx(x = a)$ by Axioms 1 and 2. Thus, $\mathfrak{B}^* \models e \neq a$ for all $a \in A$, and hence $e \notin A$. Accordingly, we see that (ii) in the definition of "precise extension relative to ϕ" is satisfied. Part (iii) holds because $\mathfrak{B}^*$ omits each necessary Σ_ψ. Thus, the proof is complete once Claim A has been proved. First, however, it is very helpful to have a useful criterion for consistency of $\tau \cup D$-sentences of $\mathscr{L}(Q)$ with $T_\phi(\mathfrak{A}^*)$.

Claim B (Consistency Criterion). For any τ-formula $\theta(y, \mathbf{z})$ of $\mathscr{L}(Q)$ and $\mathbf{a}$ in A:

(i) $\theta(c, \mathbf{c_a})$ is $\mathscr{L}(Q)$-consistent with $T_\phi(\mathfrak{A}^*)$ iff $\mathfrak{A}^* \models Qy(\phi(y, \mathbf{p}) \wedge \theta(y, \mathbf{a}))$.

(ii) $T_\phi(\mathfrak{A}^*) \vdash_{\mathscr{L}(Q)} \theta(c, \mathbf{c_a})$ iff $\mathfrak{A}^* \models Q^*y(\phi(y, \mathbf{p}) \to \theta(y, \mathbf{a}))$. (Recall $Q^* = \neg Q \neg$.)

Proof of Consistency Criterion. Using Lemma 3.1.5(i) and $\neg\theta$ for θ, it is easy to see that (i) and (ii) are equivalent. Thus, we will only prove (ii). For the ($\Leftarrow$) direction, we suppose that $\mathfrak{A}^* \models \neg Qy \neg (\phi(y, \mathbf{p}) \to \theta(y, \mathbf{a}))$. Then $\neg(\neg(\phi(c, \mathbf{c_p}) \to \theta(c, \mathbf{c_a}))) \in T_\phi(\mathfrak{A}^*)$, by definition. Thus, $T_\phi(\mathfrak{A}^*) \vdash_{\mathscr{L}(Q)} \phi(c, \mathbf{c_p}) \to \theta(c, \mathbf{c_a})$. And, since $\phi(c, \mathbf{c_p}) \in T_\phi(\mathfrak{A}^*)$, we have that $T_\phi(\mathfrak{A}^*) \vdash_{\mathscr{L}(Q)} \theta(c, \mathbf{c_a})$.

Conversely, suppose $T_\phi(\mathfrak{A}^*) \vdash \theta(c, \mathbf{c_a})$. Since proofs are finite, there exist formulas $\delta_i(\mathbf{c}_{\mathbf{a}_i})$ for $i \in I$ and $\psi_j(y, \mathbf{c}_{\mathbf{a}_j})$ with $j \in J$, where both I and J are finite, such that

(1) $\qquad \mathfrak{A}^* \models \delta_i(\mathbf{a}_i), \quad \text{all} \quad i \in I;$

(2) $\qquad \mathfrak{A}^* \models \neg Qy\psi_j(y, \mathbf{a}_j), \quad \text{all} \quad j \in J;$

(3) $\qquad \{\phi(c, \mathbf{c_p})\} \cup \{\delta_i(\mathbf{c}_{\mathbf{a}_i}) : i \in I\} \cup \{\neg\psi_j(c, \mathbf{c}_{\mathbf{a}_j}) : j \in J\} \vdash_{\mathscr{L}(Q)} \theta(c, \mathbf{c_a}).$

By repeated application of the deduction theorem (2.1.1), we see that (3) implies that

$$\vdash_{\mathscr{L}(Q)} \left(\bigwedge_{i \in I} \delta_i(\mathbf{c}_{\mathbf{a}_i})\right) \to \left[\phi(c, \mathbf{c_p}) \wedge \left(\bigwedge_{j \in J} \neg\psi_j(c, \mathbf{c}_{\mathbf{a}_j})\right) \to \theta(c, \mathbf{c_a})\right].$$

By soundness (see Proposition 2.2.1), since $\mathfrak{A}^* \models \bigwedge_{i \in I} \delta_i(\mathbf{a}_i)$ by (1) this yields

(4) $$\mathfrak{A}^* \models \forall y\left[\phi(y, \mathbf{p}) \wedge \left(\bigwedge_{j \in J} \neg\psi_j(y, \mathbf{a}_j)\right) \to \theta(y, \mathbf{a})\right].$$

We now make use of the "intersection principles" of Lemma 3.1.5. Applying Lemma 3.1.5(iv) and (i) to (2) above, we obtain $\mathfrak{A}^* \models Q^*y \bigwedge_{j \in J} \neg\psi_j(y, \mathbf{a}_j)$. Combining this with (4) above, the "intersection principle" given in Lemma 3.1.5(vi) shows that

$$\mathfrak{A}^* \models Q^*y\left[\left[\phi(y, \mathbf{p}) \wedge \left(\bigwedge_{j \in J} \neg\psi_j(y, \mathbf{a}_j)\right) \to \theta(y, \mathbf{a})\right] \wedge \bigwedge_{j \in J} \neg\psi_j(y, \mathbf{a}_j).\right]$$

By Lemma 3.1.5(iii) (monotonicity) this implies $\mathfrak{A}^* \models Q^*y[\phi(y, \mathbf{p}) \to \theta(y, \mathbf{a})]$, which concludes the proof of Claim B, the "consistency criterion".

It now remains to prove Claim A. First of all, the consistency criterion implies that $T_\phi(\mathfrak{A}^*)$ is $\mathscr{L}(Q)$-consistent. Now suppose that $\mathfrak{A}^* \models \neg Qx\psi(x, \mathbf{a})$. We must show that $T_\phi(\mathfrak{A}^*)$ $\mathscr{L}(Q)$-locally omits $\Sigma_\psi = \{\psi(x, \mathbf{c_a}) \cup \{x \neq c_b : \mathfrak{A}^* \models \psi(b, \mathbf{a})\}$, in the sense of Lemma 3.1.2(ii). Thus, suppose $\exists x\theta(x, c, \mathbf{c_d})$ is consistent with $T_\phi(\mathfrak{A}^*)$, where $\mathbf{d}$ is from A. By the consistency criterion and Lemma 3.1.5(ii), we have

(5) $$\mathfrak{A}^* \models Qy\,\exists x[\phi(y, \mathbf{p}) \wedge \theta(x, y, \mathbf{d})].$$

If $\mathfrak{A}^* \models Qy\,\exists x[\phi(y, \mathbf{p}) \wedge \theta(x, y, \mathbf{d}) \wedge \neg\psi(x, \mathbf{a})]$, then by the consistency criterion, $\exists x[\theta(x, c, \mathbf{c_d}) \wedge \neg\psi(x, \mathbf{c_a})]$ is consistent with $T_\phi(\mathfrak{A}^*)$, and we're done. Otherwise, $\mathfrak{A}^* \models Q^*y\forall x[\phi(y, \mathbf{p}) \wedge \theta(x, y, \mathbf{d}) \to \psi(x, \mathbf{a})]$. Then, by the intersection principle Lemma 3.1.5(v) and its analogue for $\mathscr{L}_{\omega\omega}$, this combines with (5) to yield

(6) $$\mathfrak{A}^* \models Qy\,\exists x[[\phi(y, \mathbf{p}) \wedge \theta(x, y, \mathbf{d})] \wedge [\phi(y, \mathbf{p}) \wedge \theta(x, y, \mathbf{d}) \to \psi(x, \mathbf{a})]].$$

Applying the monotonicity principle (Lemma 3.1.5(iii)) to (6), we have

(7) $$\mathfrak{A}^* \models Qy\,\exists x[\phi(y, \mathbf{p}) \wedge \theta(x, y, \mathbf{d}) \wedge \psi(x, \mathbf{a})].$$

Now is the time to apply the main axiom of $\mathscr{L}(Q)$, namely Axiom 4. Applied to (7) this gives

(8) $$\mathfrak{A}^* \models Qx\,\exists y[\phi(y, \mathbf{p}) \wedge \theta(x, y, \mathbf{d}) \wedge \psi(x, \mathbf{a})],$$

or

(9) $$\mathfrak{A}^* \models \exists x Qy[\phi(y, \mathbf{p}) \wedge \theta(x, y, \mathbf{d}) \wedge \psi(x, \mathbf{a})].$$

But (8) is impossible, since it implies that $\mathfrak{A}^* \models Qx\psi(x, \mathbf{a})$—a contradiction of the assumption. Thus, there exists a witness $e \in A$ for (9) above. Then $\mathfrak{A}^* \models Qy[\phi(y, \mathbf{p}) \wedge \theta(e, y, \mathbf{d})]$ which further implies $\mathfrak{A}^* \models Qy\,\exists x(\phi(y, \mathbf{p}) \wedge \theta(x, y, \mathbf{d}) \wedge \neg x \neq e)$, by monotonicity. But applying the consistency criterion we see that $\exists x(\theta(x, c, \mathbf{c_d}) \wedge \neg x \neq c_e)$ is $\mathscr{L}(Q)$-consistent with $T_\phi(\mathfrak{A}^*)$, as desired. ☐

Remark. In [Ke], $\mathfrak{B}^*$ is defined to be a precise extension of $\mathfrak{A}^*$ if it is a precise extension relative to every formula. By iterating the Main Lemma ω times in an appropriate manner, we may construct such an extension. Although this would slightly simplify the proof of the completeness theorem (3.2.3), such a notion of extension is not as useful for $\mathscr{L}(\mathrm{aa})$ in Section 4 and for $\mathscr{L}(Q^2)$ in Section 5.

The final lemma needed for the proof of the completeness theorem tells us that a careful iteration of the Main Lemma produces the desired model.

3.2.2 Lemma (Union of Chain Lemma). *Assume that $\langle \mathfrak{A}_\alpha^* : \alpha < \omega_1 \rangle$ is a chain of countable weak models for $\mathscr{L}(Q)$, with the following properties.*

(i) *For all $\alpha < \omega_1$, $\mathfrak{A}_{\alpha+1}^*$ is a precise extension of $\mathfrak{A}_\alpha^*$ relative to ϕ, for some ϕ.*
(ii) *For each formula $\phi(x)$ with parameters in some A_α, $\{\beta < \omega_1 : \mathfrak{A}_{\beta+1}^*$ is a precise extension of $\mathfrak{A}_\beta^*$ relative to $\phi\}$ is uncountable.*
(iii) *The chain is continuous, that is, $\mathfrak{A}_\lambda^* = \bigcup_{\alpha<\lambda} \mathfrak{A}_\alpha^*$ for limit $\lambda < \omega_1$.*

Then by setting $\mathfrak{A} = \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha$, we have $\mathfrak{A} \models \phi$ iff $\mathfrak{A}_\alpha^* \models \phi$ for all sentences ϕ with parameters in A_α, where $\alpha < \omega_1$ is arbitrary.

Proof. The proof is by induction on the length of ϕ. For atomic ϕ, it is clear, and both the $\neg$ and $\vee$ steps are trivial. Now notice that $\alpha < \beta < \omega_1$ implies $\mathfrak{A}_\alpha^* \prec^w \mathfrak{A}_\beta^*$, by Proposition 2.2.2 (the elementary chain theorem for weak models). The case $\phi = \exists y \psi(y)$ then follows in the usual way.

Finally, suppose that ϕ is $Qx\psi$. If $\mathfrak{A}_\alpha^* \models Qx\psi$, then by (ii) above, there exists an uncountable set $X \subseteq \omega_1$ such that for all $\beta \in X$, $\mathfrak{A}_{\beta+1}^* \models \psi(a)$ for some $a \in A_{\beta+1} - A_\beta$. This implies that $\mathfrak{A} \models \psi(a)$ for some $a \in A_{\beta+1} - A_\beta$. Thus, $\mathfrak{A} \models Qx\psi$. Conversely, if $\mathfrak{A}_\alpha^* \models \neg Qx\psi$, then by (i), it follows by induction on β (and the definition of precise extension relative to a formula) that $\mathfrak{A}_\beta^* \models \psi(a)$ implies $a \in A_\alpha$. By the inductive hypothesis, this translates into: $\mathfrak{A} \models \psi(a)$ implies $a \in A_\alpha$. Since A_α is countable, we must have that $\mathfrak{A} \models \neg Qx\psi$. $\square$

3.2.3 Theorem (Completeness Theorem for $\mathscr{L}(Q)$). *Suppose T is a set of τ-sentences of $\mathscr{L}(Q)$, where τ is a countable vocabulary. Then T is $\mathscr{L}(Q)$-consistent iff T has a model.*

Proof. We have already shown soundness. For the other direction, we suppose T is $\mathscr{L}(Q)$-consistent. We wish to define a chain $\langle \mathfrak{A}_\alpha^* : \alpha < \omega_1 \rangle$ which satisfies the hypotheses of Lemma 3.2.2, the "union of chain lemma." It will be convenient to require $A_\alpha \subseteq \omega_1$ for all $\alpha < \omega_1$. For then we will have that $\bigcup_{\alpha<\omega_1} A_\alpha \subseteq \omega_1$, and the following construction will indeed witness each $Qx\phi$ uncountably many times.

More precisely, we start with any partition of ω_1 into uncountable sets X_ϕ, where ϕ ranges over formulas $\phi(x)$ with parameters in ω_1. Let us define $\mathfrak{A}_\alpha^*$ by induction on α. First, let $\mathfrak{A}_0^*$ be a countable weak model for $\mathscr{L}(Q)$ which satisfies T, by the weak completeness theorem (2.2.3). We may require $A_0 = \omega$. For successor stages $\alpha + 1$, we apply the Main Lemma (3.2.1). Let $\mathfrak{A}_{\alpha+1}^*$ be a precise extension of $\mathfrak{A}_\alpha^*$ relative to ϕ, where $\alpha \in X_\phi$ (unless the parameters of ϕ do not lie inside A_α, in which case set $\mathfrak{A}_{\alpha+1}^* = \mathfrak{A}_\alpha^*$). Finally, set $\mathfrak{A}_\lambda^* = \bigcup_{\alpha<\lambda} \mathfrak{A}_\alpha^*$ for limit $\lambda < \omega_1$. Also set $\mathfrak{A} = \bigcup \{\mathfrak{A}_\alpha : \alpha < \omega_1\}$. By Lemma 3.2.2, we have that $\mathfrak{A} \models \phi(\mathbf{a})$ iff $\mathfrak{A}_\alpha^* \models \phi(\mathbf{a})$, for all $\alpha < \omega_1$ and $\mathbf{a}$ in A_α. In particular, since $\mathfrak{A}_0^* \models \phi$ for all $\phi \in T$, we have that $\mathfrak{A}$ is a model of T. $\square$

3.2.4 Corollary. *$\mathscr{L}(Q)$ is countably compact.* $\square$

Before continuing with an extension of the completeness theorem to $\mathscr{L}_{\omega_1\omega}(Q)$ and omitting types in $\mathscr{L}(Q)$, we will examine a corollary to the Main Lemma, as was promised in Section 1. This result appears as Corollary 3.6.1 of [Ke].

3.2.5 Theorem (Essentially due to Keisler–Morley [1968: 4.2, 2.2]). *Let* $\mathfrak{A} = (A, E)$ *be a countable model of* ZF, *possibly excepting power set. For all* $a \in A$, *set* $a_E = \{b: \langle b, a \rangle \in E\}$.

(i) *There exists* $\mathfrak{B} = (B, F) \succ \mathfrak{A}$ *such that for all* $a \in A$, $a_F = a_E$, *and the ordinals of* $\mathfrak{B}$ *are* ω_1*-like.*
(ii) *For every regular cardinal a of* $\mathfrak{A}$, *there exists* $\mathfrak{B} = (B, F) \succ \mathfrak{A}$ *such that* $b_E = b_F$ *for all* bEa, *but* $\langle a_F, F \upharpoonright a_F \rangle$ *is* ω_1*-like.*

In fact, for (ii) *it is not necessary that* $\mathfrak{A}$ *satisfy the collection schema.*

Proof. (i) We expand $\mathfrak{A}$ to a weak model $\mathfrak{A}^*$ for $\mathscr{L}(Q)$ by interpreting Q as "for unboundedly many." For every ε-formula ϕ of $\mathscr{L}(Q)$, let ϕ^+ be the result of replacing each quantifier of the form "Qx" by "there exist arbitrarily large x", that is, $\forall y\, \exists x(x \notin y \wedge \cdots)$ (where y is chosen not to conflict with other variables of ϕ). We then set $R^{\mathfrak{A}^*}_{Qx\phi(x,\mathbf{y})} = \{\mathbf{a}: \mathfrak{A} \models (Qx\phi)^+(\mathbf{a})\}$. As in the proof of Proposition 2.3.2, an easy induction on complexity of $\phi \in \mathscr{L}(Q)$ shows that $\mathfrak{A}^* \models \phi(\mathbf{a})$ iff $\mathfrak{A} \models \phi^+(\mathbf{a})$ for all ϕ and $\mathbf{a}$. It is then easy to check that $\mathfrak{A}^*$ is a countable weak model for $\mathscr{L}(Q)$: the axiom of collection is used to verify Axiom 4.

By the Main Lemma (3.2.1), $\mathfrak{A}^*$ has a precise extension relative to "x is an ordinal". Iterating, we thus obtain a chain $\langle \mathfrak{A}^*_\alpha : \alpha < \omega_1 \rangle$ with $\mathfrak{A}^*_0 = \mathfrak{A}^*$, such that $\mathfrak{A}^*_{\alpha+1}$ is a precise extension of $\mathfrak{A}^*_\alpha$ relative to "x is an ordinal" for all $\alpha < \omega_1$, and $\mathfrak{A}^*_\lambda = \bigcup_{\alpha<\lambda} \mathfrak{A}^*_\alpha$ for all limit $\lambda < \omega_1$. Set $\mathfrak{B} = \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha$; then $\mathfrak{B}$ is the desired model.

(ii) The proof here is the same, except for two changes. This time, ϕ^+ is obtained by replacing each quantifier Qx by $\forall y \in a\, \exists x \in a\, (y \in x \wedge \cdots)$, and the expansion $\mathfrak{A}^*$ of $\mathfrak{A}$ is defined accordingly. Also, in this situation we require that $\mathfrak{A}^*_{\alpha+1}$ be a precise extension of $\mathfrak{A}^*_\alpha$ relative to $x \in a$. These changes made, the proof of (i) goes through. ☐

A rather similar development concerning linear orders appears in Jervell [1975].

3.3. Omitting Types in $\mathscr{L}(Q)$

The next goal in this section is to get an omitting types theorem. Further on, in Section 3.4 we will discuss applications.

3.3.1 Definition ([Ke]). Let T be a set of τ-sentences, and $\Sigma(\mathbf{x})$ a set of τ-formulas (with free variables contained in the finite sequence $\mathbf{x}$), of $\mathscr{L}(Q)$. T is said to *strongly omit* Σ if the following condition is met. Let $\overline{Qy}$ be an arbitrary quantifier string of the form $Q_1 y_1 \ldots Q_n y_n$, where $Q_i \in \{\exists, Q\}$ for $1 \leq i \leq n$. We call such a $\overline{Qy}$ *a quexistential string.* Then, for every sentence of the form $\overline{Qy}\, \exists \mathbf{x} \phi$ which is consistent with T, there exists $\sigma \in \Sigma$ such that $\overline{Qy}\, \exists \mathbf{x}(\phi \wedge \neg \sigma)$ is consistent with T.

A weak model $\mathfrak{A}^*$ is said to *strongly omit* $\Sigma(\mathbf{x})$, where Σ may have parameters in A, if whenever $\mathfrak{A}^* \models \overline{Qy}\, \exists\mathbf{x}\phi$ with $\overline{Qy}$ a quexistential string, where ϕ may have parameters in A, then $\mathfrak{A}^* \models \overline{Qy}\, \exists\mathbf{x}(\phi \wedge \neg\sigma)$ for some $\sigma \in \Sigma$.

For applications to logics such as $\mathscr{L}^{<\omega}$ (in Section 5), it is helpful to consider certain extensions of $\mathscr{L}(Q)$.

3.3.2 Definition. A logic $\mathscr{L}$ with concrete syntax is a *reasonable extension* of $\mathscr{L}(Q)$ if it meets the following criteria.

(i) $\mathscr{L}$ is closed under Q: if $\phi \in \mathscr{L}(\tau)$ then $Qx\phi \in \mathscr{L}(\tau)$.

(ii) Every formula ϕ of $\mathscr{L}(\tau \cup C)$ with $C \cap \tau = \varnothing$, is $\psi(f)$ for some $\psi \in \mathscr{L}(\tau)$ and some f. (This is needed for the proof of the Main Lemma (3.2.1); it enables the proof of Proposition 3.1.2(ii) to go forward.)

(iii) The notions of free variable, substitution, and rank—frvar(ϕ), $\phi(f)$, $r(\phi)$ from Section 2.1—obey the obvious inductive clauses for Q.

(iv) Every axiom schema (1–4) of $\mathscr{L}(Q)$ is an axiom schema of $\mathscr{L}$. In particular, there is a notion of change of free variable to which Axiom 3 applies, as does $\exists x\phi(x) \leftrightarrow \exists y\phi(y)$.

3.3.3 Remark. The notion of "precise extension relative to ϕ", Lemma 3.1.2(ii), the quantifier manipulations of Lemma 3.1.5, and the Main Lemma (3.2.1) with Claims A and B, extend in the natural way to any reasonable extension of $\mathscr{L}(Q)$. That this is actually the case can be verified in a routine way. Accordingly, we will use these extended versions.

3.3.4 Lemma. *Fix a countable vocabulary τ. Let $\mathfrak{A}^*$ be a countable weak model for any reasonable extension $\mathscr{L}$ of $\mathscr{L}(Q)$. Suppose that $\Sigma(\mathbf{x})$ is any set of formulas in the finite sequence $\mathbf{x}$ of free variables, where Σ may have parameters in A. For every formula $\delta = \overline{Qy}\, \exists\mathbf{x}\psi(\mathbf{x}, \mathbf{y}, \mathbf{u})$, where ψ is parameter-free, such that $\mathbf{u}$ is disjoint from $\mathbf{x}$ and $\mathbf{y}$ and $\overline{Qy}$ is a quexistential string, let*

$$\Sigma^{\delta}(\mathbf{u}) = \{\overline{Qy}\, \exists\mathbf{x}\psi(\mathbf{x}, \mathbf{y}, \mathbf{u})\} \cup \{\neg\overline{Qy}\, \exists\mathbf{x}[\psi(\mathbf{x}, \mathbf{y}, \mathbf{u}) \wedge \neg\sigma(x)] : \sigma \in \Sigma\}.$$

If A^* omits each such $\Sigma^{\delta}(\mathbf{u})$, then A^* strongly omits Σ.

Proof. The proof of this result follows immediately from the definitions. □

To prove the omitting types theorem we will follow the pattern of the completeness theorem proof. That is, we will obtain a weak model, iterate a "main lemma" ω_1 times, and then take the union. Hence, we will need:

3.3.5 Lemma ("Main Lemma" for Omitting Types). *Suppose $\mathfrak{A}^*$ is a countable weak model for $\mathscr{L}(\tau)$, τ countable, where $\mathscr{L}$ is a reasonable extension of $\mathscr{L}(Q)$. Let $\{\Sigma_n : n < \omega\}$ be a countable family of sets of τ-formulas of $\mathscr{L}$, possibly with parameters in A, each in a finite sequence $\mathbf{x}_n$ of free variables. Assume that $\mathfrak{A}^*$ strongly omits Σ_n for all $n < \omega$. Then for all $\phi(x, \mathbf{p})$, there is a precise extension of $\mathfrak{A}^*$ relative to ϕ which strongly omits each Σ_n.*

Proof. The proof is an extension of the proof of the Main Lemma (3.2.1), and we refer to that argument below. Form the theory $T_\phi(\mathfrak{A}^*)$ and the sets Σ_ψ, as before. By Claim A (from the proof of Lemma 3.2.1), $T_\phi(\mathfrak{A}^*)$ locally omits each set Σ_ψ. Suppose for the moment that $T_\phi(\mathfrak{A}^*)$ also locally omits each set Σ_n^δ, as defined in Lemma 3.3.4. Then as before, we apply the Extension Lemma (2.2.6) to obtain a precise extension $\mathfrak{B}^*$ of $\mathfrak{A}^*$ relative to ϕ, which omits each Σ_n^δ. By Lemma 3.3.4, we see that $\mathfrak{B}^*$ strongly omits each Σ_n.

It now remains to show that $T_\phi(\mathfrak{A}^*)$ locally omits each Σ_n^δ, say δ is $\overline{Qy}\,\exists\mathbf{x}\psi(\mathbf{x}, \mathbf{y}, \mathbf{u})$, where $\overline{Qy}$ is a quexistential string and $\mathbf{u}$ is disjoint from $\mathbf{x}$ and $\mathbf{y}$. Suppose that $\exists\mathbf{u}\theta(\mathbf{u}, \mathbf{c_a}, c)$ is consistent with $T_\phi(\mathfrak{A}^*)$. If

$$\exists\mathbf{u}[\theta(\mathbf{u}, \mathbf{c_a}, c) \wedge \neg\overline{Qy}\,\exists\mathbf{x}\psi(\mathbf{x}, \mathbf{y}, \mathbf{u})]$$

is consistent with $T_\phi(\mathfrak{A}^*)$, our argument is done. Otherwise, $\exists\mathbf{u}[\theta(\mathbf{u}, \mathbf{c_a}, c) \wedge \overline{Qy}\,\exists\mathbf{x}\psi(\mathbf{x}, \mathbf{y}, \mathbf{u})]$ is consistent with $T_\phi(\mathfrak{A}^*)$. By "quantifier shuffling" as discussed in Lemma 3.1.5(ii), $\exists\mathbf{u}\,\overline{Qy}\,\exists\mathbf{x}[\theta(\mathbf{u}, \mathbf{c_a}, c) \wedge \psi(\mathbf{x}, \mathbf{y}, \mathbf{u})]$ is consistent with $T_\phi(\mathfrak{A}^*)$. We now apply the consistency criterion (that is, Claim B in the proof of Lemma 3.2.1) to obtain

$$\mathfrak{A}^* \models Qz[\phi(z, \mathbf{p}) \wedge \exists\mathbf{u}\,\overline{Qy}\,\exists\mathbf{x}[\theta(\mathbf{u}, \mathbf{a}, z) \wedge \psi(\mathbf{x}, \mathbf{y}, \mathbf{u})]].$$

By using the last part of Definition 3.3.2(iv), we may replace $\mathbf{u}, \mathbf{y}, \mathbf{x}$ if necessary so that these are disjoint from the free variables of ϕ. Then, by using quantifier shuffling again, we have that

$$\mathfrak{A}^* \models Qz\,\exists\mathbf{u}\,\overline{Qy}\,\exists\mathbf{x}[\phi(z, \mathbf{p}) \wedge \theta(\mathbf{u}, \mathbf{a}, z) \wedge \psi(\mathbf{x}, \mathbf{y}, \mathbf{u})].$$

But $Qz\,\exists\mathbf{u}\,\overline{Qy}$ is also a quexistential string; and so, since $\mathfrak{A}^*$ strongly omits Σ_n, there exists $\sigma \in \Sigma_n$ such that

$$\mathfrak{A}^* \models Qz\,\exists\mathbf{u}\,\overline{Qy}\,\exists\mathbf{x}[\phi(z, \mathbf{p}) \wedge \theta(\mathbf{u}, \mathbf{a}, z) \wedge \psi(\mathbf{x}, \mathbf{y}, \mathbf{u}) \wedge \neg\sigma(\mathbf{x})].$$

By using quantifier shuffling again, we obtain

$$\mathfrak{A}^* \models Qz[\phi(z, \mathbf{p}) \wedge \exists\mathbf{u}\,\overline{Qy}\,\exists\mathbf{x}[\theta(\mathbf{u}, \mathbf{a}, z) \wedge \psi(\mathbf{x}, \mathbf{y}, \mathbf{u}) \wedge \neg\sigma(\mathbf{x})]].$$

And applying the consistency criterion once more, we see that

$$\exists\mathbf{u}\,\overline{Qy}\,\exists\mathbf{x}[\theta(\mathbf{u}, \mathbf{c_a}, c) \wedge \psi(\mathbf{x}, \mathbf{y}, \mathbf{u}) \wedge \neg\sigma(\mathbf{x})]$$

is consistent with $T_\phi(\mathfrak{A}^*)$. Again using quantifier shuffling, we have that

$$\exists\mathbf{u}[\theta(\mathbf{u}, \mathbf{c_a}, c) \wedge \overline{Qy}\,\exists\mathbf{x}[\psi(\mathbf{x}, \mathbf{y}, \mathbf{u}) \wedge \neg\sigma(\mathbf{x})]]$$

is consistent with $T_\phi(\mathfrak{A}^*)$, and the proof is complete. $\square$

3.3.6 Theorem (Omitting Types Theorem for $\mathscr{L}(Q)$). *Suppose that $\mathscr{L}$ is a reasonable extension of $\mathscr{L}(Q)$ and that τ is countable. Suppose also that T is a consistent*

τ-theory of $\mathscr{L}(Q)$ which strongly omits sets $\Sigma_n(\mathbf{x}_n)$ ($n < \omega$) from $\mathscr{L}(\tau)$. Then T has a model which omits each Σ_n.

Proof. As was shown in the proof of Lemma 3.3.5, it follows that T $\mathscr{L}(Q)$-locally omits the sets Σ_n^δ of Lemma 3.3.4. By the weak omitting types theorem (2.2.5), there is a countable weak model $\mathfrak{A}^*$ for $\mathscr{L}(Q)$ which omits each Σ_n^δ. Thus, by Lemma 3.3.4, $\mathfrak{A}^*$ strongly omits each Σ_n.

We now partition ω_1 into disjoint uncountable sets X_ϕ, where ϕ ranges over formulas with parameters in ω_1. We proceed, as in the proof of the completeness theorem (3.2.3), to construct a chain $\langle \mathfrak{A}_\alpha^* : \alpha < \omega_1 \rangle$, with the additional requirement that each $\mathfrak{A}_\alpha^*$ strongly omits each Σ_n. Set $\mathfrak{A}_0^* = \mathfrak{A}^*$, where we may assume that $A_0 \subseteq \omega_1$; and, in fact, each $A_\alpha \subseteq \omega_1$. For limit λ, set $\mathfrak{A}_\lambda^* = \bigcup_{\alpha<\lambda} \mathfrak{A}_\alpha^*$; then it is clear from the elementary chain theorem for weak models (2.2.2) that $\mathfrak{A}_\lambda^*$ is still a weak model for $\mathscr{L}(Q)$ which strongly omits each Σ_n. For successor stages $a + 1$, we choose ϕ so that $\alpha \in X_\phi$. We may thus apply the main lemma for omitting types, Lemma 3.3.5, to obtain $\mathfrak{A}_{\alpha+1}^*$ as a precise extension of $\mathfrak{A}_\alpha^*$ relative to ϕ, which still strongly omits each Σ_n.

Set $\mathfrak{A} = \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha$. Using the Union of Chain Lemma 3.2.2, we see that $\mathfrak{A} \models \phi(\mathbf{a})$ iff $\mathfrak{A}_\alpha^* \models \phi(\mathbf{a})$ for all ϕ and for all $\mathbf{a}$ in A_α (all $\alpha < \omega_1$). Since $\mathfrak{A}_0^* \models T$, then we must also have that $\mathfrak{A} \models T$. In order to see that $\mathfrak{A}$ omits Σ_n, we suppose that $\mathbf{a}$ is a sequence from A with $|\mathbf{a}| = |\mathbf{x}_n|$. We may, of course, choose $\alpha < \omega_1$ so that $\mathbf{a} \in A_\alpha^{<\omega}$. Then, since $\mathfrak{A}_\alpha^* \models \exists\mathbf{x}(\mathbf{x} = \mathbf{a})$ (that is, $\exists\mathbf{x} \bigwedge_i x_i = a_i$), and $\mathfrak{A}_\alpha^*$ strongly omits Σ_n, we may then choose $\sigma \in \Sigma_n$ such that $\mathfrak{A}_\alpha^* \models \exists\mathbf{x}(\mathbf{x} = \mathbf{a} \wedge \neg\sigma(\mathbf{x}))$. That is to say, $\mathfrak{A}_\alpha^* \models \neg\sigma(\mathbf{a})$. Then, $\mathfrak{A} \models \neg\sigma(\mathbf{a})$ and the argument is done. $\square$

3.3.7 Remarks. At this point we should make a few remarks on some of the developments we have examined.

(i) The converse of Theorem 3.3.6 also holds for complete theories T, as the reader may verify. Hint: Use the fact that $\exists$ and Q commute with countable disjunctions.

(ii) Bruce [1978b] has improved the omitting types theorem for $\mathscr{L}(Q)$ by showing that the notion of strong omitting may be replaced by an equivalent notion, a notion in which the quexistential string $\overline{Qy}$ may be required to consist only of quantifiers Qy_i (not $\exists y_i$). His proof is a direct one which uses forcing for $\mathscr{L}(Q)$. An alternate syntactic argument can be found in Kaufmann [1979], where there is also an extension of Theorem 3.3.6 which produces models of $\bigwedge_n \neg \overline{Qx_n} \bigwedge \Sigma_n(\mathbf{x}_n)$ in which $\overline{Qx_n}$ may have Q quantifiers in addition to $\exists$ quantifiers. Finally, we remark that these results extend, in fact, to families of $<2^\omega$ sets of formulas, by a corresponding result for first-order logic by Shelah [1978a; Conclusion 5.17B, p. 208]. In this connection the reader should also see Lemma VIII.8.2.2.

3.4. Other Topics

3.4.1 The Infinitary Case. Before we undertake the exposition of the topics to which this section is devoted, we will observe that the reader should also consult Chapter VIII for a discussion of $\mathscr{L}_{\omega_1\omega}$ without Q. That said, we will begin our formal discussion by noting that the logic $\mathscr{L}_{\omega_1\omega}(Q)$ is formed from $\mathscr{L}(Q)$ by allowing the new rule of forming countably infinite conjunctions as long as the resulting formula has only finitely many free variables. In our development we will take $\bigvee$ as a defined symbol. The axioms and rules of inference include those of $\mathscr{L}(Q)$, together with the universal closures of all formulas of the form

$$(\bigwedge) \qquad \bigwedge \Phi \to \phi \quad \text{for all} \quad \phi \in \Phi.$$

The added infinitary rule of inference is

$$\frac{\Gamma \vdash \overline{Q^*y}(\phi \to \theta) \quad \text{all} \quad \theta \in \theta}{\Gamma \vdash \overline{Q^*y}(\phi \to \bigwedge \theta)}$$

for any quexistential string $\overline{Qy}$, where $\overline{Q^*y}$ is formed by replacing Q by Q^* and $\exists$ by $\forall$, in $\overline{Qy}$.

A *fragment* is a set of formulas of $\mathscr{L}_{\omega_1\omega}(Q)$ which is closed under the finitary formula-building operations. In [Ke], these axioms and rules are proved complete for $\mathscr{L}_{\omega_1\omega}(Q)$ and its countable admissible fragments. Observe that for the latter, we show by induction on proofs that if $T \vdash \phi$, then there is a proof *in the fragment* of ϕ from T. Keisler's argument has been abstracted in Barwise [1981] and, roughly speaking, it asserts that for many logics, the omitting types theorem implies a completeness and omitting types theorem for a corresponding infinitary logic. For the details on this, the reader should see Section VIII.6.6. Furthermore, the reader who wishes to examine Keisler's argument in *this* chapter may find it for $\mathscr{L}$(aa) in the proof of Theorem 4.3.4.

The following theorem is interesting even for first-order logic, and a well-written proof of it can be found in Section 5 of [Ke], as well as (in its essentials) in Keisler [1971a, Theorem 45]. As an exercise the reader should prove the analog of this result for $\mathscr{L}_{\omega_1\omega}$(aa) as defined in Section 4 of this chapter.

3.4.2 Theorem. *Let T be a consistent set of sentences of the countable fragment $\mathscr{L}_{\mathscr{A}}(Q)$. Suppose that T has an uncountable model which realizes uncountably many complete $\mathscr{L}_{\mathscr{A}}(Q)$-types in k variables, some $k < \omega$. Then there is a family $\{\mathfrak{A}_f : f \in {}^{(\omega_1)}2\}$ of non-isomorphic models of T. In fact, if $f \neq g$, then $\mathfrak{A}_f$ realizes an $\mathscr{L}_{\mathscr{A}}(Q)$-type which is omitted in $\mathfrak{A}_g$. In particular, a consistent countable theory of $\mathscr{L}_{\omega\omega}$ with uncountably many complete types has 2^{ω_1} models of power ω_1.* □

The next theorem is quite striking and its proof is beyond the scope of this chapter. For extensions of this result see Section XX.3.

3.4.3 Theorem (Shelah [1975c, Theorem 5.7]). *Assume $\diamondsuit_{\omega_1}$, or even (as in later work) $2^\omega < 2^{\omega_1}$. If T is a countable consistent theory of $\mathscr{L}_{\omega_1\omega}(Q)$ containing $Qx(x = x)$ with fewer than 2^{ω_1} models of power ω_1, then T has a model of power ω_2.* □

This result stands in contrast to the situation for $\mathscr{L}$(aa). For more on this the reader should see Remark 4.1.2(v).

The study of admissible fragments $\mathscr{L}_{\mathscr{A}}(Q)$ has been advanced by the work of Harnik–Makkai [1979], and these advances were based on the earlier work of Gregory [1973] and Ressayre [1977]. As concerns Gregory [1973], the reader should consult Section VIII.7.3 of the present volume. The idea is to provide an axiomatization of $\mathscr{L}_{\mathscr{A}}(Q)$ based on the notion "if ϕ holds then ψ is countable." Proofs in this direction involve $\Sigma_{\mathscr{A}}$-saturated models.

Another direction that the study of $\mathscr{L}_{\mathscr{A}}(Q)$ has taken is that of the Robinson-style forcing of Krivine–McAloon [1973] and Bruce [1978b]. Extra predicates are used in the former development, while the latter requires no extra predicates at all. In Bruce–Keisler [1979] one can find applications to the study of "decidable" weak models for $\mathscr{L}_{\mathscr{A}}(Q)$, where the model has domain α (with $\mathscr{A} = L_\alpha$) and Q means "for unboundedly many." This idea of using L_α has been extended in Wimmers [1982] to $\mathscr{L}$(aa) and $\mathscr{L}^{<\omega}$ (see Sections 4 and 5).

In the next two sections some countably compact extensions of $\mathscr{L}(Q)$ are considered.

4. *Filter Quantifiers Stronger Than Q_1: Completeness, Compactness, and Omitting Types*

In this section we will examine extensions of $\mathscr{L}(Q_1)$ that are formed by adding "filter quantifiers" over $P_{\omega_1}(A)$ = the set of countable subsets of A. We will mainly concentrate on $\mathscr{L}$(aa), or "stationary logic". Just as Q_1 refers to the family of uncountable sets, the aa quantifier ("almost all") refers to the family of *closed unbounded* subsets of ω_1, a basic family of study in set theory. For a discussion of closed unbounded sets and their largeness properties, the reader should see Kunen [1980]. This logic was introduced in a slightly different form in Shelah [1975d], where countable compactness and abstract completeness (recursive enumerability for theories) are proved. These properties are also implicit in Schmerl [1976] and, later, in Dubiel [1977a]. The proofs of these properties are related to the argument for $\mathscr{L}$(aa) in Section II.3.2. In a manner analogous to that of Keisler's 1970 paper (see Section 3) as compared to that of Fuhrken [1964] and Vaught [1964], Barwise–Makkai [1976] introduced an explicit set of axioms for $\mathscr{L}$(aa). Their completeness proof and an omitting types theorem can be found in Barwise–Kaufmann–Makkai [1978][2] and Kaufmann [1978a]. These notions form the main part of the present section. We will conclude our exposition with a discussion of some extensions of $\mathscr{L}$(aa).

[2] Henceforth referred to as [BKM].

4.1. Preliminaries

4.1.1 Definition (Stationary Logic ($\mathscr{L}$(aa)) and the Closed Unbounded (cub) Filter). Let τ be any vocabulary. A *τ-formula* of $\mathscr{L}$(aa) is a formula which is built up from atomic τ-formulas and formulas $s_i(x_j)$, by using first-order formation rules and the following rule: If ϕ is a formula so is aa $s_i\ \phi$. The defined quantifier stat is also useful, and, formally stat $s\ \phi$ is $\neg$aa $s\ \neg\phi$.

To define satisfaction, we interpret aa by the cub filter $D(A)$ on $P_{\omega_1}(A)$, an interpretation that is due to Kueker [1972] and Jech [1973]. A collection X of countable subsets of A is *cub* if X is *closed* under unions of countable chains and *unbounded* in $P_{\omega_1}(A)$; that is to say, $(\forall s \in P_{\omega_1}(A))(\exists s' \in X)(s \subseteq s')$. Then $D(A)$ is the filter generated by the cub subfamilies of $P_{\omega_1}(A)$. Satisfaction may now be defined by induction on formulas, with the new clause:

$$\mathfrak{A} \models \text{aa } s\ \phi(s) \quad \text{iff} \quad \{s \in P_{\omega_1}(A) \colon \mathfrak{A} \models \phi(s)\} \in D(A).$$

A sublogic of $\mathscr{L}$(aa) is $\mathscr{L}_{\text{pos}}$ or "positive logic", where one forms aa $s\ \phi$ only if s occurs only positively in ϕ and $\phi \in \mathscr{L}_{\text{pos}}$. For more on this the reader should see Example 3 of Section II.2.2 and Remark 4.1.2(iii) below.

4.1.2 Remarks. We will now gather some facts which serve to clarify the definition just given.

(i) Suppose $|A| = \omega_1$ and $\langle A_\alpha \colon \alpha < \omega_1\rangle$ is a *filtration* of A, that is, we have $A_\alpha = \bigcup\{A_{\beta+1} \colon \beta < \alpha\}$ and $A = \bigcup_\alpha A_\alpha$. Then, for all $X \subseteq P_{\omega_1}(A)$, $X \in D(A)$ iff $\{\alpha < \omega_1 \colon A_\alpha \in X\}$ contains a closed unbounded subset of ω_1. It follows then that if A has domain A, then $\mathfrak{A} \models$ aa $s\ \phi$ iff $\{\alpha \colon \mathfrak{A} \models \phi(A_\alpha)\}$ contains a cub subset of ω_1, and $\mathfrak{A} \models$ stat $s\ \phi$ iff $\{\alpha \colon \mathfrak{A} \models \phi(A_\alpha)\}$ is stationary in ω_1.

(ii) Exercise: For all A, the cub filter on $P_{\omega_1}(A)$ is closed under countable intersections. In fact, even more than this is true, as the reader can confirm by examining the proof of Proposition 4.1.4.

(iii) If s occurs only positively in $\phi(s, \ldots)$ and $\mathfrak{A} \models \phi(t, \mathbf{p})$ for some $t \in P_{\omega_1}(A)$, then $\mathfrak{A} \models \phi(t', \mathbf{p})$ for all $t' \supseteq t$; and, hence, $\mathfrak{A} \models$ aa $s\ \phi$. Hence, $\mathscr{L}_{\text{pos}}$ can be defined using $\exists s$ in place of aa s.

(iv) $\mathscr{L}_{\text{pos}}$ contains $\mathscr{L}(Q_1)$, since $Qx\phi \leftrightarrow \neg$aa $s\ \forall x(\phi(x) \rightarrow s(x))$.

(v) The class of ω_1-like linear orders which continuously embed ω_1, whose members are sometimes called *strongly ω_1-like*, is axiomatized in $\mathscr{L}$(aa) by: "$<$ is a linear order" $\wedge$ $Qx(x = x)$ $\wedge$ aa $s\ \exists x$("$s = \{y \colon y < x\}$"). This is easy to see using (i) above. Hence, Shelah's non-categoricity theorem for $\mathscr{L}(Q_1)$ (Theorem 3.4.3) fails for $\mathscr{L}$(aa). In fact, we just add "$<$ is dense with least element" to get a categorical sentence. The class is not $\mathscr{L}_{\text{pos}}$-axiomatizable: a back-and-forth argument such as is used in Example 6.1.2 shows that all ω_1-like dense linear orders with first element are $\mathscr{L}_{\text{pos}}$-$\equiv$. This example naturally suggests that one could restrict to strongly ω_1-like linear orders and then obtain a first-order version of $\mathscr{L}$(aa). The reader should also see Section II.3.2 for more on this.

Other properties of linear orders can be expressed in $\mathscr{L}$(aa). The following offer two interesting examples in $\mathscr{L}_{\text{pos}}$. $\mathfrak{A}$ is separable iff $\mathfrak{A} \models$ aa s (s is dense), that is to say $\mathfrak{A} \models$ aa $s\ \forall x\ \forall y\ (x < y \rightarrow \exists z(s(z) \wedge x < z \wedge z < y))$, which belongs to $\mathscr{L}_{\text{pos}}$. $\mathfrak{A}$ has cofinality ω iff $\mathfrak{A} \models \text{aa}\, s$ (s is cofinal); that is, $\mathfrak{A} \models \text{aa}\, s\, \forall x\, \exists y \in s\, (x < y)$. In fact, Shelah [1975d] has proved full compactness for such a cofinality quantifier; see Section XVIII.1.3 and Theorem II.3.2.3. None of these classes is axiomatizable in $\mathscr{L}(Q_1)$: see Theorem 6.3.3, Proposition II.7.2.5, and Theorem II.7.2.6.

(vi) Keisler's original counterexample to interpolation in $\mathscr{L}(Q_1)$ shows that the following class $\mathscr{K}$ of models is not $\mathscr{L}(Q_1)$-axiomatizable (see also Section VI.3.1 and II.4.2.8): $\mathscr{K} = \{\mathfrak{A}: \mathfrak{A} = (A, E)$, where E is an equivalence relation on A with countably many equivalence classes$\}$. However, $\mathscr{K}$ is axiomatizable in $\mathscr{L}_{\text{pos}}$ by the sentence "E is an equivalence relation" $\wedge$ aa $s\, \forall x\, \exists y(s(y) \wedge E(x, y))$.

(vii) It is shown in [BKM] that $\mathscr{L}(\text{aa}) \not\leq \mathscr{L}_{\infty\infty}$. In fact, there does not exist κ such that $\mathfrak{A} \equiv_{\infty\kappa} \mathfrak{B} \Rightarrow \mathfrak{A} \equiv_{\mathscr{L}(\text{aa})} \mathfrak{B}$ (Kaufmann [1984]). This should come as no surprise, given Kueker's game-theoretic description of the aa quantifier: If $X \subseteq P_{\omega_1}(A)$, then $X \in D(A)$ iff $\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \ldots \forall x_n\, \exists y_n \ldots (\{x_i: i < \omega\} \cup \{y_i: i < \omega\} \in X)$. (See also Chapter X of the present volume for a discussion of game quantification.)

Another sense in which $\mathscr{L}(\text{aa})$ is strictly stronger than $\mathscr{L}(Q)$ is the sense of Shelah's theorem which asserts that its Hanf number exceeds $\beth_\omega$, the Hanf number for $\mathscr{L}(Q)$; see Theorem V.3.3.11. More on this can be found in Shelah–Kaufmann [198?]. The idea is that, in a sense, $\mathscr{L}(\text{aa})$ can express well-ordering for sufficiently large structures. Notice that the constructions to follow produce models of power at most ω_1.

Alternatively, one can define $D(A)$ to be the set of subsets of κ of power less than κ. This idea was successfully applied to abelian group theory in Eklof–Mekler [1981].

4.1.3 Axioms of $\mathscr{L}(\text{aa})$. For any formula ϕ, call ψ a *quasi-universal closure* of ϕ, if ψ has no free first- or second-order variables, and ψ results by prefixing ϕ with quantifiers of the form aa s and $\forall x$.

The axioms of $\mathscr{L}(\text{aa})$ consist of the quasi-universal closures of the following.

(FO) All axioms and axiom schemas of first-order logic.

(0) $\text{aa}\, s_i \phi(s_i, \ldots) \leftrightarrow \text{aa}\, s_j \phi(s_j, \ldots)$ (s_j not occurring in $\phi(s_i)$).

(1) $\neg\, \text{aa}\, s\, (x \neq x)$.

(2) $\text{aa}\, s\, (x \in s)$.

$\text{aa}\, s_j\, (s_i \subseteq s_j)$ for $i \neq j$.

(3) $\text{aa}\, s\, \phi \wedge \text{aa}\, s\, \psi \rightarrow \text{aa}\, s\, (\phi \wedge \psi)$.

(4) $\text{aa}\, s\, (\phi \rightarrow \psi) \rightarrow (\text{aa}\, s\, \phi \rightarrow \text{aa}\, s\, \psi)$.

(5) $\forall x\, \text{aa}\, s\, \phi(x, s, \ldots) \rightarrow \text{aa}\, s\, \forall x\, (s(x) \rightarrow \phi(x, s, \ldots))$

(6) $\phi \rightarrow \text{aa}\, s\, \phi$, if s is not free in ϕ.

The only rule of inference is *modus ponens*. The reader will observe the similarity here to Keisler's axioms for $\mathscr{L}(Q_1)$ in Definition 3.1.1. In [BKM] there is a rule of aa-generalization, a rule which we do not need because we have taken quasi-universal closures in forming the axioms.

4.1.4 Proposition (Soundness). *If* $T \vdash \phi$ *in* $\mathscr{L}(\mathrm{aa})$, *then* $\mathfrak{A} \models \phi$ *for all* $\mathfrak{A}$.

Proof. It suffices to verify the validity of axioms (1)–(5), since all the others are obviously valid. Axiom 1 says $\varnothing \notin D(A)$; and Axiom 2 is equally clear since $\{t \in P_{\omega_1}(A): s \subseteq t\} \in D(A)$ for all $s \in P_{\omega_1}(A)$. Axioms 3 and 4 are valid because $D(A)$ is a filter. Finally, Axiom 5 is valid because $D(A)$ is closed under *diagonal intersections*, that is, we have that if $\{X_a: a \in A\} \subseteq D(A)$, then $\Delta\{X_a: a \in A\} = \{s \in P_{\omega_1}(A): (\forall a \in s)\, s \in X_a\} \in D(A)$. In fact, the diagonal intersection of cub families is cub, as the reader may verify. $\square$

In order to apply the results of Section 2 on weak models to our development, we may now state the following proposition by way of analogy to Proposition 3.1.2 for $\mathscr{L}(Q_1)$. The proof of this result is routine and will therefore be omitted.

4.1.5 Definition. (i) *The logic* $\mathscr{L}(\mathrm{aa})$ *with the above notion of proof is a logic with concrete syntax in the sense of Section* 2.1, *when we are restricted to formulas in which no second-order variable* s_i *occurs free.*

(ii) *The notion of "*$\mathscr{L}(\mathrm{aa})$*-locally omits" in Definition* 2.2.4 *is equivalent to the usual notion. That is, whenever* $\exists x\phi$ *is* $\mathscr{L}(\mathrm{aa})$*-consistent with* T, *so is* $\exists x(\phi \wedge \neg\sigma)$ *for some* $\sigma \in \Sigma$. $\square$

In light of the above, we may speak of weak models $\mathfrak{A}^*$ for $\mathscr{L}(\mathrm{aa})(\tau)$ when τ is a countable vocabulary. That is to say, we have $\mathfrak{A}^* = \langle \mathfrak{A}, R^{\mathfrak{A}^*}_{aa\, s\phi}\rangle_{\phi \in \mathscr{L}(\mathrm{aa})(\tau)}$. Recall now that τ^+ refers to the vocabulary of $\mathfrak{A}^*$. The reader may have guessed our strategy by now. We will require a main lemma which will show how to witness formulas stat $s\ \phi$ (recall that this means $\neg \mathrm{aa}\ s \neg\ \phi$), much as we witnessed formulas $Qx\phi$ in the $\mathscr{L}(Q_1)$ case. Since s is a second-order variable, we propose to witness stat $s\,\phi(s)$ by having $\phi(A)$ hold. This approach differs slightly from the one in [BKM], where 2-sorted structures are used with interpretations for first-order *and* second-order variables. Instead, we add a predicate symbol for A.

4.2. *Proving the Completeness Theorem for* $\mathscr{L}(\mathrm{aa})$

We begin this section with

4.2.1 Definition. Suppose that τ is any vocabulary and that $\mathfrak{A}^*$ is a countable weak model for $\mathscr{L}(\mathrm{aa})(\tau)$. Let P_A be a unary relation symbol not in τ. We say that

$\mathfrak{B}^*$ is a *precise extension of* $\mathfrak{A}^*$ *relative to* ϕ if $\phi(s)$ is a formula of $\mathscr{L}(\mathrm{aa})(\boldsymbol{\tau})$ with parameters in A, $\mathfrak{B}^*$ *is a* $(\boldsymbol{\tau} \cup P_A)^+$-structure, and

(i) $\mathfrak{A}^* \prec^w \mathfrak{B}^* \upharpoonright \boldsymbol{\tau}^+$;
(ii) if $\mathfrak{A}^* \models \mathrm{stat}\ s\ \phi$, then $\mathfrak{B}^* \models \phi(P_A)$; that is to say, $\mathfrak{B}^* \models (\phi(P_A))^*$ (see Section 2);
(iii) whenever $\mathfrak{A}^* \models \mathrm{aa}\ s\ \psi(s)$ for $\mathrm{aa}\ s\ \psi$ a sentence with parameters in $\mathfrak{A}^*$, then $\mathfrak{B}^* \models \psi(P_A)$;
(iv) $(P_A)^{\mathfrak{B}^*} = A$.

4.2.2 Main Lemma ([BKM, 3.4]). *Suppose that A^* is a countable weak model for $\mathscr{L}(\mathrm{aa})(\boldsymbol{\tau})$ and that $\phi(s, \mathbf{p})$ is a formula of $\mathscr{L}(\mathrm{aa})(\boldsymbol{\tau})$ with parameters $\mathbf{p}$ in A. Then there is a precise extension of $\mathfrak{A}^*$ relative to ϕ.*

Proof. We may assume that $\mathfrak{A}^* \models \mathrm{stat}\ s\ \phi(s, \mathbf{p})$, or else we may replace ϕ by $\forall x(x = x)$. Let $C_A = \{c_a : a \in A\}$ be a set of new constant symbols, and set

$$T_\phi(\mathfrak{A}^*) = \{\theta(\mathbf{c_a}) : \mathfrak{A}^* \models \theta(\mathbf{a})\} \cup \{\phi(P_A, \mathbf{c_p})\} \cup \{\psi(P_A, \mathbf{c_a}) : \mathfrak{A}^* \models \mathrm{aa}\ s\ \psi(s, \mathbf{c_a})\},$$

where $\mathbf{c_a} = \langle c_{a_1} \ldots c_{a_n} \rangle$ if $\mathbf{a} = \langle a_1 \ldots a_n \rangle$. Also set

$$\Sigma_A(x) = \{P_A(x)\} \cup \{x \neq c_a : a \in A\}.$$

Claim A. $T_\phi(\mathfrak{A}^*)$ is an $\mathscr{L}(\mathrm{aa})(\boldsymbol{\tau} \cup P_A)$-consistent theory which $\mathscr{L}(\mathrm{aa})(\boldsymbol{\tau} \cup \{P_A\})$-locally omits $\Sigma_A(x)$.

As in the proof of Lemma 3.2.1, the Main Lemma for $\mathscr{L}(Q_1)$, let us see how the result follows from Claim A. Now the Extension Lemma (2.2.6) (or formally, Lemma 7.2.3) gives us a countable weak model $\mathfrak{B}^*$ for $\mathscr{L}(\mathrm{aa})(\boldsymbol{\tau} \cup \{P_A\})$ such that $\mathfrak{A}^* \prec^w \mathfrak{B}^* \upharpoonright \boldsymbol{\tau}^+$, $\mathfrak{B}^* \models \theta(\mathbf{a})$ whenever $\mathfrak{A}^* \models \theta(\mathbf{a})$, $\mathfrak{B}^* \models \phi(P_A, \mathbf{p})$, and $\mathfrak{B}^* \models \psi(P_A, \mathbf{a})$ whenever $\mathfrak{A}^* \models \mathrm{aa}\ s\ \psi(s, \mathbf{a})$. So (i) through (iii) hold in the definition of precise extension relative to ϕ (Definition 4.2.1). Now Lemma 2.2.6 also allows us to choose $\mathfrak{B}^*$ so that it omits Σ_A, and this guarantees $(P_A)^{\mathfrak{B}^*} \subseteq A$. Since $\mathfrak{A}^* \models \mathrm{aa}\ s(a \in s)$ for all $a \in A$ (Axiom 2), we have that $P(c_a) \in T_\phi(\mathfrak{A}^*)$. So $\mathfrak{B}^* \models P_A(a)$; and hence $A \subseteq (P_A)^{\mathfrak{B}^*}$. Thus, (iv) holds, and $\mathfrak{B}^*$ is the desired precise extension of $\mathfrak{A}^*$ relative to ϕ.

In order to prove Claim A we will use the analogue of Claim B in the proof of Lemma 3.2.1. The proof is essentially the same once we observe that, for every $\boldsymbol{\tau}$-formula $\theta(s, \ldots)$ and every set Γ of $\mathscr{L}(\mathrm{aa})(\boldsymbol{\tau})$-sentences, if $\Gamma \vdash \theta(P_A, \ldots)$ in $\mathscr{L}(\mathrm{aa})(\boldsymbol{\tau} \cup \{P_A\})$, then $\Gamma \vdash \mathrm{aa}\ s\ \theta(s, \ldots)$ in $\mathscr{L}(\mathrm{aa})(\boldsymbol{\tau})$. This follows from an induction, using Axiom 4.

Claim B (Consistency Criterion). For any formula $\theta(s, \mathbf{z})$ of $\mathscr{L}(\mathrm{aa})(\boldsymbol{\tau})$ and $\mathbf{a}$ in A,

(i) $\theta(P_A, \mathbf{c_a})$ is $(\boldsymbol{\tau} \cup \{P_A\})$-consistent with $T_\phi(\mathfrak{A}^*)$ iff $\mathfrak{A}^* \models \mathrm{stat}\ s(\phi(s, \mathbf{c_p}) \wedge \theta(s, \mathbf{c_a}))$.
(ii) $T_\phi(\mathfrak{A}^*) \vdash \theta(P_A, \mathbf{c_a})$ in $\mathscr{L}(\mathrm{aa})(\boldsymbol{\tau} \cup \{P_A\})$ iff $\mathfrak{A}^* \models \mathrm{aa}\ s(\phi(s, \mathbf{p}) \rightarrow \theta(s, \mathbf{a}))$.

It remains to prove Claim A. The consistency criterion implies that $T_\phi(\mathfrak{A}^*)$ is consistent. Now suppose $\exists x\theta(x, P_A, \mathbf{c_a})$ is consistent with $T_\phi(\mathfrak{A}^*)$. If $\exists x(\theta(x, P_A, \mathbf{c_a}) \wedge \neg P_A(x))$ is consistent with $T_\phi(\mathfrak{A}^*)$, then our work is done. Otherwise, we have that $\exists x(\theta(x, P_A, \mathbf{c_a}) \wedge P_A(x))$ is consistent with $T_\phi(\mathfrak{A}^*)$. Thus,

$$\mathfrak{A}^* \models \text{stat } s\, \exists x[\phi(s, \mathbf{p}) \wedge \theta(x, s, \mathbf{a}) \wedge s(x)]$$

by Claim B. Rewriting this as

$$\mathfrak{A}^* \models \neg \text{aa } s\, \forall x \in s \neg\, [\phi(s, \mathbf{p}) \wedge \theta(x, s, \mathbf{a})],$$

we see that Axiom 5 implies that $\mathfrak{A}^* \models \neg\forall x \text{ aa } s \neg[\phi(s, \mathbf{p}) \wedge \theta(x, s, \mathbf{a})]$. That is to say, we have that

$$\mathfrak{A}^* \models \text{stat } s[\phi(s, \mathbf{p}) \wedge \theta(e, s, \mathbf{a})]$$

for some $e \in A$. Then, by using the consistency criterion again, we have that $\theta(c_e, P_A, \mathbf{c_a})$ is consistent with $T_\phi(\mathfrak{A}^*)$. And, hence, $\exists x(\theta(x, P_A, \mathbf{c_a}) \wedge \neg x \neq c_e)$ is also. $\square$

4.2.3 Lemma (Union of Chain Lemma). *Assume that $\langle \mathfrak{A}^*_\alpha : \alpha < \omega_1 \rangle$ is a chain of countable weak models for $\mathscr{L}$(aa) with the following properties.*

(i) *$\mathfrak{A}^*_\alpha$ is a countable weak model for $\mathscr{L}(\text{aa})(\mathbf{\tau}_\alpha)$ for all $\alpha < \omega_1$, where $\mathbf{\tau}_\alpha = \mathbf{\tau} \cup \{P_{A_\beta} : \beta < \alpha\}$.*
(ii) *For all $\alpha < \omega_1$, $\mathfrak{A}^*_{\alpha+1}$ is a precise extension of $\mathfrak{A}^*_\alpha$ relative to ϕ, for some ϕ.*
(iii) *For all $\alpha < \omega_1$ and for every formula $\phi(s, \mathbf{x})$ of $\mathscr{L}(\text{aa})(\mathbf{\tau}_\alpha)$ and for all parameters **a** from A_α, the set $\{\beta < \omega_1 : \mathfrak{A}^*_{\beta+1}$ is a precise extension of $\mathfrak{A}^*_\beta$ relative to $\phi(s, \mathbf{a})\}$ is stationary in ω_1.*
(iv) *The chain is continuous: For all limit $\lambda < \omega_1$ and $\alpha < \lambda$, $\mathfrak{A}^*_\lambda \upharpoonright \mathbf{\tau}^+_\alpha = \bigcup \{\mathfrak{A}^*_\beta \upharpoonright \mathbf{\tau}^+_\alpha : \alpha \le \beta < \lambda\}$.*

*Set $\mathfrak{A} = \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha$. That is, for all $\delta < \omega_1$, $\mathfrak{A} \upharpoonright \mathbf{\tau}_\delta = \bigcup \{\mathfrak{A}_\alpha \upharpoonright \mathbf{\tau}_\delta : \delta \le \alpha < \omega_1\}$. Then, for all $\alpha < \omega_1$, $\mathscr{L}(\text{aa})(\mathbf{\tau}_\alpha)$-formulas $\phi(\mathbf{x})$, and **a** in A_α, $\mathfrak{A} \models \phi(\mathbf{a})$ iff $\mathfrak{A}^*_\alpha \models \phi(\mathbf{a})$.*

Proof. The argument is by induction on the length of ϕ. Notice that $\alpha < \beta < \omega_1$ implies that $\mathfrak{A}^*_\alpha \prec^w \mathfrak{A}^*_\beta \upharpoonright \mathbf{\tau}^+_\alpha$, by Proposition 2.2.2. We first show that $\alpha < \beta < \omega_1$ implies that $(P_{A_\alpha})^{\mathfrak{A}^*_\beta} = A_\alpha$, by induction on β. For $\beta = \alpha + 1$, this is part of the definition of precise extension. Now, $\mathfrak{A}^*_\alpha \models \text{aa } s \text{ aa } t(s \subseteq t)$ by Axiom 2. So $\mathfrak{A}^*_{\alpha+1} \models \text{aa } t(P_{A_\alpha} \subseteq t)$; and, hence, $\mathfrak{A}^*_\gamma \models \text{aa } t(P_{A_\alpha} \subseteq t)$ for all $\gamma \ge \alpha + 1$. Then $\mathfrak{A}^*_{\gamma+1} \models P_{A_\alpha} \subseteq P_{A_\gamma}$ so $(P_{A_\alpha})^{\mathfrak{A}^*_{\gamma+1}} = (P_{A_\alpha})^{\mathfrak{A}^*_\gamma}$, which is A_α by the inductive hypothesis. Limit stages of the induction are clear and we have verified that $(P_{A_\alpha})^{\mathfrak{A}^*_\beta} = A_\alpha$ for all $\alpha < \beta < \omega_1$.

Clearly, $\mathfrak{A} \models \phi(\mathbf{a})$ iff $\mathfrak{A}^*_\alpha \models \phi(\mathbf{a})$ for atomic ϕ. The $\vee$ and $\neg$ steps are trivial, while the $\exists$ step presents no problems. For $\phi(\mathbf{a}) = \text{aa } s\, \psi(s, \mathbf{a})$, suppose that $\mathfrak{A}^*_\alpha \models \phi(\mathbf{a})$. Then $\mathfrak{A}^*_\beta \models \phi(\mathbf{a})$ for all $\beta > \alpha$, so that $\mathfrak{A}^*_{\beta+1} \models \psi(P_{A_\beta}, \mathbf{a})$ for all $\beta > \alpha$. By the inductive hypothesis, we have that $\mathfrak{A} \models \psi(P_{A_\beta}, \mathbf{a})$ for all $\beta > \alpha$.

Hence, $\mathfrak{A} \models \text{aa}\, s\, \psi(s, \mathbf{a})$. As to other direction, we suppose that $\mathfrak{A}_\alpha^* \models \neg \text{aa}\, s\, \psi(x, \mathbf{a})$. That is, we suppose that $\mathfrak{A}_\alpha^* \models \text{stat}\, s \neg \psi(s, \mathbf{a})$. Then, $\mathfrak{A}_\beta^* \models \text{stat}\, s \neg \psi(s, \mathbf{a})$ for all $\beta \geq \alpha$. But hypothesis (iii) implies that $\{\beta \geq \alpha \colon \mathfrak{A}_{\beta+1}^* \models \neg \psi(A_\beta, \mathbf{a})\}$ is stationary in ω_1. Since this set equals $\{\beta \geq \alpha \colon \mathfrak{A} \models \neg \psi(A_\beta, \mathbf{a})\}$ by the inductive hypothesis, we must have that $\mathfrak{A} \models \text{stat}\, s \neg \psi(s, \mathbf{a})$ by Remark 4.1.2(ii). That is, $\mathfrak{A} \models \neg \text{aa}\, s\, \psi(s, \mathbf{a})$. ☐

4.2.4 Theorem (Completeness Theorem for $\mathscr{L}(\text{aa})$ [BKM]). *Suppose T is a set of sentences of $\mathscr{L}(\text{aa})(\tau)$, where τ is a countable vocabulary. Then T is $\mathscr{L}(\text{aa})(\tau)$-consistent iff T has a model.*

Proof. The direction $\Leftarrow$ is Proposition 4.1.4 (Soundness). Now suppose that T is $\mathscr{L}(\text{aa})(\tau)$-consistent. For each $\alpha \leq \omega_1$, set $\tau_\alpha = \tau \cup \{P_{A_\beta} \colon \beta < \alpha\}$. By a theorem of Ulam [1930] (see, for instance, Kunen [1980, p. 79]), there is a partition of ω_1 into disjoint stationary sets X_ϕ, where ϕ ranges over formulas $\phi(s)$ of $\mathscr{L}(\text{aa})(\tau_{\omega_1})$ with parameters in ω_1. Define $\langle \mathfrak{A}_\alpha^* \colon \alpha < \omega_1 \rangle$, each $\mathfrak{A}_\alpha^*$ a countable weak model for $\mathscr{L}(\text{aa})(\tau_\alpha)$, by induction on α as follows, where $A_\alpha \subseteq \omega_1$ for all $\alpha < \omega_1$. Let $\mathfrak{A}_0^*$ be a countable weak model for $\mathscr{L}(\text{aa})(\tau)$ which satisfies T, and for each α, if $\alpha \in X_\phi$ let $\mathfrak{A}_{\alpha+1}^*$ be a precise extension of $\mathfrak{A}_\alpha^*$ relative to ϕ, by the Main Lemma (4.2.2). We take unions at limits.

We now set $\mathfrak{A} = \bigcup_{\alpha < \omega_1} \mathfrak{A}_\alpha$. Since $\mathfrak{A}_0^* \models T$, then $\mathfrak{A} \models T$ by the union of chain Lemma 4.2.3. ☐

4.2.5 Corollary. (i) *$\mathscr{L}(\text{aa})$ is countably compact.*

(ii) *Every consistent countable theory of $\mathscr{L}(\text{aa})$ has a model of power at most $\aleph_1$.* ☐

In connection with this corollary, it is interesting to observe that whether or not $\mathscr{L}(\text{aa})$-elementary submodels must exist is independent. This fact has been proved by Harrington, Kunen, and Shelah (see [BKM], Footnote 2, p. 221).

As is true for $\mathscr{L}(Q_1)$, the study of $\mathscr{L}(\text{aa})$ was partly motivated by the study of end extension of linear orders and models of set theory. By analogy with Theorem 3.2.5(ii), we could reverse history by proving a relativized version of the Main Lemma (4.2.2) to obtain the following theorem of Hutchinson [1976a].

4.2.6 Theorem. *Let $\mathfrak{A} = (A, E)$ be a countable model of* ZF, *possibly excepting power set and the collection schema. For every regular cardinal a of $\mathfrak{A}$, there exists $\mathfrak{B} = (B, F) \succ \mathfrak{A}$ such that $b_E = b_F$ for all bEa, but $\langle a_F - a_E, F \restriction a_F - a_E \rangle$ has a least element. Moreover we may require $\langle a_F, F \restriction a_F \rangle$ to be ω_1-like and embed ω_1 continuously.*

Hint of Proof. Define $\mathfrak{A}^* \models \text{aa}\, s\, \varphi$ iff $\mathfrak{A} \models \exists C$ (C is cub in a and $\forall \gamma \in C(\varphi(\gamma))$). ☐

4.2.7 Remark. Probably the closest known analogue of Theorem 3.2.5(i) is obtained by adding a quantifier "aa α" to the language of set theory, as studied independently by Kaufmann [1983] and Kakuda [1980]. See, for example, Kaufmann [1983, 2.16 and 5.8]. A combination of Peano arithmetic and $\mathscr{L}(Q^2)$, a topic that is discussed in Section 5, has been studied in Macintyre [1980], in Morgenstern [1982], and in Schmerl–Simpson [1982]. The reader should also see Schmerl [1982] for an extension.

4.3. Omitting Types and Infinitary Completeness

As in Section 3, we now extend the completeness theorem to obtain omitting types and infinitary completeness theorems. In fact, the proofs are direct descendents of Keisler's proofs for $\mathscr{L}(Q)$.

4.3.1 Definition ([BKM]). Let T be a set of $\mathscr{L}(\mathrm{aa})(\tau)$-sentences and $\Sigma(\mathbf{x}, \mathbf{t})$ a set of $\mathscr{L}(\mathrm{aa})(\tau)$-formulas in finitely many free variables $x_1 \ldots x_m$, $t_1 \ldots t_n$. Let S be any quantifier string composed of quantifiers stat s_i and stat t_i, where $i < j$ implies that stat t_i occurs only before stat t_j. T strongly omits Σ if for every such S and every formula $S\,\exists x\,\phi(\mathbf{x}, \mathbf{s}, \mathbf{t})$ which is $\mathscr{L}(\mathrm{aa})(\tau)$-consistent with T, then $S\,\exists\mathbf{x}(\phi \wedge \neg\sigma)$ is consistent with T for some $\sigma \in \Sigma$. (Notice that we've fixed an ordering $\langle t_1, \ldots, t_n\rangle$ of the second-order free variables of Σ.) We say that $\mathfrak{A}$ *omits* Σ *if* $\mathfrak{A} \models \mathrm{aa}\, t_1 \ldots \mathrm{aa}\, t_n \forall\mathbf{x} \bigvee_{\delta \in \Sigma} \neg\sigma(\mathbf{x}, \mathbf{t})$.

4.3.2 Theorem (Omitting Types Theorem for $\mathscr{L}(\mathrm{aa})$ [BKM, 4.2]). *Let τ be countable and suppose that T is a consistent τ-theory of $\mathscr{L}(\mathrm{aa})$ which strongly omits sets $\Sigma_n(\mathbf{x}_n, \mathbf{t}_n)$, where $n < \omega$. Then T has a model which omits each Σ_n. The converse also holds if T is complete.*

Hint of Proof. Let us merely remark that Lemmas 3.3.4 and 3.3.5 have straightforward translations into $\mathscr{L}(\mathrm{aa})$, and the case $t_n = \varnothing$ for all n follows just as Theorem 3.3.6 follows for $\mathscr{L}(Q_1)$. ☐

4.3.3 Definition. The *logic* $\mathscr{L}_{\omega_1\omega}(\mathrm{aa})$ is formed from $\mathscr{L}(\mathrm{aa})$ by allowing the new rule of forming countably infinite conjunctions, as long as the resulting formula has only finitely many free variables. The new axioms are the quasi-universal closures of

$$(\textstyle\bigwedge) \qquad \bigwedge \Phi \to \phi \quad \text{for all} \quad \phi \in \Phi;$$

and the new rules of inference are

$$\frac{\Gamma \vdash S^*(\phi \to \theta) \quad \text{all} \quad \theta \in \theta}{\Gamma \vdash S^*(\phi \to \bigwedge \theta)}$$

for any quantifier string S consisting only of quantifiers of the form stat s or $\exists x$. Here, S^* results from S by changing each stat to aa and each $\exists$ to $\forall$.

"Countable fragment" is defined as for $\mathscr{L}(Q_1)$ in Definition 3.4.1), as is the notion of $\mathscr{L}_{\mathscr{A}}(\mathrm{aa})$-consistency, that is, consistency with respect to proofs consisting of $\mathscr{L}_{\mathscr{A}}$(aa)-formulas. If the fragment is admissible, then the following theorem can be extended to give Barwise completeness and compactness.

4.3.4 Theorem (Completeness and Omitting Types Theorems for $\mathscr{L}_{\omega_1\omega}$(aa), [BKM, 4.6]). *If T is a consistent theory of a countable fragment $\mathscr{L}_{\mathscr{A}}$(aa), then T has a model. If in addition T strongly omits sets Σ_n,* where $n < \omega$, *then T has a model which omits each Σ_n.*

Proof. We will reduce to finitary logic in a manner analogous to that of Definition 2.1.2, except that we do not need to eliminate the "aa" quantifier, since we already have a completeness theorem for $\mathscr{L}$(aa). Rather, we will replace each infinitary conjunction by an atomic formula. Formally, we define a map "prime" (′) from formulas of $\mathscr{L}_{\mathscr{A}}$(aa) to $\mathscr{L}$(aa)($\boldsymbol{\tau}$), where

$$\boldsymbol{\tau} = [\text{vocabulary of } \mathscr{L}_{\mathscr{A}}(\text{aa})] \cup \{R_{\bigwedge\Phi}(\mathbf{x}, \mathbf{t}) \colon \bigwedge \Phi(\mathbf{x}, \mathbf{t}) \in \mathscr{L}_{\mathscr{A}}(\text{aa})\}.$$

We set $\phi' = \phi$ for atomic ϕ; and we set $(\phi \wedge \psi)' = \phi' \wedge \psi'$, $(\neg\phi)' = \neg\phi'$, $(\exists x\phi)' = \exists x\phi'$; and $(\text{aa}\ s\ \phi)' = \text{aa}\ s\ \phi'$; and, finally,

$$(\bigwedge \Phi)' = R_{\bigwedge\Phi}(x_1 \cdots x_m, t_1 \ldots t_n),$$

where $x_1 \ldots x_m$ (resp. $t_1 \ldots t_n$) enumerates the first-order (resp. second-order) free variables of Φ in order of subscript. "Prime" almost has an inverse, "minus": namely, $\phi^- = \phi$ for atomic $\phi \in \mathscr{L}_{\mathscr{A}}$(aa), and "minus" commutes with the finitary connectives and quantifiers; and

$$[R_{\bigwedge\Phi}(\tau_1 \ldots \tau_m, s_1 \ldots s_n)]^- = \bigwedge \Phi\begin{pmatrix} x_1 & \ldots & x_m & t_1 & \ldots & t_n \\ \tau_1 & & \tau_m & s_1 & & s_n \end{pmatrix},$$

where $\Phi, x_1 \ldots x_m, t_1 \ldots t_n$ are as above. It is clear that $(\phi')^- = \phi$ for all $\phi \in \mathscr{L}_{\mathscr{A}}$(aa).

Let $T' = \{\phi' : T \vdash \phi \text{ and } \phi \in \mathscr{L}_{\mathscr{A}}(\text{aa})\} \cup \{S^*[(\phi(\mathbf{x}, \mathbf{t})^-)' \leftrightarrow \phi(\mathbf{x}, \mathbf{t})] : \phi \in \mathscr{L}(\text{aa})(\boldsymbol{\tau})$, S^* consists of quantifiers aa t, $\forall x\}$. ☐

Claim 1. *For every $\phi \in \mathscr{L}_{\mathscr{A}}$(aa, $T \vdash \phi$ iff $T' \vdash \phi'$.* The forward implication is clear. For the converse direction, we verify that if p is a proof from axioms of T' in $\mathscr{L}$(aa)($\boldsymbol{\tau}$) and p^- results from p by replacing each formula ψ in p by ψ^-, then p^- is a proof in $\mathscr{L}_{\mathscr{A}}$(aa) from axioms of T. We omit the details of the argument. So if $T' \vdash \phi'$, then $T \vdash (\phi')^-$. That is, $T \vdash \phi$.

Claim 2. *If T strongly omits Σ, then T' strongly omits $\Sigma' = \{\sigma' : \sigma \in \Sigma\}$.* For suppose that $S\,\exists\mathbf{x}\,\phi$ is consistent with T', for appropriate S. Then $S\,\exists\mathbf{x}\,\phi^-$ is consistent with T, by Claim 1. So for some $\sigma \in \Sigma$, $S\,\exists\mathbf{x}\,(\phi^- \wedge \neg\sigma)$ is consistent with T. And, hence, by using Claim 1 again we have that $S\,\exists\mathbf{x}\,((\phi^-)' \wedge \neg\sigma')$ is consistent with T'. But since $S^*\,\forall\mathbf{x}\,[(\phi^-)' \leftrightarrow \phi]$ is an axiom of T', we must have that $S\,\exists\mathbf{x}\,(\phi \wedge \neg\sigma')$ is consistent with T', as desired.

Claim 3. *T strongly omits the set $\Sigma_{\bigwedge \Phi} = \{\neg \bigwedge \Phi\} \cup \Phi$, for each $\bigwedge \Phi \in \mathscr{L}_{\mathscr{A}}(\mathrm{aa})$.* To see this, suppose that we are given a sentence $S\, \exists \mathbf{x}\, \psi(\mathbf{x}, \mathbf{s}, \mathbf{t})$ which witnesses that T does not strongly omit $\Sigma_{\bigwedge \Phi}$. Then, for all $\phi \in \Phi$, $T \vdash S^* \forall \mathbf{x}\, (\psi \to \phi)$. By the infinitary rule of inference, we deduce that $T \vdash S^* \forall \mathbf{x}\, (\psi \to \bigwedge \Phi)$. But also $\neg \bigwedge \Phi \in \Sigma_{\bigwedge \Phi}$. Hence, by choice of $S\, \exists \mathbf{x}\, \psi$, $T \vdash S^* \forall \mathbf{x}\, (\psi \to \neg \bigwedge \Phi)$. It thus follows that $S\, \exists \mathbf{x}\, \psi$ is not consistent with T, which contradicts our choice of this sentence.

Now by Claim 1, T' is consistent, and by Claims 2 and 3, T' strongly omits $(\Sigma_n)'$ and each $(\Sigma_{\bigwedge \Phi})'$ for $\bigwedge \Phi \in \mathscr{L}_{\mathscr{A}}(\mathrm{aa})$. Let $\mathfrak{A}$ be a model of T' which omits each $(\Sigma_n)'$ and each $(\Sigma_{\bigwedge \Phi})'$, by Theorem 4.3.2 (the omitting types theorem for $\mathscr{L}(\mathrm{aa})$). The theorem now follows from the following claim.

Claim 4. *For all $\phi(\mathbf{x}, \mathbf{t}) \in \mathscr{L}_{\mathscr{A}}(\mathrm{aa})$, $\mathfrak{A} \models \mathrm{aa}\, \mathbf{t}\, \forall \mathbf{x}\, [\phi(\mathbf{x}, \mathbf{t}) \leftrightarrow \phi'(\mathbf{x}, \mathbf{t})]$.* The proof is by induction on ϕ. Although the details are left as an exercise, some hints are put forward in the following discussion. For the $\bigwedge$ step, one should at some point observe: $\models \bigwedge_{i \in I} \mathrm{aa}\, \mathbf{t}\, \theta_i \leftrightarrow \mathrm{aa}\, \mathbf{t} \bigwedge_{i \in I} \theta_i$ if I is countable; and

$$\mathfrak{A} \models \mathrm{aa}\, \mathbf{t}\, \forall \mathbf{x} \left[\bigwedge_{\phi \in \Phi} \phi' \to R_{\bigwedge \Phi}(\ldots) \right],$$

since $\mathfrak{A}$ omits $(\Sigma_{\bigwedge \Phi})'$. For the "aa" step, $\mathrm{aa}\, s\, \phi(\mathbf{x}, \mathbf{t}, \mathbf{s})$, one uses $\models \mathrm{aa}\, \mathbf{t}\, \mathrm{aa}\, s\, \forall \mathbf{x}\, \psi \to \mathrm{aa}\, \mathbf{t}\, \forall \mathbf{x}\, \mathrm{aa}\, s\, \psi$, where ψ is the formula $\phi(\mathbf{x}, \mathbf{t}, \mathbf{s}) \leftrightarrow \phi'(\mathbf{x}, \mathbf{t}, \mathbf{s})$, together with

$$\models \mathrm{aa}\, s\, [\phi \leftrightarrow \phi'] \to [\mathrm{aa}\, s\, \phi \leftrightarrow \mathrm{aa}\, s\, \phi']. \quad \square$$

4.4. Other Filters

The completeness and compactness theorems of $\mathscr{L}(\mathrm{aa})$ were extended by Kaufmann [1981] to logics $\mathscr{L}^{\mathscr{F}}(\mathrm{aa}, \mathrm{M})$. The quantifier M (="most") is interpreted using a filter $\mathscr{F}$ on ω_1 which contains every cub subset of ω_1. That is, for $|A| = \omega_1$ and any filtration $\langle A_\alpha : \alpha < \omega_1 \rangle$ of A, $\mathfrak{A} \models \mathrm{M}\, s\, \phi(s)$ iff $\{\alpha < \omega_1 : \mathfrak{A} \models \phi(A_\alpha)\} \in \mathscr{F}$. Thus, for example, M is really just aa if $\mathscr{F}$ is just $\mathscr{F}^{\mathrm{cub}} = \{X \subseteq \omega_1 : X \supseteq Y$ for some cub $Y\}$. Some of these results are summarized in:

4.4.1 Theorem *Suppose that $\mathscr{F}$ is a countably complete filter on ω_1. Then $\mathscr{L}^{\mathscr{F}}(\mathrm{aa}, \mathrm{M})$ is countably compact and recursively enumerable for consequence. Moreover, for countably complete $\mathscr{F}$ and $\mathscr{G}$, $\mathscr{L}^{\mathscr{F}}(\mathrm{aa}, \mathrm{M})$ and $\mathscr{L}^{\mathscr{G}}(\mathrm{aa}, \mathrm{M})$ have the same valid sentences iff either*

(i) *$\mathscr{F} = \mathscr{G} = \mathscr{F}^{\mathrm{cub}}$; or*
(ii) *$\mathscr{F} \neq \mathscr{F}^{\mathrm{cub}}$, $\mathscr{G} \neq \mathscr{F}^{\mathrm{cub}}$, but $\mathscr{F}$ and $\mathscr{G}$ are both closed under diagonal intersections; or*
(iii) *no diagonal intersection from $\mathscr{F}$ is empty, and the same holds for $\mathscr{G}$, but $\mathscr{F}$ and $\mathscr{G}$ are not closed under diagonal intersections; or*
(iv) *neither $\mathscr{F}$ nor $\mathscr{G}$ belongs to the classes described in* (i), (ii), *or* (iii) *above.* $\square$

Other filters $\mathscr{F}$ give compact logics. As an example, countable compactness holds for any regular ultrafilter $\mathscr{F}$ on ω_1 such that $\mathscr{F} \supseteq \mathscr{F}^{\text{cub}}$ (see Kaufmann [1981, 3.14]). We know of no filter $\mathscr{F} \supseteq \mathscr{F}^{\text{cub}}$, in fact, for which $\mathscr{L}^{\mathscr{F}}(\text{aa}, \text{M})$ is not countably compact. What about filters $\mathscr{F} \not\supseteq \mathscr{F}^{\text{cub}}$? In [BKM, 7.1], we find the "eventual filter" $\mathscr{F}^{\text{ev}} = \{X \subseteq P_{\omega_1}(\omega_1)$: for some s_0, we have $s \in X$ for all $s \supseteq s_0$ $(s \subseteq \omega_1)\}$. By [BKM, 7.2] the corresponding logic $\mathscr{L}^{\mathscr{F}}(\text{M})$ is not countably compact. However, if $\mathscr{H}$ is the filter generated by all collections of the form $\{s - F : s \in X, F$ is finite, $F \subseteq \omega_1\}$, then the resulting logic $\mathscr{L}^{\mathscr{H}}(\text{M})$is countably compact and axiomatizable even though $\mathscr{L}^{\mathscr{H}}(\text{aa}, \text{M})$ is not (see Kaufmann [1981b, Example C, p. 189]).

5. *Extensions of $\mathscr{L}(Q_1)$ by Quantifiers Asserting the Existence of Certain Uncountable Sets*

5.1. *Preliminaries*

In Section 4 we considered an extension $\mathscr{L}(\text{aa})$ of $\mathscr{L}(Q_1)$ in which one could quantify over countable sets. A simple piece of $\mathscr{L}(\text{aa})$, $\mathscr{L}_{\text{pos}}$, was presented in which we can assert $\exists s\phi(s)$ when s occurs only positively in ϕ. A related logic is "negative logic" $\mathscr{L}_{\text{neg}}$, which is defined below. Now $\mathscr{L}_{\text{neg}}$ is not countably compact (Theorem 5.1.2), which is perhaps surprising, since it looks like a rather small extension of the logic $\mathscr{L}^{<\omega}$ of Magidor–Malitz [1977a][3], which *is* countably compact assuming $\diamondsuit$ (Corollary 5.2.6). That done, we will examine some related quantifiers of Malitz–Rubin [1980] and of Shelah [1978d].

5.1.1 Definition. The logic $\mathscr{L}_{\text{neg}}$ is formed from the atomic formulas by closing under $\neg$, $\vee$, $\exists x$, and a second-order quantifier $\exists X$: if X is a unary relation symbol which occurs only negatively in ϕ, where $\phi \in \mathscr{L}_{\text{neg}}$, then $\exists X\phi \in \mathscr{L}_{\text{neg}}$. Hence, we allow $\forall X\phi$ when X occurs only positively in ϕ and $\phi \in \mathscr{L}_{\text{neg}}$. The interpretation of $\exists X$ is given by: $A \models \exists X\phi$ iff $(A, X) \models \phi$ for some uncountable $X \subseteq A$. Notice that $\mathscr{L}_{\text{neg}}$ contains $\mathscr{L}(Q_1)$, since $Qx\phi(x) \leftrightarrow \exists X\, \forall x(X(x) \to \phi(x))$.

5.1.2 Theorem (Stavi and Malitz, Independently). *The class*

$$\{(A, <): (A, <) \cong (\omega_1, \in)\}$$

is RPC *in* $\mathscr{L}_{\text{neg}}$. *Hence*, $\mathscr{L}_{\text{neg}}$ *is not countably compact.*

Proof. Let ϕ be the conjunction of the following: a sufficiently large finite amount of set theory (for the argument below); $\forall x(U(x) \leftrightarrow x \in \omega_1)$; and the sentence ψ,

[3] Henceforth we will write [M[2]] for Magidor–Malitz [1977].

where ψ says that U is ω_1-like in the real world and that every uncountable subset of U (in the real world) contains a subset internal to the model. Formally, ψ is

$$Qx(x \in \omega_1) \wedge \forall\alpha \in \omega_1 \neg Qx(x \in \alpha)$$
$$\wedge \forall X[X \subseteq \omega_1 \rightarrow \exists y(|y| = \omega_1 \wedge y \subseteq X)].$$

Now if $(A, E) \models \phi$ and some reasonable set theory holds in (A, E), then $(\omega_1^{\mathfrak{A}}, E \restriction \omega_1^{\mathfrak{A}})$ is well-ordered. For, in (A, E) we let X be any strictly increasing ω_1-sequence that is cofinal in $\omega_1^{\mathfrak{A}}$. If $y \subseteq X$, then y is also well-ordered. Now, choose $y \in A$, witnessing ψ. Then the transitive collapse of y in $\mathfrak{A}$, which must be ω_1 in $\mathfrak{A}$, is well-ordered. □

Note: Suppose that there is an "almost disjoint" family of $\aleph_2$ subsets of ω_1, that is, every pair has countable intersection; this, of course, is the case if CH holds. Then the argument above shows that although ϕ has an uncountable model, the argument above shows that ϕ has no model of power at most $\aleph_1$. Hence, the following logic is *properly* contained in $\mathscr{L}_{\text{neg}}$, as the reader can easily verify by using the exercise which immediately precedes Proposition 1.3.1.

5.1.3 Definition ([M²]). The logic $\mathscr{L}^{<\omega} = \mathscr{L}(Q, Q^2, Q^3, \ldots, Q^n, \ldots)$ is obtained by closing the atomic formulas under $\neg$, $\vee$, $\exists x$, and the quantifiers Q^n: If ϕ is a formula of $\mathscr{L}^{<\omega}$ so is $Q^n x_1 x_2 \ldots x_n \phi$. The semantics are defined with the new rule: $\mathfrak{A} \models Q^n x_1 \ldots x_n \phi(\mathbf{x})$ iff for some uncountable $X \subseteq A$, $\mathfrak{A} \models \phi(a_1 \ldots a_n)$ for all distinct $a_1, \ldots, a_n \in X$; that is, "there is an uncountable homogeneous set for ϕ". This is really a definition of $\mathscr{L}(Q_1, Q_1^2, \ldots, Q_1^n, \ldots)$. A compactness theorem for the $\aleph_\alpha$-interpretation is proved in Shelah [1981a] for $\alpha = \lambda^+$, assuming $\diamond_\alpha$ and $\diamond_\lambda$; see Section V.8. See also Remark 4.2.7 and VII.1, 2, and 5 for "applied" results on the $\mathscr{L}(Q_\alpha^n)$. Notice that if $i < j$ then Q^i is definable in terms of Q^j, that is:

$$Q^i x_1 \ldots x_i \phi(x_1 \ldots x_i) \leftrightarrow Q^j x_1 \ldots x_j \phi(x_1 \ldots x_i).$$

Let $\mathscr{L}(Q^n)$ denote the restriction of $\mathscr{L}^{<\omega}$ to the quantifiers $Q, Q^2, \ldots, Q^n$.

Recall that in $\mathscr{L}$(aa) we may axiomatize the class of models $\mathfrak{A} = (A, E)$ such that E is an equivalence relation on A with only countably many equivalence classes. This is also possible in $\mathscr{L}(Q^2)$ using the sentence $\neg Q^2 xy \neg E(x, y)$. Hence, $\mathscr{L}(Q^2)$ is also a proper extension of $\mathscr{L}(Q)$. In fact, Garavaglia [1978b] has shown in ZFC that $\mathscr{L}(Q^n)$-equivalence does not imply $\mathscr{L}(Q^{n+1})$-equivalence, and it is shown in Rubin–Shelah [1983] that $\{\mathfrak{A} : \mathfrak{A} \models \neg Q^{n+1} x_1 \ldots x_{n+1} R(x_1 \ldots x_{n+1})\}$ is not the class of reducts of models of a countable $\mathscr{L}(Q^n)$-theory, assuming $\diamond_{\omega_1}$. However, while satisfiability is absolute in $\mathscr{L}$(aa) (by the completeness theorem), this is not the case for $\mathscr{L}(Q^2)$:

5.1.4 Example ([M²]). A *Suslin-like tree* is an ω_1-tree $(T, <, \leq)$ (also see Section V.3.3) such that:

(i) there is no branch; that is, $\neg Q^2 xy(x < y \vee y < x)$, and
(ii) there is no uncountable antichain; that is, $\neg Q^2 xy(\neg x < y \wedge \neg y < x)$.

It is easy to see that there exists a Suslin-like tree iff there exists a Suslin tree. However, the latter is independent of ZFC. Therefore, satisfiability of $\mathscr{L}(Q^2)$ sentences is not absolute for models of ZFC. ▯

5.2. *The Magidor–Malitz Completeness Theorem*

The next goal is to prove completeness for $\mathscr{L}(Q^2)$. The same idea works for $\mathscr{L}^{<\omega}$, although the notation there is more involved. That being so, we will only indicate how the argument for $\mathscr{L}(Q^2)$ extends to $\mathscr{L}(Q^3)$ (in Section 5.2.5), rather than to all of $\mathscr{L}^{<\omega}$. Sections 5.2.1 through 5.2.6 are adapted from [M²].

5.2.1 Axioms of $\mathscr{L}(Q^2)$. An *acceptable* vocabulary τ is a vocabulary which contains a $(|\mathbf{z}| + 1)$-ary predicate symbol $P_{\phi, x, y, \mathbf{z}}$ for all formulas ϕ which do not contain constants, and for all distinct x, y, $\mathbf{z}$. We will feel free to write P_ϕ for $P_{\phi, x, y, \mathbf{z}}$, when the variables are understood. The axioms for $\mathscr{L}(Q^2)$ include the universal closures of [0]–[6] below. Notice that [0]–[4] are exactly the $\mathscr{L}(Q)$ schemas (see Definition 3.1.1). Fix an acceptable vocabulary.

[0] All first-order axiom schemas.

[1] $\neg Qx(x = y \vee x = z)$.

[2] $\forall x[\phi \to \psi] \to (Qx\phi \to Qx\psi)$.

[3] $Qx\phi(x) \leftrightarrow Qy\phi(y)$, where $\phi(x)$ is a formula in which y does not occur.

[4] $Qy\,\exists x\phi \to \exists xQy\phi \vee Qx\,\exists y\phi$.

[5] "Witnessing schema": this axiom schema says that $P_{\theta, x_1, x_2, \mathbf{z}}(x, \mathbf{a})$ provides a witness to $Q^2x_1x_2\,\theta(x_1, x_2, \mathbf{a})$:

$$[Q^2x_1x_2\,\theta(x_1, x_2, \mathbf{y}) \to QxP_{\theta, x_1, x_2, \mathbf{y}}(x, \mathbf{y})]$$
$$\wedge\ [QxP_{\theta, x_1, x_2, \mathbf{y}}(x, \mathbf{y}) \to \forall x_1\,\forall x_2[P_{\theta, x_1, x_2, \mathbf{y}}(x_1, \mathbf{y})$$
$$\wedge\ P_{\theta, x_1, x_2, \mathbf{y}}(x_2, \mathbf{y}) \wedge x_1 \neq x_2 \to \theta(x_1, x_2, \mathbf{y})]].$$

And, finally, there is the following schema, a schema that is both difficult to describe and hard to look at (hence the name "Medusan"). For now, think of it as saying that ψ produces a homogeneous set for θ. What this actually means will become clearer in the proof of soundness which follows. Moreover, the origin of these axioms will be explained in the proof of completeness.

[6] "Medusan axioms": Let $\overline{Qy}$ be a quexistential string, that is, a string of quantifiers of the form Qy_i or $\exists y_i$. Also let $\overline{Q^*y}$ be the result of replacing each Qy_i and $\exists y_i$ by Q^*y_i and $\forall y_i$, respectively. Then

$$\overline{Qy}\,\exists x\psi(x, \mathbf{y}) \wedge \overline{Q^*y}\,\forall x[\psi(x, \mathbf{y}) \to \overline{Q^*y'}\,\forall x'(\psi(x', \mathbf{y}')$$
$$\to x' \neq x \wedge \theta(x', x) \wedge \theta(x, x'))] \to Q^2x_1x_2\,\theta(x_1, x_2)$$

is an axiom, whenever all variables in the list x, $\mathbf{y}$, x', $\mathbf{y}'$ are distinct.

The rule of inference is *modus ponens*, and, as usual, we can check that universal generalization is a derived rule.

Clearly every τ-structure may be expanded to a σ-structure for some acceptable $\sigma \supseteq \tau$ so that schema [5] holds, where we may assume that no P_θ is in τ. Hence, soundness follows from

5.2.2 Proposition. *The "Medusan axioms" [6] are valid.*

Proof. Suppose $\overline{Qy}$ is quexistential and

$$(1) \qquad \mathfrak{A} \models \overline{Qy}\, \exists x \psi(x, \mathbf{y}),$$

$$(2) \qquad \mathfrak{A} \models \overline{Q^*y}\, \forall x[\psi(x, \mathbf{y}) \to \eta(x)],$$

where $\eta(x)$ is $\overline{Q^*y'}\, \forall x'[\psi(x', \mathbf{y}') \to \theta(x, x') \wedge \theta(x', x) \wedge x \neq x']$. We will construct a homogeneous set $\{x_\alpha : \alpha < \omega_1\}$ for θ by induction on α with inductive hypotheses (a) $\mathfrak{A} \models \eta(x_\beta)$, (b) $\mathfrak{A} \models \theta(x_\beta, x_\gamma) \wedge \theta(x_\gamma, x_\beta) \wedge x_\beta \neq x_\gamma$, all $\gamma < \beta \leq \alpha$. To define x_0, notice that (1) and (2), together with an appropriate intersection principle (Lemma 3.1.5), combine to yield $\mathfrak{A} \models \exists \mathbf{y}\, \exists x[(\psi(x, \mathbf{y}) \to \eta(x)) \wedge \psi(x, \mathbf{y})]$. It follows then that $\mathfrak{A} \models \exists x \eta(x)$; choose x_0 such that $\mathfrak{A} \models \eta(x_0)$.

Now, suppose that we have x_β for all $\beta < \alpha$, where the inductive hypotheses hold for all $\beta < \alpha$. Then (a) implies that $\mathfrak{A} \models \eta(x_\beta)$ for all $\beta < \alpha$; that is, for all $\beta < \alpha$, we have

$$(3) \qquad \mathfrak{A} \models \overline{Q^*y}\, \forall x[\psi(x, \mathbf{y}) \to \theta(x, x_\beta) \wedge \theta(x_\beta, x) \wedge x \neq x_\beta].$$

(1) through (3) yield, by "intersecting",

$$\mathfrak{A} \models \exists \mathbf{y}\, \exists x[\psi(x, \mathbf{y}) \wedge [\psi(x, \mathbf{y}) \to \eta(x)]$$
$$\wedge \bigwedge_{\beta < \alpha} [\psi(x, \mathbf{y}) \to \theta(x, x_\beta) \wedge \theta(x_\beta, x) \wedge x \neq x_\beta]].$$

This implies that

$$(4) \qquad \mathfrak{A} \models \exists x(\eta(x) \wedge \bigwedge_{\beta < \alpha} [\theta(x, x_\beta) \wedge \theta(x_\beta, x) \wedge x \neq x_\beta]).$$

Pick any witness to (4) and call it x_α. Then the inductive hypotheses are preserved.

Inductive hypothesis (b) guarantees that $\{x_\alpha : \alpha < \omega_1\}$ is an uncountable homogeneous set for θ. ☐

As for $\mathscr{L}(Q_1)$ and $\mathscr{L}(\mathrm{aa})$, it is convenient to observe:

5.2.3 Lemma. *$\mathscr{L}(Q^2)$ is a reasonable extension of $\mathscr{L}(Q)$ (see Definition 3.3.2) if we are restricted to acceptable vocabularies (defined in Section 5.2.1).* ☐

Example 5.1.4 shows that the set of valid sentences of $\mathscr{L}(Q^2)$ is not absolute. Therefore, we will need an added set-theoretic hypothesis in order to prove that the axioms, already proved sound in Proposition 5.2.2, are also complete. The following well-known principal of Jensen is a consequence of $V = L$.

> $\diamondsuit$: There is a sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$ with $S_\alpha \subseteq \alpha$ for all $\alpha < \omega_1$, such that for all $X \subseteq \omega_1$, $\{\alpha : X \cap \alpha = S_\alpha\}$ is stationary.

We will call such a sequence a *$\diamondsuit$-sequence.*

5.2.4 Theorem (Completeness Theorem for $\mathscr{L}(Q^2)$). *Assume $\diamondsuit$. Let τ be a countable acceptable vocabulary (see Section 5.2.1) and suppose that T is an $\mathscr{L}(Q^2)(\tau)$-consistent set of τ-sentences of $\mathscr{L}(Q^2)$. Then T has a model.*

To prepare for the proof of this theorem, we will give a fairly detailed outline in the following discussion. We will build an ω_1-chain of weak models, much as we did for $\mathscr{L}(Q)$. The "witnessing schema" [5] will guarantee that sentences $Qx_1x_2\,\theta(x_1, x_2)$ which hold in some $\mathfrak{A}_\alpha^*$ will also hold in $\mathfrak{A}$. The key problem is to guarantee that when some $\mathfrak{A}_\alpha^*$ satisfies $\neg Q^2x_1x_2\,\theta$, then this also holds in $\mathfrak{A}$. So, we must in a sense "kill off" all potential uncountable homogeneous sets for such θ. The following diagram summarizes the plan of the proof, as explained further below. Notice the similarity to Jensen's construction of a Suslin tree from $\diamondsuit$. An arrow indicates that the lower box is intended to make the upper box true.

Goal: To "kill" all potential uncountable homogeneous sets for θ.

$\uparrow$

Keep each S_α from growing into an uncountable homogeneous set for θ.

$\uparrow$

Suffices to omit (in $\mathfrak{A}$) a type $\Sigma_{\theta,\alpha}^0(x)$ which says that $S_\alpha \cup \{x\}$ is a homogeneous set for θ.

Instead, it suffices to omit a slightly bigger type $\Sigma_{\theta,\alpha}(x)$ (as we will see).

$\uparrow$

It suffices that $\mathfrak{A}_\gamma^*$ strongly omit $\Sigma_{\theta,\alpha}(x)$ for all $\gamma \geq \alpha$.

$\uparrow$

Suffices that $\mathfrak{A}_\alpha^*$ strongly omit $\Sigma_{\theta,\alpha}$, which follows from the Medusan Axioms.

As before, the interesting stages of the construction are the successor stages. Suppose we already have $\mathfrak{A}_\beta^*$ and want to get $\mathfrak{A}_{\beta+1}^*$. We form essentially the same theory $T_\phi(\mathfrak{A}_\beta^*)$ as in the proof of the Main Lemma 3.2.1, for appropriate ϕ. The consistency criterion still holds. Keeping countable sets from expanding can be accomplished just as before, by omitting certain types. It makes sense that we *also* omit types to keep homogeneous sets countable, as follows.

Suppose that a set S_β is a homogeneous set for a formula $\theta(x_1, x_2)$, where $\mathfrak{A}_\beta^* \models \neg Qx_1x_2\theta(x_1, x_2)$. Here, S_β is the βth member of a fixed $\diamond$-sequence. How can we keep S_β from expanding to an uncountable homogeneous set for θ? We would like to omit the type

$$\Sigma_{\theta,\beta}^0(x) = \{x \neq a \wedge \theta(x, a) : a \in S_\beta\},$$

where, for the sake of simplicity, we will suppose that θ is symmetric; that is, $\models \theta(x_1, x_2) \leftrightarrow \theta(x_2, x_1)$. As before, it will suffice that $\mathfrak{A}_\beta^*$ strongly omit $\Sigma_{\theta,\beta}^0$. In this way, we can keep strongly omitting this type at later stages.

How can it be that $\mathfrak{A}_\beta^*$ does *not* strongly omit $\Sigma_{\theta,\beta}^0$? That means that there is some $\psi(x, \mathbf{y})$ for which (1) and (2) below hold in $\mathfrak{A}_\beta^*$:

(1) $\quad \overline{Qy}\, \exists x \psi(x, \mathbf{y})$

(2) $\quad \overline{Q^*y}\, \forall x(\psi(x, \mathbf{y}) \to (x \neq a \wedge \theta(x, a)))$ for all $a \in S_\beta$.

However, this is not enough. A bigger type than $\Sigma_{\phi,\beta}^0$ might be easier to strongly omit—that is, *failure* to strongly omit a *bigger* type might have *stronger* consequences. Regard (2) as a formula $\eta(a)$—then $\eta(a)$ holds for all $a \in S_\beta$. If we had chosen $\Sigma_{\theta,\beta}^0$ so that it included $\eta(x)$, we would then have $\mathfrak{A}_\beta^*$ satisfying

(3) $\quad \overline{Q^*y}\, \forall x(\psi(x, \mathbf{y}) \to \eta(x))$.

But, "(1) $\wedge$ (3) $\to Qx_1x_2\theta$" is an instance of the Medusan Axioms (6). Hence $\mathfrak{A}_\beta^* \models Qx_1x_2\theta$, a contradiction.

So $\mathfrak{A}_\beta^*$ *does* strongly omit any type containing $\Sigma_{\theta,\beta}^0$ which also contains every formula $\eta(x)$ having the property possessed by our "η" above, that is $\mathfrak{A}_\beta^* \models \eta(a)$ for all $a \in S_\beta$. So set

$$\Sigma_{\theta,\beta} = \Sigma_{\theta,\beta}^0 \cup \{\delta(x) : \text{for all } a \in S_\beta,\ \mathfrak{A}_\beta^* \models \delta(a)\}.$$

As previously mentioned, we can continue strongly omitting this type in models $\mathfrak{A}_\gamma^*$ for $\gamma \geq \beta$. Hence, as in the previous proof for $\mathscr{L}(Q)$, $\mathfrak{A}^*$ omits $\Sigma_{\theta,\beta}$, where $\mathfrak{A}^* = \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha^*$.

Suppose that $\mathfrak{A}_{\alpha_0}^* \models \neg Q^2x_1x_2\theta$; we want to show that $\mathfrak{A} \models \neg Q^2x_1x_2\theta$. To this purpose, we will suppose not, and choose $S \subseteq A$ so that $\mathfrak{A} \models \theta(a, b)$ for all distinct $a, b \in S$. $\diamond$ will give us $\alpha \geq \alpha_0$ for which $(\mathfrak{A}_\alpha^*, S_\alpha) \prec (\mathfrak{A}^*, S)$. We may check that every $a \in S - A_\alpha$ realizes $\Sigma_{\theta,\alpha}$ in $\mathfrak{A}^*$. On the other hand, since $\mathfrak{A}_{\alpha_0}^* \prec^w \mathfrak{A}_\alpha^*$, $\mathfrak{A}_\alpha^* \models \neg Q^2x_1x_2\theta$, which implies (by construction) that $\mathfrak{A}^*$ *omits* $\Sigma_{\theta,\alpha}$!

Proof of Theorem 5.2.4. Partition $\omega_1 = \bigcup \{X_\phi: \phi$ is a formula of $\mathscr{L}(Q^2)$ with parameters in $\omega_1\}$. For all $\alpha < \omega_1$, choose ϕ_α so that $\alpha \in X_{\phi_\alpha}$. Fix a $\diamondsuit$-sequence $\langle S_\alpha: \alpha < \omega_1\rangle$. We build a chain $\langle \mathfrak{A}^*_\alpha: \alpha < \omega_1\rangle$ of countable weak models satisfying the following inductive hypotheses on α:

(a) $\mathfrak{A}^*_0 \models T$.
(b) If $\alpha = \beta + 1$, then $\mathfrak{A}^*_\alpha$ is a precise extension of $\mathfrak{A}^*_\beta$ relative to ϕ_β.
(c) If α is a limit, then $\mathfrak{A}^*_\alpha = \bigcup_{\beta<\alpha} \mathfrak{A}^*_\beta$.
(d) $A_\alpha = \omega \cdot (1 + \alpha)$
(e) For each $\delta < \alpha$ and formula $\theta(x_1, x_2)$ with parameters in A_δ, *if* S_δ is a homogeneous set for θ in $\mathfrak{A}^*_\delta$ (that is, $\forall x_1 x_2 \in S_\delta, \mathfrak{A}^*_\delta \models \theta(x_1, x_2) \vee x_1 = x_2$), and $\mathfrak{A}^*_\delta \models \neg Q x_1 x_2 \theta$, *then* $\mathfrak{A}^*_\alpha$ strongly omits

$$\Sigma_{\theta,\delta}(x) = \{x \neq a \wedge \theta(x, a) \wedge \theta(a, x): a \in S_\delta\} \\ \cup \{\eta(x): \text{for all } a \in S_\delta,\ \mathfrak{A}^*_\delta \models \eta(a)\}.$$

$\mathfrak{A}^*_0$ is constructed by applying the weak completeness theorem (2.2.3). For limit α, it's easy to see that, by setting $\mathfrak{A}^*_\alpha = \bigcup_{\beta<\alpha} \mathfrak{A}^*_\beta$, we preserve the inductive hypotheses.

For the successor step $\alpha = \beta + 1$, we want to use the Main Lemma 3.3.5 from the proof of the omitting types theorem for $\mathscr{L}(Q)$. Hence, it suffices to see that $\mathfrak{A}^*_\beta$ strongly omits all of the sets given in inductive hypothesis (e) for α. For $\delta < \beta$, $\mathfrak{A}^*_\beta$ strongly omits $\Sigma_{\theta,\delta}(x)$ whenever S_δ is a homogeneous set for θ in $\mathfrak{A}^*_\delta$, by the inductive hypothesis. So we are left with the problem of showing that for any formula $\theta(x_1, x_2)$ with parameters in A_β, if S_β is a homogeneous set for θ in $\mathfrak{A}^*_\beta$ and $\mathfrak{A}^*_\beta \models \neg Q^2 x_1 x_2 \theta(x_1, x_2)$, then $\mathfrak{A}^*_\beta$ strongly omits $\Sigma_{\theta,\beta}(x)$.

To obtain a contradiction, we suppose not. Then, for some formula $\psi(x, \mathbf{y})$ with parameters in $\mathfrak{A}^*_\beta$ and some quexistential $\overline{Qy}$, $\overline{Qy}\, \exists x \psi$ witnesses this supposition. Hence, we have

$$\text{(1)} \qquad \mathfrak{A}^*_\beta \models \overline{Qy}\, \exists x \psi(x, \mathbf{y});$$

$$\text{(2)} \qquad \mathfrak{A}^*_\beta \models \overline{Q^*y}\, \forall x[\psi(x, \mathbf{y}) \to x \neq a \wedge \theta(x, a) \wedge \theta(a, x)] \quad \text{for each} \quad a \in S_\beta.$$

By (2), the formula $\eta(x) \in \Sigma_{\theta,\beta}(x)$, where

$$\text{(3)} \qquad \eta(x) \equiv \overline{Q^*y}'\, \forall x'[\psi(x', \mathbf{y}') \to x' \neq x \wedge \theta(x', x) \wedge \theta(x, x')].$$

So, by choice of $\overline{Qy}\, \exists x \psi$, we have

$$\text{(4)} \qquad \mathfrak{A}^*_\beta \models \overline{Q^*y}\, \forall x[\psi(x, \mathbf{y}) \to \eta(x)].$$

By the Medusan axiom schema [6], together with (1), (3), and (4) above, $\mathfrak{A}^*_\beta \models Q^2 x_1 x_2 \theta$. This contradicts our assumption and the successor step is thus complete. Hence, the induction is also complete.

Set $\mathfrak{A}^* = \bigcup_{\alpha<\omega_1} \mathfrak{A}^*_\alpha$. Since $\mathfrak{A}^*_0 \models T$, the proof will be finished once the following claim has been established.

Claim. For every $\alpha < \omega_1$, sequence $\mathbf{a}$ of members of A_α, and formula $\phi(\mathbf{y})$ of $\mathscr{L}(Q^2)$,

$$\mathfrak{A}_\alpha^* \models \phi(\mathbf{a}) \quad \text{iff} \quad \mathfrak{A} \models \phi(\mathbf{a}).$$

The proof is by induction on the number of Q^2 quantifiers occurring in ϕ, and within a fixed such number, by induction on the complexity of ϕ. All of the inductive steps except Q^2 work just as in the proof of the union of chain Lemma 3.2.2. Let us therefore focus on the Q^2 step.

Using the witnessing schema [5], the direction ($\Rightarrow$) is easy. For the converse, we suppose that $\mathfrak{A} \models Q^2 x_1 x_2 \theta(x_1, x_2, \mathbf{a})$; say $S \subseteq A$, S uncountable and for all $x_1, x_2 \in S$, $x_1 \neq x_2$ implies that $\mathfrak{A} \models \theta(x_1, x_2, \mathbf{a})$. Let C_1 and C_2 be the following cub subsets of ω_1:

$$C_1 = \{\alpha < \omega_1 : \omega \cdot (1 + \alpha) = \alpha\} = \{\alpha < \omega_1 : A_\alpha = \alpha\};$$

$$C_2 = \{\alpha < \omega_1 : (\mathfrak{A}_\alpha^*, S \cap A_\alpha) \prec (\mathfrak{A}^*, S)\}.$$

Also, define a set

$$E = \{\alpha < \omega_1 : S \cap \alpha = S_\alpha\};$$

then E is stationary by choice of $\langle S_\alpha : \alpha < \omega_1 \rangle$. Choose $\delta \in C_1 \cap C_2 \cap E$ such that $\mathbf{a} \in A_\delta^{<\omega}$ and pick $b \in S - A_\delta$.

Subclaim 1. $\mathfrak{A}^* \models \bigwedge \Sigma_{\theta(x_1, x_2, \mathbf{a}), \delta}(b)$.

Proof. Choose $\sigma \in \Sigma_{\theta(x_1, x_2, \mathbf{a})}$. If σ is $x \neq c \wedge \theta(x, c, \mathbf{a}) \wedge \theta(c, x, \mathbf{a})$ for some $c \in S_\delta$, then $c \in S$. So this follows from the choice of S, since $\mathfrak{A}^* \models \theta(b, c, \mathbf{a})$ iff, by the inductive hypothesis, $\mathfrak{A} \models \theta(b, c, \mathbf{a})$, which is true for all distinct $b, c \in S$. Otherwise, σ is $\eta(x)$ for some η holding in $\mathfrak{A}_\delta^*$ of every element of S_δ. But $S_\delta = S \cap \delta = S \cap A_\delta$, since $\delta \in E \cap C_1$. Thus, since $\delta \in C_2$, and since $S_\delta = S \cap A_\delta$ implies that $\mathfrak{A}_\delta^* \models \forall x(x \in S \to \eta(x))$, we have $\mathfrak{A}^* \models \forall x(x \in S \to \eta(x))$. Therefore, $\mathfrak{A}^* \models \eta(b)$. $\square$

Subclaim 2. $\mathfrak{A}_\delta^* \models Q x_1 x_2 \theta(x_1, x_2)$.

Proof. Suppose not. By inductive hypothesis (e), that would imply that $\mathfrak{A}_\beta^*$ strongly omits $\Sigma_{\theta(x_1, x_2, \mathbf{a}), \delta}$ for all $\beta > \delta$. But then $\mathfrak{A}^*$ *strongly omits* and hence *omits* $\Sigma_{\theta(x_1, x_2, \mathbf{a}), \delta}$, contradicting subclaim 1.

Now $(\mathfrak{A}_\beta^* : \beta < \omega_1)$ is an $\prec^w$-elementary chain, by construction. Thus, by subclaim 2, $\mathfrak{A}_\alpha^* \models Q^2 x_1 x_2 \theta(x_1, x_2)$. $\square$

5.2.5 The Case $\mathscr{L}(Q^3)$. The goal here is to lift the preceding results to $\mathscr{L}(Q^3)$, so that one can believe in a corresponding set of results for $\mathscr{L}^{<\omega}$ without going through all the ghastly notation which might otherwise be required. For $\mathscr{L}(Q^3)$ we need extra witnessing axioms of the form

$$\begin{aligned}&\forall \mathbf{y}[Q^3 x_1 x_2 x_3 \theta(x_1, x_2, x_3, \mathbf{y}) \to QzP_\theta(z, \mathbf{y})] \wedge\\ &\forall \mathbf{y}[QzP_\theta(z, \mathbf{y}) \to \forall x_1 x_2 x_3 (P_\theta(x_1, \mathbf{y})\\ &\qquad \wedge P_\theta(x_2, \mathbf{y}) \wedge P_\theta(x_3, \mathbf{y}) \wedge x_1 \neq x_2 \wedge x_1 \neq x_3\\ &\qquad \wedge x_2 \neq x_3 \to \theta(x_1, x_2, x_3, \mathbf{y}))].\end{aligned}$$

Furthermore, we need extra Medusan axioms, and these are described below.

To prove the completeness theorem, we proceed as in the situation for $\mathscr{L}(Q^2)$. As before, we want to omit a type $\Sigma_{\theta,\alpha}$, whenever $\mathfrak{A}^*_\alpha \models \neg Q^3 x_1 x_2 x_3 \theta(x_1, x_2, x_3)$. By analogy, we have

$$\begin{aligned}\Sigma_{\theta,\alpha} = &\{\theta'(x, a, b)\colon a, b \in S_\alpha, a \neq b\}\\ &\cup \{\eta(x)\colon \text{for all but at most one } a \in S_\alpha, \mathfrak{A}^*_\alpha \models \eta(a)\};\end{aligned}$$

here $\theta'(x_1, x_2, x_3)$ is $\bigwedge \{\theta(x_{i_1}, x_{i_2}, x_{i_3})\colon i \text{ a permutation}\}$. If $\mathfrak{A}^*_\alpha$ does not strongly omit $\Sigma_{\theta,\alpha}$, then for some $\overline{Qy}\,\exists x\psi$ where $\overline{Qy}$ is quexistential, $\mathfrak{A}^*_\alpha \models \overline{Qy}\,\exists x\psi$ and $\mathfrak{A}^*_\alpha \models \overline{Q^*y}\,\forall x(\psi(x, \mathbf{y}) \to x \neq a \wedge x \neq b \wedge \theta'(x, a, b))$ for all distinct $a, b \in S_\alpha$. Thus, we may write $\mathfrak{A}^*_\alpha \models \eta_1(a, b)$, where

$$\eta_1(x, b) \equiv \overline{Q^*y}^1\, \forall x^1(\psi(x^1, \mathbf{y}^1) \to x^1 \neq x \wedge x^1 \neq b \wedge \theta'(x^1, x, b)).$$

Accordingly, $[x \neq b \wedge \eta_1(x, b)] \in \Sigma_{\theta,\alpha}$ for all $b \in S_\alpha$; then

$$(1) \qquad \mathfrak{A}^*_\alpha \models \overline{Q^*y}\,\forall x(\psi(x, \mathbf{y}) \to x \neq b \wedge \eta_1(x, b)), \quad \text{all} \quad b \in S_\alpha.$$

Now set

$$\eta_2(x) \equiv \overline{Q^*y}^2\, \forall x^2(\psi(x^2, \mathbf{y}^2) \to x^2 \neq x \wedge \eta_1(x^2, x)).$$

Then (1) says that $\mathfrak{A}^*_\alpha \models \eta_2(b)$ for all $b \in S_\alpha$. Thus, $\eta_2(x) \in \Sigma_{\theta,\alpha}$. Again, using our choice of $\overline{Qy}\,\exists x\psi$,

$$\mathfrak{A}^*_\alpha \models \overline{Q^*y}\,\forall x(\psi(x, \mathbf{y}) \to \eta_2(x)).$$

This yields $\mathfrak{A}^*_\alpha \models Q^3 x_1 x_2 x_3 \theta(x_1, x_2, x_3)$—a contradiction—*if* we make the following an axiom:

$$[\overline{Qy}\,\exists x\psi \wedge \overline{Q^*y}\,\forall x(\psi(x, \mathbf{y}) \to \eta_2(x))] \to Q^3 x_1 x_2 x_3 \theta(x_1, x_2, x_3),$$

where η_2 is as defined above.

As before, at stages $\beta \geq \alpha$ we can strongly omit $\Sigma_{\theta,\alpha}(x)$, if $\mathfrak{A}^*_\alpha \models \neg Q^3 x_1 x_2 x_3 \theta$. And, as before, this does the job.

Are the new axioms valid? Suppose that $\mathfrak{A}$ is a model of the hypotheses of a new Medusan axiom. Define $\{x_\alpha : \alpha < \omega_1\}$ subject to the following inductive hypotheses on α. For all $\alpha_1 < \alpha_2 < \alpha_3 \leq \alpha$:

(i) $\mathfrak{A} \models \eta_2(x_{\alpha_3})$.
(ii) $\mathfrak{A} \models \eta_1(x_{\alpha_2}, x_{\alpha_3})$.
(iii) $\mathfrak{A} \models \theta'(x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_3})$.

Details are straightforward extensions of those given in Theorem 5.2.2 for $\mathscr{L}(Q^2)$. □

5.2.6 Corollary ([M²]). *Assume* $\diamond$. *Then* $\mathscr{L}^{<\omega}$ *is countably compact and recursively enumerable for consequence.*

Proof for $\mathscr{L}(Q^2)$. This result follows from the completeness theorem, since every countable vacabulary τ can be expanded to an acceptable τ' which is still countable and is recursive in τ. □

There is no known *explicit* set of axioms for $\mathscr{L}(Q^2)$, that is, axioms which do not require τ to be acceptable, even assuming $\diamond$. Shelah has recently shown that in a certain sense, no finite set of schema axiomatizes the set of validities of $\mathscr{L}(Q^2)$; see Shelah–Steinhorn [1982].

The following theorem bears on the sensitivity of $\mathscr{L}^{<\omega}$ to the axioms in the metatheory. We are indebted to Ken Kunen for supplying the following theorem and proofs. In this connection we note that it would also be interesting to find a complete set of axioms under MA + $\neg$CH.

5.2.7 Theorem (Kunen). (i) *One cannot prove in* ZFC + SH (*Suslin's hypothesis*) *that adding* "SH" *to the axioms for* $\mathscr{L}(Q^2)$ (*in Section* 5.2.1) *results in a complete axiomatization for* $\mathscr{L}(Q^2)$.

(ii) *One cannot prove in* ZFC + $\neg$SH *that the usual axioms of* $\mathscr{L}(Q^2)$ (*see Section* 5.2.1) *are complete.*

Proof. (i) Otherwise, satisfiability for $\mathscr{L}(Q^2)$-sentences is absolute for models of ZFC + SH. However, in $\mathscr{L}(Q^2)$ one can assert that a partial order is c.c.c. On the one hand, Con(ZFC + SH + CH) by Jensen, and CH $\to \exists\mathbb{P}$ ($\mathbb{P}$ is c.c.c and $\mathbb{P} \times \mathbb{P}$ is not c.c.c.), by Laver and Galvin. While on the other hand, MA + $\neg$CH $\to$ SH and MA + $\neg$CH $\to \forall\mathbb{P}$ ($\mathbb{P}$ is c.c.c. $\to \mathbb{P} \times \mathbb{P}$ is c.c.c.).

(ii) In Kunen–Van Douwen [1982] we find that by iterating c.c.c. forcing, we may obtain the consistency of ZFC + $\neg$CH + $\neg$SH +

(*) Whenever $A_\alpha \subset \mathbb{Q}$ for $\alpha < \omega_1$ satisfy $\forall \alpha < \beta < \omega_1$ ($A_\beta - A_\alpha$ is bounded $\wedge A_\alpha - A_\beta$ is unbounded), there is an $X \subset \omega_1$ such that $|X| = \omega_1$ and $\forall \alpha, \beta \in X$ ($\alpha < \beta \to A_\beta \not\subset A_\alpha$).

Observe that here, $S \subseteq \mathbb{Q}$ is *bounded* iff $\exists q(S \subset (-\infty, q))$. However, there is a sentence ϕ of $\mathscr{L}(Q^2)$ which has a model iff $\neg(*)$. ϕ is consistent with the usual axioms because CH $\Rightarrow \neg(*)$. □

It is shown in [M^2] that the axioms for $\mathscr{L}^{<\omega}$ remain complete in some model of $\neg\diamond$, namely when one adds $\aleph_2$ Cohen reals to a model of $\diamond$. Nevertheless, Shelah has recently found a model of set theory in which $\mathscr{L}(Q^2)$ is not countably compact.

5.3. Other Related Logics

An extension of $\mathscr{L}^{<\omega}$ has been proved to be countably compact (assuming $\diamond$) in Malitz–Rubin [1980]. In this logic, for example, we can say that $\{\langle x, y\rangle: \phi(x, y)\}$ contains an equivalence relation with uncountably many uncountable equivalence classes as follows, where X^2 ranges over uncountable sets of uncountable sets.

$$\begin{aligned}&(\exists X^2)(\forall X_1^1 \in X^2)(\forall X_2^1 \in X^2)(\forall x_1 \in X_1^1)(\forall x_2 \in X_1^1)(\forall x_3 \in X_2^1)\\ &\qquad[\phi(x_1, x_2) \wedge \neg\phi(x_1, x_3)].\end{aligned}$$

Here, it is understood that distinct variables are intended to represent distinct things. More generally, we allow "descending quantifier strings", which begin with $(\exists X^n)$ (some n) and contain various $(\forall X_j^i \in X_k^{i+1})$ for $i < n$, such that each such X_k^{i+1} is either X^n or else appears in an earlier quantifier $(\forall X_k^{i+1} \in X_l^{i+2})$. Here, $X_j^0 = x_j$ ranges over $K^0(A) = A$, and X_j^{i+1} ranges over

$$\{Z \subseteq K^n(A): |Z| \geq \omega_1\} = K^{n+1}(A).$$

Perhaps a simpler logic, which is equivalent—at least if one has a pairing function—allows quantifiers $(\exists X^n)(\forall x_s: s \in T)\ \phi(\langle x_s: s \in T\rangle)$, where T is any finite subtree of ω^n (ordered by inclusion). This quantifier is interpreted as follows: X^n is a tree of height n having uncountably many elements of level 0 as well as uncountably many immediate successors of each element of level $<(n-1)$; and x_s is to be $< x_t$ whenever $s < t$, with x_s ranging over elements of level $|s| - 1$.

Another version of this quantifier is defined in Rubin–Shelah [1983]. Moreover, it is there proved that one has a strict hierarchy of these quantifiers.

The following definition gives a simplified version of the quantifier "there is a branch" from Shelah [1978d], the general version being found in Section V.8 of the present volume. Shelah's quantifier is, in fact, *fully* compact, whereas the following version is not (and this for the same reason that $\mathscr{L}(Q_1)$ is not). It is also interesting to note that this logic is a countably compact piece of $\mathscr{L}(Q^2)$ for which $\diamond$ is not needed, since satisfiability is absolute. Intuitively, it seems then that one has the equation

$$\frac{\mathscr{L}(\text{"there is a branch"})}{\mathscr{L}(Q^2)} \simeq \frac{\text{Aronszajn tree}}{\text{Suslin tree}}.$$

And, of course, $\diamond$ (or something at least) is needed to construct a Suslin tree, but not to construct an Aronszajn tree.

5.3.1 Definition. $\mathscr{L}(Q^B)$ is the logic formed from $\mathscr{L}(Q_1)$ by adding an additional quantifier Q^B: if $\phi(x, y)$ is a formula of $\mathscr{L}(Q^B)$ then so is $Q^B xy\phi$. Write $\eta^{\mathfrak{A}}$ for $\{\mathbf{a}: \mathfrak{A} \models \eta(\mathbf{a})\}$. Then the new inductive clause for satisfaction is

$$\mathfrak{A} \models Q^B xy\, \phi(x, y) \quad \text{iff} \quad \begin{array}{l} \langle \text{field}(\phi^{\mathfrak{A}}), \phi^{\mathfrak{A}} \rangle \text{ is a tree satisfying} \\ \forall y \neg Qx\phi(x, y), \text{ such that there is an} \\ \text{uncountable branch through this tree.} \end{array}$$

5.3.2 Theorem (Shelah [1978d]). *$\mathscr{L}(Q^B)$ is countably compact and recursively enumerable for consequence.*

Proof. An approximate idea of the proof is to place $\mathscr{L}(Q^B)$ inside $\Delta(\mathscr{L}(Q))$ (defined in Section II.7.2), more or less, in a generic extension of universe. We then may use the absoluteness of $\mathscr{L}(Q)$-satisfiability. Fix a vocabulary τ. We define maps $\phi \mapsto \phi^\exists$ and $\phi \mapsto \phi^\forall$ from $\mathscr{L}(Q^B)(\tau)$ to $\mathscr{L}(Q)(\tau')$, where $\tau' = \tau \cup S$ for some set S of new relation symbols (these relation symbols will be determined below). The approximate idea here is that if the world were perfect, then ϕ would be equivalent to $\exists \mathbf{X}\phi^\exists$ and to $\forall \mathbf{Y}\phi^\forall$, where $\mathbf{X}$ and $\mathbf{Y}$ are the new relation symbols in $\phi^\exists$ and $\phi^\forall$, respectively.

$\phi^\exists$ and $\phi^\forall$ are defined by induction on the following depth $r(\phi)$ of the quantifiers Q^B and Q in ϕ: $r(\phi) = 0$ for ϕ atomic, $r(\neg\psi) = r(\forall x\psi) = r(\psi)$, $r(\phi \vee \psi) = \max(r(\phi), r(\psi))$, $r(Qx\phi) = r(\phi) + 1$ and $r(Q^B xy\phi) = r(\phi) + 2$. Set $\phi^\exists = \phi^\forall = \phi$ for ϕ atomic. Now, suppose that $\phi^\exists$ and $\phi^\forall$ are defined for $r(\phi) < n$. We then define $\phi^\exists$ and $\phi^\forall$ for $r(\phi) = n$ by induction on ϕ. If ϕ is $\neg\psi$, then $\phi^\exists$ is $\neg(\psi^\forall)$ and $\phi^\forall$ is $\neg(\psi^\exists)$. Suppose that ϕ is $\theta \vee \psi$. Then, of course, $\phi^\exists$ is $\theta^\exists \vee \psi^\exists$. To define $\phi^\forall$, where ϕ is $\theta \vee \psi$, we first make the new relation symbols of $\theta^\forall$ disjoint from those of $\psi^\forall$, say by suffixing a "0" on those of $\theta^\forall$ and a "1" on those of $\psi^\forall$. Call these modified formulas θ' and ψ', and set $(\theta \vee \psi)^\forall = \theta' \vee \psi'$. The next case is $\phi = \forall x\psi$. Then $\phi^\forall$ is $\forall x\psi^\forall$. For $\phi^\exists$, we first consider the choice schema

$$\forall x\, \exists Y \eta(x, Y, \ldots) \to \exists Y'\, \forall x \eta(x, (Y')_x, \ldots),$$

where $\eta(x, (Y')_x, \ldots)$ denotes the result of replacing each occurrence of the form $Y(\mathbf{y})$ by an occurrence of $Y'(x, \mathbf{y})$ in η. Then $\phi^\exists = \forall x\psi^\exists((X^1)_x, \ldots, (x^n)_x)$, where $X^1, \ldots, X^n$ are the new relation symbols occurring in $\psi^\exists$. (Observe that this idea appears in the proof of Theorem II.7.2.4(a).) The next step in our development is to define $(Qx\theta)^\exists$ as $QxX(x) \wedge [\forall x(X(x) \to \theta)]^\exists$ by using the rules above. Similarly, we have that $(Qx\theta)^\forall$ is $[Q^*xX(x) \to [\exists x(X(x) \wedge \theta)]^\forall] \wedge Qx(x = x)$.

Finally, we wish to define $\eta^\exists$ and $\eta^\forall$, when η is $Q^B xy\phi(x, y)$. $\eta^\exists$ is easy to define, since it simply is ["X is an uncountable branch of the tree $\langle \text{field}(\phi), \phi \rangle$" $\wedge$ $\forall y \neg Qx\phi(x, y)]^\exists$. In order to define $\eta^\forall$, we imagine that if a given ranked tree does *not* have a branch, then there is an order-preserving map from that tree into the rationals. Thus, $\eta^\forall$ is the following formula, where R and S are binary relation symbols not in τ which do not occur in η: $\neg$["R is an order-preserving function from the tree $\{\langle x, y \rangle : \phi(x, y)\}$ to the countable linear order S" $\vee$ $\neg$"ϕ is a tree" $\vee$ $\exists y Qx\phi(x, y)]^\exists$.

It is routine to verify by induction on ϕ that for all $\phi \in \mathscr{L}(Q^B)$, we have that

(1) $\models \phi^\exists \to \phi$ and $\models \phi \to \phi^\forall$.

Now we claim that for any set Γ of sentences of $\mathscr{L}(Q^B)$, if $\Gamma^\exists = \{\phi^\exists : \phi \in \Gamma\}$, then:

(2) Γ is satisfiable iff $\Gamma^\exists$ is satisfiable.

Since $\phi^\exists \in \mathscr{L}(Q)$ for all $\phi \in \mathscr{L}(Q^B)$, the theorem follows from (2) above. The direction ($\Leftarrow$) follows immediately from (1) above. For the proof of ($\Rightarrow$), we suppose that $\mathfrak{A} \models \Gamma$. Let $(U, <)$ be the disjoint sum of all trees P such that $\forall y \neg QxPxy$, with field contained in $\mathfrak{A}$, that do not contain an uncountable branch. Then $(U, <)$ does not contain an uncountable branch. By Baumgartner–Malitz–Reinhardt [1970], there is a c.c.c. partial order which generically adds an order-preserving map from $(U, <)$ into the rationals. An easy induction shows that the predicate "$\mathfrak{A} \models \phi[s]$" is absolute for the generic extension, since no new branches are added. Thus, Γ remains satisfiable. Moreover, in the generic extension: $\exists \mathbf{X}\phi^\exists \leftrightarrow \phi$, $\phi \leftrightarrow \forall \mathbf{Y}\phi^\forall$ are valid in $\mathfrak{A}$ for all $\phi \in \mathscr{L}(Q^B)$, as one can again check by induction on ϕ. Hence, we have that $\Gamma^\exists$ is satisfiable in the generic extension. Thus, $\Gamma^\exists$ is $\mathscr{L}(Q)$-consistent in the generic extension, which implies that $\Gamma^\exists$ is $\mathscr{L}(Q)$-consistent in V, since consistency is finitary. By the completeness theorem for $\mathscr{L}(Q)$, $\Gamma^\exists$ is satisfiable (in V). ☐

6. *Interpolation and Preservation Questions*

In this section we will survey some of the results, methods, and questions that are related to definability properties of $\mathscr{L}(Q_1)$ and (to some extent, at least) its extensions $\mathscr{L}(\mathrm{aa})$ and $\mathscr{L}^{<\omega}$. In Section 6.4 we will consider such properties for the "weak models" of Section 2.3.

6.1. *Preservation of $\mathscr{L}$-equivalence Under Products and Unions, for $\mathscr{L} = \mathscr{L}(Q_\alpha)$, $\mathscr{L}(\mathrm{aa})$, $\mathscr{L}^{<\omega}$*

The following theorem is proved in Lipner [1970].

6.1.1 Theorem. *Suppose that $\{\mathfrak{A}_i : i \in I\}$ and $\{\mathfrak{B}_i : i \in I\}$ are finite families of τ-structures, where $\mathfrak{A}_i \equiv_{\mathscr{L}(Q_\alpha)} \mathfrak{B}_i$ for all $i \in I$. Then $\prod \{\mathfrak{A}_i : i \in I\} \equiv_{\mathscr{L}(Q_\alpha)} \prod \{\mathfrak{B}_i : i \in I\}$. And if τ has no function symbols, then the disjoint unions $\dot{\bigcup} \{\mathfrak{A}_i : i \in I\}$ and $\dot{\bigcup} \{\mathfrak{B}_i : i \in I\}$ are also $\mathscr{L}(Q_\alpha)$-equivalent.* ☐

One method of proof is the method of *back-and-forth systems*, also known as "Ehrenfeucht games": see Section II.4.2. This method has also been used in

Vinner [1972]. Badger[1977] has also given an appropriate back-and-forth criterion for $\mathscr{L}^{<\omega}$. In that work, we also find—in spite of this criterion—that $\mathscr{L}(Q^2)$-equivalence is not preserved by finite direct products. The reader should see Definition II.4.2.2 for material on back-and-forth systems in a more general setting.

By assuming an appropriate combinatorial hypothesis on $\aleph_\alpha$, Lipner has also proved Theorem 6.1.1 for other powers of I. For $\aleph_\alpha = 2^\omega$, this theorem also holds for any index set I which is not at least as large as some measurable cardinal, and this even if we only assume $\mathfrak{A}_i$ and $\mathfrak{B}_i$ are $\mathscr{L}_{\omega\omega}$-equivalent for all $i \in I$ (Flum [1975a, Theorem 2.10]). Related results on preservation of $\mathscr{L}(Q_\alpha)$-equivalence by reduced products, where $\aleph_\alpha = 2^\omega$ can also be found in Flum [1975a].

We will now turn to $\mathscr{L}(\mathrm{aa})$. There are several back-and-forth criteria for $\mathscr{L}(\mathrm{aa})$-equivalence. These results were developed independently by Caicedo [1978], Makowsky (see Makowsky–Shelah [1981, Section 2]), Kaufmann [1978a], and Seese and Weese [1982]. Nevertheless, the following example shows that $\mathscr{L}(\mathrm{aa})$-equivalence is not preserved by disjoint unions. A similar argument can also be given to show that it is not preserved by finite products.

6.1.2 Example (Shelah). Let S be a stationary subset of ω_1 with stationary complement, and set $(A, <)$ equal to the ordered sum $\sum \{X_\alpha : \alpha < \omega_1\}$ where $X_\alpha = \mathbb{Q}$ if $\alpha \in S$, otherwise $X_\alpha = 1 + \mathbb{Q}$, and $<$ results from replacing each α by X_α. Similarly, let $(B, <)$ be the ordered sum $\sum \{Y_\alpha : \alpha < \omega_1\}$ where $Y_\alpha = 1 + \mathbb{Q}$ if $\alpha \in S$, otherwise $Y_\alpha = \mathbb{Q}$. A back-and-forth argument establishes that $(A, <)$ and $(B, <)$ are $\mathscr{L}_{\infty\omega}(\mathrm{aa})$-equivalent. [*Hint*: let $X'_\alpha = \bigcup \{X_\gamma : \gamma < \alpha\}$, and $Y'_\alpha = \bigcup \{Y_\gamma : \gamma < \alpha\}$. By induction on $\phi(\mathbf{s}, \mathbf{x})$ show that if $\alpha_0 < \cdots < \alpha_{n-1}$ and $\beta_0 < \cdots < \beta_{n-1}$, where $\alpha_i \in S$ iff $\beta_i \notin S$, and if f is a partial isomorphism from $(\mathfrak{A}, X'_{\alpha_0}, \ldots, X'_{\alpha_{n-1}})$ to $(\mathfrak{B}, Y'_{\alpha_0}, \ldots, Y'_{\alpha_{n-1}})$, then $\mathfrak{A} \models \phi(X'_{\alpha_0}, \ldots, X'_{\alpha_{n-1}}, \operatorname{domain} f)$ iff $\mathfrak{B} \models \phi(Y'_{\alpha_0}, \ldots, Y'_{\alpha_{n-1}}, \operatorname{range} f)$.] However, if $(A', <')$ is a disjoint copy of $(A, <)$, then $(A' \cup A, <' \cup <)$ and $(A' \cup B, <' \cup <)$ are *not* $\mathscr{L}(\mathrm{aa})$-equivalent. For, the following sentence θ holds in the former but not in the latter: $\theta \equiv \operatorname{stat} s\, \exists x \exists y\, \forall z (s(z) \leftrightarrow z <' x \vee z < y)$.

To take care of this problem, Kaufmann [1978a] defines and Eklof–Mekler [1979] further studies the notion of finitely determinate structure. Roughly speaking, such a structure is one in which we do not have disjoint definable stationary sets. In fact, we might say that the aa quantifier is self-dual on such structures. More precisely, we have

6.1.3 Definition. A structure $\mathfrak{A}$ is *finitely determinate* if it satisfies all formulas of the form $\mathrm{aa}\, s_1 \ldots \mathrm{aa}\, s_n\, \forall \mathbf{x}\, [\operatorname{stat} t\, \phi(\mathbf{x}, \mathbf{s}, t) \rightarrow \mathrm{aa}\, t\, \phi(\mathbf{x}, \mathbf{s}, t)]$.

We observe that many familiar structures are finitely determinate, for example, $(\mathbb{R}, <)$ and all modules are proved to be finitely determinate in Eklof–Mekler [1979].

Using back-and-forth systems for finitely determinate structures (see Kaufmann [1978a] or Eklof–Mekler [1979]), we can prove an analogue of Theorem 6.1.1.

6.1.4 Theorem (Kaufmann [1978a]). *Suppose that* $\{\mathfrak{A}_i : i \in I\}$ *and* $\{\mathfrak{B}_i : i \in I\}$ *are families of finitely determinate* τ*-structures such that* $\mathfrak{A}_i \equiv_{\mathscr{L}(\mathrm{aa})} \mathfrak{B}_i$ *for all* $i \in I$.

(i) *If* I *is finite then* $\prod \{\mathfrak{A}_i : i \in I\}$ *and* $\prod \{\mathfrak{B}_i : i \in I\}$ *are* $\mathscr{L}(\mathrm{aa})$*-equivalent and finitely determinate.*

(ii) *If* τ *is relational then the disjoint unions* $\bigcup \{\mathfrak{A}_i : i \in I\}$ *and* $\bigcup \{\mathfrak{B}_i : i \in I\}$ *are* $\mathscr{L}(\mathrm{aa})$*-equivalent and finitely determinate.* □

6.1.5 Remark. Shelah has recently shown that every countable consistent theory of $\mathscr{L}(Q_1)$ has a finitely determinate model; see Mekler–Shelah [198?].

A number of variants of Theorems 6.1.4 and 6.1.1 have been proved. Aside from some obvious extensions to $\mathscr{L}_{\infty\omega}(Q_\alpha)$ and to $\mathscr{L}_{\infty\omega}(\mathrm{aa})$, we may also consider other operations on structures, such as direct sums (see Eklof–Mekler [1979] or Kaufmann [1978a, III.3.11]). Moreover, Seese [1981b] and Mekler [1984] have used ordered sums to prove theorems such as Seese's theorems that every ordinal $(\alpha, \in)$ is finitely determinate, and that the $\mathscr{L}(\mathrm{aa})$-theory of ordinals is decidable; see also Section VII.4.

6.2. *Preservation of $\mathscr{L}(Q_1)$-sentences by Extensions, and Related Problems*

Among the definability problems that might be raised for $\mathscr{L}(Q_1)$, one that has received some attention (see, for example, Bruce [1978a]) is:

6.2.1 Question. Classify those sentences ϕ of $\mathscr{L}(Q_1)$ such that whenever $\mathfrak{A} \models \phi$ and $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{B} \models \phi$. Such ϕ are said to be *preserved by extensions*.

Of course, the Łos–Tarski theorem for first-order logic establishes that the class of existential sentences is the answer if one restricts to $\mathscr{L}_{\omega\omega}$. The natural generalization is the class of quasi-existential or "quexistential" sentences:

6.2.2 Definition (Bruce [1978a]). A formula ϕ of $\mathscr{L}(Q)$ is *quexistential* if it is in prenex form, with only Q and $\exists$ quantifiers.

One may easily verify, as Bruce has noted, that every quexistential sentence is preserved by extensions. Although Question 6.2.1 remains open, the natural conjecture was proved false in Baldwin–Miller [1982] if one restricts to the class of models of a given theory. The general result, given below, is new and due to Shelah:

6.2.3 Example (Shelah). There is a sentence ϕ of $\mathscr{L}(Q_1)$ which is preserved by extensions but which is not equivalent to a quexistential sentence.

Proof. Let ψ be the conjunction of:

(a) $<$ is a linear order of the universe such that every proper initial segment is countable.
(b) $<^*$ is a linear order of the universe.
(c) $Qx(x = x) \rightarrow \exists x[\{y: y < x\}$ is dense for $<^*]$.

Then ψ is preserved by submodels. This is not difficult to see, since every suborder of a separable linear order is separable. It suffices to show, then, that $\neg\psi$ $(=\phi)$ is not equivalent to a quexistential sentence. This, in turn, follows from the existence of models $\mathfrak{A} = (A, <_A, <_A^*)$ and $\mathfrak{B} = (B, <_B, <_B^*)$ such that $\mathfrak{A} \models \neg\psi$ and $\mathfrak{B} \models \psi$, and for every quexistential sentence θ, if $\mathfrak{A} \models \theta$ then $\mathfrak{B} \models \theta$. We will construct models $\mathfrak{A}$ and $\mathfrak{B}$ with the following properties:

$(1)_A$ $(\mathfrak{A}, <_A^*)$ is a non-separable linear order.

$(2)_A$ $(\mathfrak{A}, <_A^*)$ is ω_1-dense, that is, it satisfies

$$\forall x\, \forall y(x <^* y \rightarrow Qz(x <^* z <^* y)).$$

$(3)_A$ $(\mathfrak{A}, <_A)$ is ω_1-like.

$(1)_B$ $(\mathfrak{B}, <_B^*)$ is an ω_1-dense subset of $\mathbb{R}$ of power ω_1.

$(2)_B$ If $a <_B b$ and $c <_B^* d$ then for some e, $a <_B e <_B b$ and $c <_B^* e <_B^* d$; and there is no $<_B$-least element.

$(3)_B$ *$(\mathfrak{B}, <_B)$ is ω_1-like.*

Assume for the moment that such models have indeed been constructed. Then $\mathfrak{A} \models \neg\psi$ and $\mathfrak{B} \models \psi$. In fact, $\{x: x <_B b\}$ is $<_B^*$-dense for all $b \in B$, by $(2)_B$ above. We thus claim that for every quexistential formula $\theta(\mathbf{x})$ and finite partial isomorphism $\{\langle a_i, b_i\rangle : i \in I\}$, if $\mathfrak{A} \models \theta(\mathbf{a})$ then $\mathfrak{B} \models \theta(\mathbf{b})$ also. This is easy to show by induction on complexity of θ. We use $(2)_B$ for the $\exists$ step. As to the Q step, if $a_1 <_A^* a_2 <_A^* \cdots <_A^* a_{n-1}$ and $\mathfrak{A} \models Qx\theta(x, \mathbf{a})$, then for some $j < n$ there exist uncountably many x such that $\mathfrak{A} \models \theta(x, \mathbf{a}) \wedge a_j <^* x <^* a_{j+1}$ (where $a_0 = -\infty$ and $a_n = \infty$). Since almost all these x are $<_A$-greater than every a_i by $(3)_A$, then every x with $b_j <_B^* x <_B^* b_{j+1}$ which is $<_B$-greater than every b_j, will satisfy $\theta(x, \mathbf{b})$ in $\mathfrak{B}$, by the inductive hypothesis. By $(1)_B$ and $(3)_B$ above, we have that $\mathfrak{B} \models Qx\theta(x, \mathbf{b})$.

It now remains to construct such models $\mathfrak{A}$ and $\mathfrak{B}$. $(A, <_A^*)$ is any ω_1-dense cofinal subset of $\mathbb{R} \cdot \omega_1$, of power ω_1. Then $<_A$ is any ω_1-like ordering of A. Also $(B, <_B^*)$ is easy to choose so that $(1)_B$ holds. The construction of $<_B$ so that $(2)_B$ and $(3)_B$ hold is left to the reader, with the hint that it proceeds ω steps at a time, and that it suffices to consider rational numbers c and d in $(2)_B$ above. $\square$

An analogous question is raised in Bruce [1978a] for $\mathscr{L}(\mathrm{aa})$. One might conjecture that the sentences preserved by extensions are the "generalized Σ" sentences, that is, those sentences in prenex form with no universal quantifiers. In fact such sentences are preserved by extensions (Bruce [1978a, Theorem 3.1]). But here again equality eludes us. For, Theorem 3.5 of Baldwin–Miller [1982] states that the class of separable dense linear orders is defined by some sentence ϕ for which $\neg\phi$ is not equivalent to a generalized Σ sentence, and yet $\mathrm{Mod}(\phi)$ is closed under substructures.

Of course, there are other preservation questions we might raise, and they are all open. For example, Bruce has conjectured that by analogy to first-order logic, a sentence is preserved by unions of ω-chains iff it is of the form $Q_1^* x_1 \ldots Q_n^* x_n \phi(x)$, where each $Q_i^* \in \{\forall, Q^*\}$ and ϕ is preserved by extensions. Here again, one direction is easy. Bruce points out that interpolation properties can be useful in proving such theorems—see, for example, Section 6.4(1). Thus, let us turn next to the interpolation problem.

6.3. The Interpolation Problem for Extensions of $\mathscr{L}(Q_1)$

Recall that the interpolation property (even Δ-interpolation) fails for $\mathscr{L}(Q_1)$; see Remark 4.1.2 (vi). However, research has been stimulated by questions such as the following, which was raised by Feferman and others (also see Makowsky–Shelah–Stavi [1976, §3]).

6.3.1 Question. Is there an extension of $\mathscr{L}(Q_1)$ which is countably compact and satisfies the interpolation property?

Shelah has recently announced that it is relatively consistent with ZFC that the answer here should be affirmative, and he has also recently shown [1982a] that every valid implication in $\mathscr{L}^{\mathrm{cf}\,\omega}$ (see Section II.2.4) has an interpolant in $\mathscr{L}(\mathrm{aa})$. A topological result of Caicedo [1981b, 1.3] is that interpolation holds for the restriction of $\mathscr{L}(Q_1)$ to monadic vocabularies. Since space is limited here, we will not prove any of the (admittedly limited) number of positive results. Instead, we will indicate some obstacles to the interpolation property by way of presenting a few examples. This will also provide us with a rationale for becoming more familiar with the expressive powers of the logics we have been discussing.

6.3.2 Lemma (Badger [1977]; see also Ebbinghaus [1975b]). *Let κ be a cardinal and suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are linear orders which are κ-dense, that is, they satisfy $\forall x\, \forall y[x < y \to Q_\kappa z\,(x < z < y)]$. Then $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathscr{L}_{\infty\omega}^{<\omega}$-equivalent in the κ-interpretation.*

Proof. A routine induction on formulas ϕ of $\mathscr{L}_{\infty\omega}^{<\omega}$ shows that for every partial isomorphism f from $\mathfrak{A}$ to $\mathfrak{B}$, $\mathfrak{A} \models_\kappa \phi[s]$ iff $\mathfrak{B} \models_\kappa \phi[f \circ s]$ for every assignment s of the free variables of ϕ into domain(f). The key observation here is that if $-\infty = a_0 < a_1 < \cdots < a_k = \infty$ and $\mathfrak{A} \models_\kappa Q^n x_0 \cdots x_{n-1} \theta(\mathbf{x}, \mathbf{a})$, then for some $i < k$,

$\mathfrak{A} \models_\kappa Q^n\mathbf{x}(\bigwedge_{j<n} a_i < x_j < a_{i+1} \wedge \theta(\mathbf{x}, \mathbf{a}))$; and, hence, for all one–one $\mathbf{x}$ in A, $\mathfrak{A} \models_\kappa (\bigwedge_{j<n} a_i < x_j < a_{i+1} \to \theta(\mathbf{x}, \mathbf{a}))$. We argue similarly for $\mathfrak{B}$. ▯

6.3.3 *Theorem* (Based on Makowsky–Shelah–Stavi [1976, Theorem 2.15]). *$\mathscr{L}^{<\omega}_{\infty\omega}$ does not allow Δ-interpolation for $\mathscr{L}(Q_1)$.*

Proof. Let $\mathscr{K}$ be the class of separable ω_1-dense linear orders without endpoints. Clearly, $\mathscr{K}$ is $\Sigma^1_1(\mathscr{L}(Q))$. Also the complement of $\mathscr{K}$ is immediately seen to be $\Sigma^1_1(\mathscr{L}(Q))$, once we observe that a dense linear order L is non-separable iff $L \times L$ has an uncountable family of pairwise disjoint open rectangles not meeting the diagonal (Kurepa [1952]). Certainly, if L is separable, then so is $L \times L$; and, for the converse, notice that for every maximal family of pairwise disjoint rectangles not meeting the diagonal $\{(a_i, b_i) \times (c_i, d_i): i \in I\}$, $\{a_i: i \in I\}$ is dense.

However, $\mathscr{K}$ is not elementary in $\mathscr{L}^{<\omega}_{\infty\omega}$. For $\mathfrak{A} = (\mathbb{R}, <)$ is separable, while $\mathfrak{B} = (\mathbb{R} \cdot \omega_1, <)$ is not separable, yet $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathscr{L}^{<\omega}_{\infty\omega}$-equivalent by Lemma 6.3.2. ▯

Badger [1977] has shown that $\mathscr{L}^{<\omega}_{\infty\omega}$ does not allow Δ-interpolation for $\mathscr{L}(Q^2)$, in every cardinal interpretation. In Badger [1980] one finds that the Beth property fails for $\mathscr{L}^{<\omega}_{\infty\omega}$, in every cardinal interpretation κ with κ regular. This partially generalizes a theorem (and its proof) of *H.* Friedman [1973], that the Beth property fails for every $\mathscr{L}_{\infty\omega}(Q_\alpha)$. In particular we can prove:

6.3.4 Corollary (Badger [1980]). *There is an implicitly definable relation of $\mathscr{L}(Q)$ which is not explicitly definable in $\mathscr{L}^{<\omega}_{\infty\omega}$ (in the ω_1-interpretation).*

Hint of proof. We may combine the proofs of Corollary 6.3.4 and of Theorem XVIII.4.3, which then say that under suitable conditions the Beth property implies interpolation. Roughly speaking, we may show by induction on formulas that for any two tree structures as in the proof of Theorem XVIII.4.3, every map which is a partial isomorphism from a subtree onto a subtree (and appropriately respects the tree order) actually does preserve $\mathscr{L}^{<\omega}_{\infty\omega}$-formulas. ▯

Following is a natural example which shows that the Δ-interpolation property does not imply the interpolation property. The original version appears in Theorem II.7.2.6 and is due to H. Friedman. Although that result involves infinitary logic, the following one, in fact, is based on it.

6.3.5 Theorem. *$\Delta(\mathscr{L}^{<\omega})$ does not allow interpolation (or the Robinson property) for $\mathscr{L}(Q)$.*

Proof. If $\kappa = \omega$ or $\kappa = \omega_1$ then the class of linear orders of cofinality κ is a PC class of $\mathscr{L}(Q)$, as we simply assert that X is a cofinal subset which is countable (if $\kappa = \omega$) or is ω_1-like (if $\kappa = \omega_1$). Since these classes are disjoint, it suffices to find two linear orders which are $\Delta(\mathscr{L}^{<\omega})$-equivalent, and whose cofinalities are ω and ω_1. Choose a structure $\mathfrak{A} = (R_\alpha, \omega_2, \in)$, where $(R_\alpha, \in)$ satisfies the same Σ_n-sentences with parameters in $(R_\alpha, \in)$ as does $(V, \in)$, by the reflection theorem; n

should be sufficiently large so that the definitions of satisfaction for $\Delta(\mathscr{L}^{<\omega})$, and of ω_2 should be, say, Σ_{n-10} (to be safe). Thus, let us say that $\omega_2 = P^A$. Choose $\mathfrak{B}_1 \prec \mathfrak{A}$ and $\mathfrak{B}_2 \prec \mathfrak{A}$ such that $\omega_1 + 1 \subseteq B_1 \cap B_2$, $(P^{\mathfrak{B}_1}, \in)$ has cofinality ω, and $(P^{\mathfrak{B}_2}, \in)$ has cofinality ω_1. That done, our proof will be complete once we have proved that for all formulas ϕ of $\Delta(\mathscr{L}^{<\omega})$:

(*) Let $\mathfrak{C} \prec \mathfrak{A}$, where $\omega_1 + 1 \subseteq C$. Then, for all $\mathbf{a}$ in $P^{\mathfrak{C}}$ and all $X_1, \ldots, X_n \in C$ $(X_i \subseteq \omega_2^{m_i})$,

$$(P^{\mathfrak{C}}, \in, X_1 \cap C, \ldots, X_n \cap C) \models \phi(\mathbf{a}) \text{ iff } (\omega_2, \in, X_1, \ldots, X_n) \models \phi(\mathbf{a}).$$

To prove (*) it suffices (see the proof of the result in Theorem II.7.2.4(i)) to show that for all Σ formulas θ of $\mathscr{L}^{<\omega}$, if $\mathbf{X} \in C$ $(\mathbf{X} \subseteq \omega_2^{<\omega})$ and $(\omega_2, \in, \mathbf{X}) \models \theta(\mathbf{a})$ then $(P^{\mathfrak{C}}, \in, \mathbf{X} \cap C) \models \theta(\mathbf{a})$. Thus, suppose that $(\omega_2, \in, \mathbf{X}) \models \exists Y \psi(\mathbf{a}, Y)$, where $\psi \in \mathscr{L}^{<\omega}$. Then, by choice of $\mathfrak{A}$, $\mathfrak{A} \models \exists Y$"$(P, \in, \mathbf{X}, Y) \models \psi(\mathbf{a}, Y)$". Since $\mathfrak{C} \prec \mathfrak{A}$ we have $\mathfrak{A} \models$ "$(P, \in, \mathbf{X}, Y) \models \psi(\mathbf{a}, Y)$" for some $Y \in C$; and then again, by choice of $\mathfrak{A}$, it follows that $(\omega_2, \in, \mathbf{X}, Y) \models \psi(\mathbf{a}, Y)$. Hence, it suffices to prove (*) for all $\phi \in \mathscr{L}^{<\omega}$. However, this is merely a straightforward induction on ϕ and is therefore left to the reader. □

For $\mathscr{L}^{<\omega}$, we have just presented counterexamples to interpolation involving separability of linear orders and countable versus uncountable cofinality, notions which are elementary in $\mathscr{L}(\mathrm{aa})$. However, the notion of whether an ω_1-like tree has a branch is elementary in $\mathscr{L}^{<\omega}$ but not in $\mathscr{L}(\mathrm{aa})$. This was observed by Shelah, and the relevant details are supplied in Makowsky–Shelah [1981]. In this connection, the reader should also see Ebbinghaus [1975b] for a related theorem. For an extension see also Example XVII.2.4.5 and Proposition XVII.2.4.6.

6.3.6 Theorem. $\mathscr{L}(\mathrm{aa})$ *does not allow interpolation for* $\mathscr{L}(Q)$. *In fact, under* MA + $\neg$CH, $\mathscr{L}(\mathrm{aa})$ *does not allow* Δ*-interpolation for* $\mathscr{L}(Q)$.

Hint of Proof. We find two ranked trees of height ω_1 (or ω_1-like) which are $\mathscr{L}(\mathrm{aa})$-equivalent, but such that one has a branch and the other does not. More precisely, we find $\mathscr{L}(\mathrm{aa})$-equivalent structures in the following disjoint PC classes of $\mathscr{L}(Q)$: the class of ω_1-like ranked trees satisfying $\exists X$ ("X is an uncountable linearly ordered subset"); and the class of models of $\exists f$ ("f is an order-preserving map into a countable linear order"). According to a theorem of Baumgartner–Malitz–Reinhardt [1970, Theorem 4], under MA + $\neg$CH, these classes are complementary in the class of ω_1-like ranked trees. □

Shelah [1982a] has announced the relative consistency of Δ-interpolation of $\mathscr{L}(\mathrm{aa})$ for $\mathscr{L}(Q)$. We should also note that by a theorem of Caicedo [1981b, 4.1], Theorem 6.3.6 implies that $\mathscr{L}_{\infty\omega}(\mathrm{aa})$ does not allow interpolation for $\mathscr{L}(Q)$.

Question 6.3.1 might be asked for the Robinson property rather than for the interpolation property. As to that case, Mundici [1981c] used uncountable vocabularies to supply a negative answer. See also Section XVIII.4.1 for a generalization.

6.4. *Interpolation and Preservation Revisited: Monotone Structures*

Consider the logic of monotone structures $(\mathfrak{A}, q)$ as defined in Section 2.3; that logic has nice properties, including not only compactness and axiomatizability (see Section 2.3), but also interpolation. Interpolation was first proved independently by Shelah (see Bruce [1978a; 3.1, 3.2]), Sgro, and Makowsky–Tulipani [1977, Corollary 3.1]. The reader should also see Chapter XV for related theorems about topological logics. A particularly straightforward way of obtaining completeness and interpolation theorems, even for countable fragments of $\mathscr{L}_{\omega_1\omega}$, is to use consistency properties: see Section VIII.3. We add the following clause: if $Qx\phi$ and $Q^*x\psi$ are in s where s belongs to a consistency property S, then for some c, $s \cup \{\phi(c), \psi(c)\} \in S$. The details involved in this development are straightforward for the reader who is familiar with consistency properties.

Some Directions Radiating from the Study of the Logic of Monotone Structures

(1) Guichard [1980] has used consistency properties to generalize Feferman's many-sorted interpolation theorem [1974a] and its application to preservation theorems, so as to obtain a preservation theorem for bounded quantifiers $Q^y x$ ("for many $x \in y$"), such as are studied in Barwise [1978b].

(2) Interpolation and countable compactness theorems can be proved for the logics $\mathscr{L}^{\mathscr{F}}(Q)$, whose structures are of the form $(A, \ldots; q_{\mathscr{F}})$, where $q_{\mathscr{F}} = \{X \subseteq \omega : \omega - X \notin \mathscr{F}\}$, with $\mathscr{F}$ being any given filter on ω which properly contains the cofinite filter; see Kaufmann [1984a]. There seems to be a connection with uniform validity (as discussed in Kueker [1978]) which has not yet been fully clarified although related work has been undertaken by S. Buechler and D. Kueker.

(3) Finally, we will mention a paper by Ebbinghaus–Ziegler [1982], a paper in which the quantifiers Q^n (as discussed in Section 5) are studied for monotone structures $(\mathfrak{A}, q)$, especially when q is an ultrafilter on A. It is proved there (Theorem 1.1) that the following are equivalent, where we write $\mathscr{L}^{\mathrm{U}}(Q^n)$ to indicate our restriction to ultrafilters:

(i) $\mathscr{L}^{\mathrm{U}}(Q^n)$ is compact;
(ii) $\mathscr{L}^{\mathrm{U}}(Q^n)$ satisfies interpolation;
(iii) $n = 1$.

The emphasis in this chapter has been on logics involving cardinality $\aleph_1$. Although monotone structures may provide some additional understanding of the area, their application to logics with cardinality (and related) quantifiers seems to be limited. Methods appropriate to higher cardinals are studied in the next chapter.

We will conclude this section with a problem: To find an extension of $\mathscr{L}(Q_1)$ which has a Lindström-type characterization (in the sense of Chapter III).

7. *Appendix* (*An Elaboration of Section* 2)

In this section we will present here the precise definitions and the proofs that were promised in section 2. We will begin by considering

7.1. *Concrete Syntax*

In the ensuing discussions, we will frequently make use of the notation $\mathscr{L} = \bigcup_\tau \mathscr{L}(\tau)$. This clear we now present

7.1.1 Definition. A logic $\mathscr{L}$ has *concrete syntax* if the following properties hold.

(i) $\mathscr{L}_{\omega\omega}(\tau) \subseteq \mathscr{L}(\tau)$ for all τ, and furthermore $\mathscr{L}$ is closed under first-order operations $\neg$, $\vee$, $\exists$ (and there is unique readability). If τ is countable, so is $\mathscr{L}(\tau)$. $\forall$, $\wedge$, $\rightarrow$, and $\leftrightarrow$ are defined symbols. Finally, $\tau_1 \subseteq \tau_2$ implies that $\mathscr{L}(\tau_1) \subseteq \mathscr{L}(\tau_2)$.

We allow the map $\tau \mapsto \mathscr{L}(\tau)$ to be a partial map, provided that $\mathscr{L}(\tau \cup C)$ exists whenever $\mathscr{L}(\tau)$ exists and C is a set of constant symbols.

(ii) There is a map *frvar* which assigns a finite set of variables to each formula ϕ of $\mathscr{L}$. Moreover, for $\phi \in \mathscr{L}_{\omega\omega}$, frvar($\phi$) is the set of free variables of ϕ. As usual, a *sentence* is a formula ϕ such that frvar(ϕ) $= \varnothing$. Finally, the map frvar obeys the obvious rules for $\neg$, $\vee$, $\exists$.

(iii) For each formula ϕ of $\mathscr{L}$ and function f mapping a finite set of variables to constants, there is a unique formula $\phi(f)$, which has the usual meaning (substitute $f(v)$ for v) if ϕ is atomic. If $\phi \in \mathscr{L}(\tau)$, then $\phi(f) \in \mathscr{L}(\tau \cup \text{range}(f))$. Moreover,

(a) $(\exists v\phi)(f) = \begin{cases} \exists v(\phi(f)), & \text{if } v \notin \text{dom}(f), \\ \exists v(\phi(f - \{\langle v, f(v)\rangle\})), & \text{if } v \in \text{dom}(f). \end{cases}$

(b) $(\neg\phi)(f) = \neg\phi(f)$, $(\phi \vee \psi)(f) = \phi(f) \vee \psi(f)$.

(c) ("Restriction rule of substitution") $\phi(f) = \phi(f \upharpoonright \text{frvar}(\phi))$.

(d) $\phi(f)(g) = \phi(f \cup g)$, if $f \cup g$ is a function.

(e) $\phi(\varnothing) = \phi$

(iv) There is a notion $\vdash_{\mathscr{L}(\tau)}$ of $\mathscr{L}(\tau)$-proof satisfying the following properties:

(a) An $\mathscr{L}(\tau)$-proof is a finite sequence of $\mathscr{L}(\tau)$-formulas. We write $\Gamma \vdash_{\mathscr{L}(\tau)} \phi$ to indicate that an $\mathscr{L}(\tau)$-proof from $\Gamma \subseteq \mathscr{L}(\tau)$ exists. That is to say, each formula in the proof is either a member of Γ, or an axiom of $\mathscr{L}(\tau)$ (where the axioms include those given below), or else follows from previous formulas in the proof by *modus ponens*. We also require that whenever $\Gamma \vdash_{\mathscr{L}(\tau \cup \{c\})} \phi(\{\langle x, c\rangle\})$ where $\Gamma \cup \{\phi\} \subseteq \mathscr{L}(\tau)$, then $\Gamma \vdash_{\mathscr{L}(\tau)} \forall x\phi$. That is, universal generalization is a derived (and not an explicit) rule.

(b) Every tautology in $\mathscr{L}(\tau)$ is an axiom of $\mathscr{L}(\tau)$, as is every $\mathscr{L}(\tau)$-formula $\forall x \neg\phi \leftrightarrow \neg\exists x\phi$.

(c) Every $\mathscr{L}(\tau)$-formula $\neg\exists y\phi(f) \rightarrow \neg\phi(f \cup \{\langle y, c\rangle\})$ is an axiom of $\mathscr{L}(\tau)$, when $y \notin \text{dom}(f)$.

(d) Every equality axiom of first-order logic which belongs to $\mathscr{L}(\tau)$, is an axiom of $\mathscr{L}(\tau)$. And, for all $\phi \in \mathscr{L}(\tau)$, and f and g such that f and g map variables to constants in τ, where $\mathrm{dom}(f) = \mathrm{dom}(g)$, we have $\vdash_{\mathscr{L}(\tau)} \phi(f) \wedge \bigwedge_{x \in \mathrm{dom}(f)} f(x) = g(x) \to \phi(g)$.

(e) If $\Gamma \cup \{\phi\} \subseteq \mathscr{L}(\tau)$ and $\tau' = \tau \cup C$ for some set C of constants, then $\Gamma \vdash_{\mathscr{L}(\tau)} \phi$ iff $\Gamma \vdash_{\mathscr{L}(\tau')} \phi$.

The final condition for a concrete syntax is:

(v) There is a "rank function" r from $\mathscr{L}$ into the ordinals such that $r(\phi)$ is less than each of $r(\exists x\phi)$, $r(\phi \vee \psi)$, $r(\neg\phi)$.

7.1.2 Remark. For the purposes of Section 2.3, we note that we may speak of a "concrete syntax" even if we omit all of the semantics of a logic. Since all of the results below rely only on the first-order semantics of weak models anyhow, they also make sense and remain true when the semantics is removed.

7.1.3 *Proof of Soundness* (Proposition 2.2.1). Assume that $\vdash_{\mathscr{L}(\tau \cup C)} \phi(f)$, where $\mathrm{range}(f) \cap \mathrm{frvar}(\phi) = \varnothing$. Using the derived rule of universal generalization, we have $\vdash_{\mathscr{L}(\tau \cup C)} \forall x_1 \ldots \forall x_n \phi$, where $\mathrm{dom}(f) = (x_1, \ldots, x_n)$. Then $\vdash_{\mathscr{L}(\tau)} \forall x_1 \cdots \forall x_n \phi$, by Definition 7.1.1 (iv) (e). Thus, we have reduced to the case $f = \varnothing$. But this is a trivial induction on the length of the proof, since *modus ponens* is the only rule of inference and every axiom is valid in $\mathfrak{A}^*$. $\square$

7.2. *Proofs of the Weak Completeness Theorem and Its Extensions*

7.2.1 *Proof of Theorem* 2.2.3 (Weak Completeness Theorem). The argument here is a straightforward Henkin argument. However, it should be observed that we do not attempt to control what sentences hold in $\mathfrak{A}^*$ other than, of course, those of the form ϕ^*. Since τ is countable, so is $\mathscr{L}(\tau)$ by Definition 7.1.1(i). Now, let C be a countable set of constant symbols disjoint from τ, and let $\{\langle \phi_n, f_n \rangle : n < \omega\}$ enumerate all $\langle \phi, f \rangle$ such that ϕ is an $\mathscr{L}(\tau)$-formula and $f : \mathrm{frvar}(\phi) \to C$. By proceeding in the usual way, we may form finite theories T_n of $\mathscr{L}(\tau \cup C)$ such that $T \cup T_n$ is $\mathscr{L}(\tau \cup C)$-consistent, such that for all n:

(i) $\phi_n(f_n) \in T_{n+1}$ or $\neg\phi_n(f_n) \in T_{n+1}$;

(ii) if ϕ_n is $\exists y \psi$ and $\phi_n(f_n) \in T_n$ then $\psi(f \cup \{\langle y, c \rangle\}) \in T_{n+1}$ for some $c \in C$.

Let $T_\omega = \bigcup_{n \in \omega} T_n$. Observe that T_ω is deductively closed. Form the Henkin model from equivalence classes from C ($[c] = [d]$ iff "$c = d$" $\in T_\omega$). For *atomic* $\mathscr{L}(\tau)$-*formulas* ϕ, define

$$(*) \qquad \mathfrak{A}^* \models \phi[\tilde{f}] \quad \text{iff} \quad \phi(f) \in T_\omega, \quad \text{whenever} \quad f : \mathrm{frvar}(\phi) \to C,$$

where $\tilde{f}(x) = [f(x)]$. For $\mathscr{L}(\tau)$-formulas ϕ which are neither atomic, nor a negation, nor a disjunction, nor of the form $\exists x\psi$, define

$$R_\phi^{\mathfrak{A}^*} = \{\langle [c_0], \ldots, [c_{n-1}]\rangle : \phi(f) \in T_\omega,$$

where

$$\operatorname{dom}(f) = \{v_{i_0}, \ldots, v_{i_{n-1}}\} = \operatorname{frvar}(\phi) \quad (i_0 < \cdots < i_{n-1})$$

and

$$f(v_{i_k}) = c_{i_k} \quad (\text{all } k < n)\}.$$

This is well defined by the equality axioms for $\mathscr{L}$, and we see that $(*)$ holds for all such ϕ also. As usual (using the rank function $r(\phi)$ so that we can carry out the induction), $(*)$ holds for formulas $\neg\psi$ and $\psi_1 \vee \psi_2$. The latter uses the "restriction rule of substitution," which is given in Definition 7.1.1(iii)(c) and which we henceforth use implicitly. Finally, for the $\exists$ step, suppose that $\mathfrak{A}^* \models \exists y\psi[\tilde{f}]$, where $\operatorname{dom}(f) = \operatorname{frvar}(\exists y\psi)$, so $y \notin \operatorname{dom}(f)$. Choose $c \in C$ such that $\mathfrak{A}^* \models \psi[\tilde{f} \cup \{\langle y, [c]\rangle\}]$. By the inductive hypothesis (since $r(\psi) < r(\exists y\psi)$), $\psi(f \cup \{\langle y, c\rangle\}) \in T_\omega$. Hence, $(\exists y\psi)(f) \in T_\omega$. For otherwise, we would have that $\neg\exists y\psi(f) \in T_\omega$, so that by an axiom and *modus ponens*, $\neg\psi(f \cup \{\langle y, c\rangle\}) \in T_\omega$, contradicting consistency of some T_n. For the other direction of $(*)$, suppose that $(\exists y\psi)(f) \in T_\omega$, where f: $\operatorname{frvar}(\phi) \to C$. Then for some n, we have that $\langle(\exists y\psi), f\rangle = \langle \phi_n, f_n\rangle$ holds. Thus, by construction, there exists $c \in C$ such that $\psi(f \cup \{\langle y, c\rangle\}) \in T_{n+1}$. By the inductive hypothesis, $\mathfrak{A}^* \models \psi(\tilde{f} \cup \{\langle y, [c]\rangle\})$. So, $\mathfrak{A}^* \models \exists y\psi[\tilde{f}]$.

By construction, $T \subseteq T_\omega$ and every $\mathscr{L}(\tau)$ axiom belongs to T_ω. By $(*)$ it now follows that $\mathfrak{A}^*$ is a weak model of T. ☐

7.2.2 *Proof of Weak Omitting Types Theorem* (2.2.5). The proof of the weak completeness theorem given above will suffice here provided we mix in some additional steps as follows. We enumerate (in type ω) all pairs $\langle \Sigma, f\rangle$ such that $\Sigma = \Sigma_n$ for some $n < \omega$ and f maps $\mathbf{x}_n$ to C. At stage $(n + 1)$ we guarantee that $\mathfrak{A}^* \models \neg\sigma[\tilde{f}]$ for some $\sigma \in \Sigma$. By $(*)$ in the proof of Theorem 2.2.3, it suffices that $\neg\sigma(f) \in T_{n+1}$ for some $\sigma \in \Sigma$. But this can be easily achieved by using the local omitting hypothesis, since T_n is consistent with T. ☐

The following technical lemma is used in Sections 3, 4, and 5, to extend weak models. It is the precise version of Lemma 2.2.6.

7.2.3 Lemma (Extension Lemma). *Assume the following hypotheses, where $\mathscr{L}$ has concrete syntax.*

(i) *τ, τ', and D are disjoint countable vocabularies.*
(ii) *$\mathfrak{A}^*$ is a countable weak model for $\mathscr{L}(\tau)$, and $D \supseteq D_A = \{d_a : a \in A\}$.*

(iii) *T is an $\mathscr{L}(\tau \cup \tau' \cup D)$-consistent set of $\mathscr{L}(\tau \cup \tau' \cup D)$-sentences; and in addition, $T = \{\phi(f): \langle \phi, f \rangle \in \Gamma\}$ for some Γ such that for all $\langle \phi, f \rangle \in \Gamma$, $\phi \in \mathscr{L}(\tau \cup \tau')$ and range $f \subseteq D$.*

(iv) *For all assignments s in A, set $\tilde{s} = \{\langle x, d_{s(x)} \rangle : x \in \operatorname{dom} s\}$; then*

$$\{\langle \phi, \tilde{s} \rangle : \mathfrak{A}^* \models \phi[s]\} \subseteq \Gamma.$$

(v) *T $\mathscr{L}(\tau \cup \tau' \cup D)$-locally omits sets $\Sigma_n(f_n) = \{\sigma(f_n): \sigma \in \Sigma_n\}$ for all $n < \omega$, where Σ_n has free variables $\mathbf{x}_n$ and $\mathbf{y}_n$ (disjoint) and $f_n: \mathbf{y}_n \to D_A$. Observe that $\mathbf{y}_n$ may be infinite.*

Then there exists a countable weak model $\mathfrak{B}^*$ for $\mathscr{L}(\tau \cup \tau')$ such that $\mathfrak{A}^* \prec^w \mathfrak{B}^* \restriction \tau^+$, and moreover there exists a function $g: D \to B$ such that $g(d_a) = a$ for all $a \in A$ and $\mathfrak{B}^* \models \phi[g \circ f]$ for all $\langle \phi, f \rangle \in \Gamma$. Finally, $\mathfrak{B}^* \models \forall \mathbf{x}_n \vee \{\neg \sigma: \sigma \in \Sigma_n\} [g \circ f_n]$ for all $n < \omega$.

Proof. The proofs of the weak completeness and weak omitting types theorems show that if we add a countable set C of new constant symbols, we may obtain an $\mathscr{L}(\tau \cup \tau' \cup D \cup C)$-theory T_ω with the following properties.

(1) $T \subseteq T_\omega$.

(2) For all $\phi \in \mathscr{L}(\tau \cup \tau')$ and all maps $f: \operatorname{frvar}(\phi) \to D \cup C$, $\phi(f) \in T_\omega$ iff $\neg \phi(f) \notin T_\omega$.

(3) For all $\exists x \phi \in \mathscr{L}(\tau \cup \tau')$ and $f: \operatorname{frvar}(\exists x \phi) \to D \cup C$, if $\exists x \phi(f) \in T_\omega$ then $\phi(f \cup \{\langle x, c \rangle\}) \in T_\omega$ for some $c \in C$.

(4) For all $f: \mathbf{x}_n \to D \cup C$, $\neg \sigma(f_n \cup f) \in T_\omega$ for some $\sigma \in \Sigma_n$.

Form the Henkin model $\mathfrak{B}^*$ from $D \cup C$. As in the proof of Theorem 2.2.3, (2) and (3) together imply that

$$(*) \qquad \mathfrak{B}^* \models \phi[\bar{f}] \quad \text{iff} \quad \phi(f) \in T_\omega$$

for all $\mathscr{L}(\tau \cup \tau')$-formulas ϕ and functions $f: \operatorname{frvar}(\phi) \to C \cup D$, where $\bar{f}(x) = [f(x)]$. In particular, since $T \subseteq T_\omega$ by (1) above, $\mathfrak{B}^* \models \phi[\bar{f}]$ for all $\langle \phi, f \rangle \in \Gamma$, by (iii). Now, if $\mathfrak{A}^* \models \phi[s]$, then $\langle \phi, \tilde{s} \rangle \in \Gamma$ so $\mathfrak{B}^* \models \phi[\tilde{s}]$. Hence, by identifying $[d_a]$ and a for all $a \in A$, we obtain $\bar{\tilde{s}} = s$ and conclude that $\mathfrak{A}^* \prec^w \mathfrak{B}^* \restriction \tau^+$. Finally, if we set $g(c) = [c]$ for all $c \in D$, then, by using (4) to see that each Σ_n is appropriately omitted, we obtain the desired conclusions. ∎