Part A

Basic Notions of Definability

Chapter I
Groundwork

In this introductory chapter we review some of the prerequisites to the theory to be studied in this book. In the first section we discuss the kinds of objects we shall study — functions and relations with natural numbers and functions of natural numbers as arguments. The second section outlines the application of topological and measure-theoretic notions to these objects. In the third section we discuss inductive definability, a notion which plays a dual role in our theory. Most of our fundamental definitions are given inductively, but in addition we shall *study* inductive definability as a means of classifying sets and relations.

The reader need not master all of Chapter I before going on to the theory proper. Subsections 1.1–1.5, 2.1–2.4, and 3.1–3.11 will suffice for a reading of Chapter II and most of Chapter III, and the other subsections may be used for reference. A list of all of the global notational conventions is on page 467.

1. Logic and Set Theory

1.1 Functions and Sequences. A function φ is a set of ordered pairs (x, y). The *domain* of φ is the set $\mathrm{Dm}\,\varphi = \{x\colon \text{for some } y\,(x, y) \in \varphi\}$, and the *image* of φ is the set $\mathrm{Im}\,\varphi = \{y\colon \text{for some } x\,(x, y) \in \varphi\}$. We often write $\varphi(x)\downarrow$ or say $\varphi(x)$ *is defined* to mean that $x \in \mathrm{Dm}\,\varphi$. Similarly, $\varphi(x)\uparrow$ means $x \notin \mathrm{Dm}\,\varphi$ and is read $\varphi(x)$ *is undefined.* If φ and ψ are two functions, we write $\varphi(x) \simeq \psi(x')$ to mean that *either* both $\varphi(x)$ and $\psi(x')$ are undefined *or* both are defined and have the same value $(\varphi(x) = \psi(x'))$. In particular, $\varphi(x) \simeq y$ means that $\varphi(x)$ is defined and has value y — that is, $(x, y) \in \varphi$. We write $\varphi(x) = y$ only in contexts where it is clear that $\varphi(x)$ is defined. If $\mathrm{Dm}\,\varphi$ and $\mathrm{Dm}\,\psi$ are both subsets of a set X, then the statement (for all $x \in X$) $\varphi(x) \simeq \psi(x)$ means simply that φ and ψ denote the same function. If the set X is clear from context, this may be written simply $\varphi(x) \simeq \psi(x)$. The *restriction of* φ *to* X is the function $\varphi \restriction X = \{(x, y)\colon x \in X \text{ and } \varphi(x) \simeq y\}$. The *image of* X *under* φ is the set $\varphi'' X = \mathrm{Im}(\varphi \restriction X)$.

We write $\varphi\colon X \to Y$ to mean φ is a function, $\mathrm{Dm}\,\varphi \subseteq X$, and $\mathrm{Im}\,\varphi \subseteq Y$. If $\mathrm{Dm}\,\varphi = X$ we say φ is *total*; otherwise, φ is *partial.* The set of all total functions $\varphi\colon X \to Y$ is denoted by ${}^{X}Y$.

If y_x denotes an element of Y whenever $x \in Z \subseteq X$, we use any of the expressions $x \mapsto y_x$, $\lambda x . y_x$ and $\langle y_x : x \in Z \rangle$ to denote the function $\{(x, y_x) : x \in Z\}$.

The *natural number* m is the set $\{0, 1, \ldots, m-1\}$ of all smaller natural numbers. The set of all natural numbers is denoted by ω.

For any set X, a *finite sequence* from X is a function $\mathbf{x}$ with domain a natural number k called the *length of* $\mathbf{x}$ ($\lg(\mathbf{x})$) and image a subset of X. Hence $\mathbf{x} \in {}^kX$. For $i < \lg(\mathbf{x})$, $\mathbf{x}(i)$ is called the i-th *component* of $\mathbf{x}$ and is usually denoted by x_i. To exhibit all the components we write $(x_0, x_1, \ldots, x_{k-1})$ for $\mathbf{x}$. Note that the *empty sequence* $\varnothing$ is the unique sequence of length 0. We make no distinction in general between X and ${}^1X$. Note that if $\mathbf{x} = (x_0, \ldots, x_{k-1})$ and $\mathbf{y} = (y_0, \ldots, y_{l-1})$ are two finite sequences from X, then $\mathbf{x} \subseteq \mathbf{y}$ just in case $\mathbf{y}$ extends $\mathbf{x}$; that is, $k \leqslant l$ and for all $i < k$, $x_i = y_i$. The operation $\mathbf{x} * \mathbf{y}$ produces the sequence $(x_0, \ldots, x_{k-1}, y_0, \ldots, y_{l-1})$. If $\varphi \in {}^\omega X$, then $\mathbf{x} * \varphi$ is the function $\psi \in {}^\omega X$ such that for $i < \lg(\mathbf{x})$, $\psi(i) = x_i$, and for $i \geqslant \lg(\mathbf{x})$, $\psi(i) = \varphi(i - \lg(\mathbf{x}))$. If $Z \subseteq X$, we sometimes write $\mathbf{x} \in Z$ to mean that for all $i < \lg(\mathbf{x})$, $x_i \in Z$. Similarly, $\mathbf{m} < n$ means that for all $i < \lg(\mathbf{m})$, $m_i < n$. If $\varphi : X \to Y$ and $\mathbf{x} \in {}^kX$, then $\varphi(\mathbf{x})$ denotes the sequence $(\varphi(x_0), \ldots, \varphi(x_{k-1}))$.

1.2 Functionals and Relations. For $k, l \in \omega$ we set ${}^{k,l}\omega = {}^k\omega \times {}^l({}^\omega\omega)$. A function $\mathsf{F} : {}^{k,l}\omega \to \omega$ is called a *functional of rank* (k, l). A functional of rank $(k, 0)$ is also called a *function of rank* k and is identified with the corresponding function $F : {}^k\omega \to \omega$. Elements of ${}^\omega\omega$ are thus total functions of rank 1.

Elements of ${}^{k,l}\omega$ are ordered pairs of the form $(\mathbf{m}, \boldsymbol{\alpha})$. However, if F is a functional of rank (k, l), we write $\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha})$ instead of $\mathsf{F}((\mathbf{m}, \boldsymbol{\alpha}))$ and think of $\mathbf{m}, \boldsymbol{\alpha}$ as a *list* of arguments $m_0, \ldots, m_{k-1}, \alpha_0, \ldots, \alpha_{l-1}$. Thus, for example, we write $\mathsf{F}(m_0, \ldots, m_{k-1}, \boldsymbol{\alpha})$ instead of $\mathsf{F}((m_0, \ldots, m_{k-1}), \boldsymbol{\alpha})$ and $\mathsf{F}(p, \mathbf{m}, \boldsymbol{\alpha}, \beta, \gamma)$ instead of of $\mathsf{F}((p) * \mathbf{m}, \boldsymbol{\alpha} * (\beta, \gamma))$. If F is a total functional of rank $(k+1, l)$, then F may also be thought of as a function from ${}^{k,l}\omega$ into ${}^\omega\omega$ whose values are given by:

$$\mathsf{F}[\mathbf{m}, \boldsymbol{\alpha}] = \langle \mathsf{F}(p, \mathbf{m}, \boldsymbol{\alpha}) : p \in \omega \rangle = \lambda p . \mathsf{F}(p, \mathbf{m}, \boldsymbol{\alpha}).$$

A subset R of ${}^{k,l}\omega$ is called a *relation of rank* (k, l). We usually write $\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha})$ instead of $(\mathbf{m}, \boldsymbol{\alpha}) \in \mathsf{R}$. A relation of rank $(k, 0)$ is called a *relation* (*on numbers*) *of rank* k and is identified with the corresponding subset R of ${}^k\omega$. In accord with the list notation for functionals, we write, for example, $\mathsf{R}(\mathbf{m}, \alpha_0, \ldots, \alpha_{l-1})$ for $\mathsf{R}(\mathbf{m}, (\alpha_0, \ldots, \alpha_{l-1}))$ and $\mathsf{R}(p, q, \mathbf{m}, r, \beta, \boldsymbol{\alpha})$ for $\mathsf{R}((p, q) * \mathbf{m} * (r), (\beta) * \boldsymbol{\alpha})$. For $\mathsf{R} \subseteq {}^{k,l}\omega$, the *complement* of R (with respect to ${}^{k,l}\omega$) is the relation $\sim\mathsf{R} = \{(\mathbf{m}, \boldsymbol{\alpha}) : (\mathbf{m}, \boldsymbol{\alpha}) \in {}^{k,l}\omega$ and $(\mathbf{m}, \boldsymbol{\alpha}) \notin \mathsf{R}\}$.

With each relation R of rank (k, l) is associated its *characteristic functional* of rank (k, l) defined by

$$\mathsf{K}_{\mathsf{R}}(\mathbf{m}, \boldsymbol{\alpha}) = \begin{cases} 0, & \text{if } \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}); \\ 1, & \text{otherwise.} \end{cases}$$

Conversely, to each functional F of rank (k, l) corresponds it graph Gr_{F} (occasionally $\mathrm{Gr}(\mathsf{F})$), a relation of rank $(k+1, l)$ defined by

$$\mathrm{Gr}_{\mathsf{F}}(n, \mathbf{m}, \boldsymbol{\alpha}) \quad \text{iff} \quad \mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n.$$

By the identification of functionals of rank $(1, 0)$ with functions from ω into ω and of relations of rank $(1, 0)$ with subsets of ω, if $A \subseteq \omega$, then $\mathsf{K}_A \in {}^{\omega}\omega$. We write, for example, $\mathsf{F}(\mathbf{m}, A, \boldsymbol{\alpha}, B)$ instead of $\mathsf{F}(\mathbf{m}, \mathsf{K}_A, \boldsymbol{\alpha}, \mathsf{K}_B)$ and thus extend functionals to admit subsets of ω as arguments.

Compositions of partial functionals and relations are taken to be defined whenever possible. For example, $\mathsf{F}(\mathsf{G}(\mathbf{m}, \boldsymbol{\alpha}), \mathbf{m}, \boldsymbol{\alpha}) \simeq n$ just in case for some $p \in \omega$, $\mathsf{G}(\mathbf{m}, \boldsymbol{\alpha}) \simeq p$ and $\mathsf{F}(p, \mathbf{m}, \boldsymbol{\alpha}) \simeq n$. Similarly, $\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}, \lambda p \,.\, \mathsf{H}(p, \mathbf{m}, \boldsymbol{\alpha})) \simeq n$ just in case for some $\beta \in {}^{\omega}\omega$, $\mathsf{H}(p, \mathbf{m}, \boldsymbol{\alpha}) \simeq \beta(p)$ for all $p \in \omega$, and $\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}, \beta) \simeq n$. Note that $\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}, \lambda p \,.\, \mathsf{H}(p, \mathbf{m}, \boldsymbol{\alpha}))$ is undefined for any $\mathbf{m}$ and $\boldsymbol{\alpha}$ for which $\lambda p \,.\, \mathsf{H}(p, \mathbf{m}, \boldsymbol{\alpha})$ is not total. For relations, we have, for example, $\mathsf{R}(\mathbf{m}, \mathsf{G}(\mathbf{m}, \boldsymbol{\alpha}), \boldsymbol{\alpha})$ is true iff for some $p \in \omega$, $\mathsf{G}(\mathbf{m}, \boldsymbol{\alpha}) \simeq p$ and $\mathsf{R}(\mathbf{m}, p, \boldsymbol{\alpha})$, and false otherwise (not undefined).

Natural numbers are said to be objects of *type* 0. Functions from ${}^{k}\omega$ into ω and subsets of ${}^{k}\omega$ are objects of *type* 1; functionals from ${}^{k,l}\omega$ into ω and subsets of ${}^{k,l}\omega$ $(l > 0)$ are objects of *type* 2. In general a function with natural number values or a relation is of *type* $n+1$ iff its arguments are objects of types at most n. In practice, the arguments of types >0 will almost always be total unary functions. Thus the objects of type 3 discussed in § VI.7 and Chapter VII are functions and relations on ${}^{k,l,l'}\omega = {}^{k}\omega \times {}^{l}({}^{\omega}\omega) \times {}^{l'}({}^{{}^{\omega}\omega}\omega)$. Elements of ${}^{k,l,l'}\omega$ are written $(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I})$, where $\mathbf{I} = (\mathsf{I}_1, \ldots \mathsf{I}_{l'-1})$. Functionals of type 3 are denoted by letters $\mathbb{F}, \mathbb{G}, \mathbb{H}, \ldots$ and relations of type 3 by $\mathbb{R}, \mathbb{S}, \mathbb{T}, \ldots$.

1.3 Logical Notation. We shall use the logical symbols $\wedge$, $\vee$, $\neg$, $\rightarrow$, and $\leftrightarrow$ as abbreviations for the expressions 'and', 'or', 'not', 'implies', and 'if and only if', respectively. Although we are not, for the most part, dealing with formalized languages, these connectives are to be understood in their usual truth-functional sense. Thus, for example, an expression of the form ——— $\rightarrow$ .... is true just in case ——— is false, or .... is true (or both).

The symbols $\exists$ and $\forall$ will be used as abbreviations for 'there exists' and 'for all', respectively. In most cases the range of the quantifier will be indicated by the type of variable following it in accord with the conventions listed on page 467. For example, an expression of the form $\exists m$ [— m —] is true just in case — m — is true for some natural number m. Similarly, the condition for equality of partial functionals is written

$$\mathsf{F} = \mathsf{G} \leftrightarrow \forall \mathbf{m} \forall \boldsymbol{\alpha} \, [\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) \simeq \mathsf{G}(\mathbf{m}, \boldsymbol{\alpha})].$$

Further restrictions on the range of a quantifier may be indicated by use of a *bounded quantifier.* For example, $(\exists m < p)$— m — is true iff — m — is true for some m among $0, 1, \ldots, p-1$; $(\forall \gamma \in \mathrm{W}) \ldots . \gamma \ldots .$ is true iff $\ldots . \gamma \ldots .$ is true for all γ belonging to W. We write $\exists ! m [— m —]$ to mean that — m — is true for exactly one m.

Parentheses, (and), and brackets, [and], are used interchangeably in sufficient quantity to ensure unique readability of expressions. In addition, a single dot . may be used to set off two parts of an expression — for example, $(\forall p < q) . R(p, \mathbf{m})$ or $\lambda p . f(p, \mathbf{m}) + g(p)$.

1.4 Sequence Coding. For each k, ${}^k\omega$ is a countable set and may thus be put in one-one correspondence with a subset of ω. Similarly for $l > 0$, ${}^l({}^\omega\omega)$ may be put in one-one correspondence with a subset of ${}^\omega\omega$. We define here some particularly simple such correspondences which we call *coding functions.*

Temporarily we let p_i denote the i-th prime number: $p_0 = 2, p_1 = 3, \ldots$. For each k we define a total function $\langle \cdot \rangle^k$ of rank k by:

$$\langle \quad \rangle^0 = 1, \text{ and for any } k > 0 \text{ and any } \mathbf{m} \in {}^k\omega,$$

$$\langle \mathbf{m} \rangle^k = \langle m_0, \ldots, m_{k-1} \rangle^k = p_0^{m_0+1} \cdot p_1^{m_1+1} \cdot \cdots \cdot p_{k-1}^{m_{k-1}+1}.$$

The unique factorization theorem of arithmetic ensures that if $\langle \mathbf{m} \rangle^k = \langle \mathbf{n} \rangle^l$, then $k = l$ and $\mathbf{m} = \mathbf{n}$. As the superscript is usually clear from the context, we shall usually omit it.

For any s, t, and $i \in \omega$, let

$$(s)_i = \text{least } m \ [m < s \wedge p_i^{m+2} \text{ does not divide } s];$$

$$\lg(s) = \text{least } k \ [k < s \wedge p_k \text{ does not divide } s];$$

$$s * t = s \cdot t', \text{ where } t' \text{ arises from } t \text{ by replacing each}$$

$$\text{factor } p_i^n \text{ in the prime decomposion of } t \text{ by } p_{\lg(s)+i}^n.$$

Then it is an arithmetical exercise to verify that for all k, all $\mathbf{m} \in {}^k\omega$, all $\mathbf{n}$, and all $i < k$, $(\langle \mathbf{m} \rangle)_i = m_i$, $\lg(\langle \mathbf{m} \rangle) = k$, and $\langle \mathbf{m} \rangle * \langle \mathbf{n} \rangle = \langle \mathbf{m} * \mathbf{n} \rangle$. We denote by Sq the set of all s such that $s = \langle \mathbf{m} \rangle$ for some $\mathbf{m}$. Note that for any k and s, s may be regarded as coding a sequence of length k, namely $((s)_0, \ldots, (s)_{k-1})$. We often regard $\mathbf{m}$ and $\langle \mathbf{m} \rangle$ as interchangeable and write, for example, $\mathbf{p} \subseteq \langle \mathbf{m} \rangle$ instead of $\mathbf{p} \subseteq \mathbf{m}$. In particular, $s \subseteq t$ iff for some $\mathbf{m}$ and $\mathbf{n}$, $s = \langle \mathbf{m} \rangle$, $t = \langle \mathbf{n} \rangle$, and $\mathbf{m} \subseteq \mathbf{n}$. For any $\beta \in {}^\omega\omega$, $\bar{\beta}(k)$ denotes the code for the sequence $\beta \upharpoonright k$ — that is, $\bar{\beta}(k) = \langle \beta(0), \ldots, \beta(k-1) \rangle$.

We next define coding functions from ${}^l({}^\omega\omega)$ into ${}^\omega\omega$:

$$\langle \quad \rangle^0 = \lambda m . 1, \text{ and for any } l > 0 \text{ and any } \boldsymbol{\alpha} \in {}^l({}^\omega\omega),$$

$$\langle \boldsymbol{\alpha} \rangle^l = \langle \alpha_0, \ldots, \alpha_{l-1} \rangle^l = \lambda m . \langle \alpha_0(m), \ldots, \alpha_{l-1}(m) \rangle.$$

Again it is obvious that if $\langle \boldsymbol{\alpha} \rangle^l = \langle \boldsymbol{\beta} \rangle^k$, then $l = k$ and $\boldsymbol{\alpha} = \boldsymbol{\beta}$, and that we may omit the superscript without ambiguity.

For any γ and $\delta \in {}^\omega\omega$ and any $j \in \omega$, let

$$(\gamma)_j = \lambda m \,.\, (\gamma(m))_j;$$

$$\lg(\gamma) = \lg(\gamma(0));$$

$$\gamma * \delta = \lambda m \,.\, \gamma(m) * \delta(m).$$

Then for all l, all $\boldsymbol{\alpha} \in {}^l({}^\omega\omega)$, all $\boldsymbol{\beta}$, and all $j < l$, $(\langle \boldsymbol{\alpha} \rangle)_i = \alpha_i$, $\lg(\langle \boldsymbol{\alpha} \rangle) = l$, and $\langle \boldsymbol{\alpha} \rangle * \langle \boldsymbol{\beta} \rangle = \langle \boldsymbol{\alpha} * \boldsymbol{\beta} \rangle$. We denote by Sq_l the set of all γ such that $\gamma = \langle \boldsymbol{\alpha} \rangle$ for some $\boldsymbol{\alpha}$. For any l and γ, γ may be regarded as coding the sequence $((\gamma)_0, \ldots, (\gamma)_{l-1})$.

It will also be useful occasionally to code ω-sequences of functions. We set

$$\langle \alpha_0, \alpha_1, \ldots, \alpha_n, \ldots \rangle = \lambda m \,.\, \alpha_{(m)_0}((m)_1), \quad \text{and}$$

$$(\gamma)^n(m) = \gamma(\langle n, m \rangle).$$

Then clearly $(\langle \alpha_0, \alpha_1, \ldots, \alpha_n, \ldots \rangle)^n(m) = \alpha_n(m)$.

1.5 Set Theory. Except where we specify otherwise, the results of this book are all theorems of ZFC, Zermelo-Fraenkel Set Theory with the Axiom of Choice. The (Generalized) Continuum Hypothesis is not assumed. We shall occasionally want to replace the full Axiom of Choice (AC) by the weaker Axiom of Dependent Choices:

$$\text{(DC)} \qquad \forall x \, \exists y \,.\, (x, y) \in X \rightarrow \exists \varphi \, \forall m \,.\, (\varphi(m), \varphi(m+1)) \in X.$$

We recall that this implies the principle of choice for countable families of non-empty sets:

$$(\mathrm{AC}_\omega) \qquad \forall m \,.\, Y_m \neq \varnothing \rightarrow \exists \psi \, \forall m \,.\, \psi(m) \in Y_m.$$

Most of our set-theoretic conventions are standard and we refer the reader to (for example) Lévy [1978] for further background. A set x is transitive iff $\forall y (y \in x \rightarrow y \subseteq x)$. x is an *ordinal* (*number*) iff x and all of its elements are transitive. For ordinals π and ρ, $\pi < \rho$ iff $\pi \in \rho$; the relation $\leqslant$ is a well-ordering on any set of ordinals. For any ordinal π, $\pi + 1$ is the set $\pi \cup \{\pi\}$, the *ordinal successor* of π. ρ is a *successor ordinal* iff $\rho = \pi + 1$ for some π; ρ is a *limit ordinal* iff $\forall \pi (\pi < \rho \rightarrow \pi + 1 < \rho)$. Every ordinal is either 0, a limit, or a successor. The natural numbers are exactly the finite ordinals and ω is the smallest limit ordinal. Or is the class of all ordinals.

For any set X of ordinals we denote by $\inf X$ the $\leqslant$-least element of X.

Although X need not have a $\leqslant$-greatest element, there is always an ordinal greater than or equal to all elements of X and we denote by $\sup X$ the least such ordinal. In fact, $\sup X$ is exactly the union of the members of X. We set also $\sup^+ X = \sup\{\pi + 1: \pi \in X\}$. Then $\sup^+ X$ is the least ordinal strictly greater than all elements of X and is the same as $\sup X$ if X has no greatest element; otherwise, $\sup X$ is the greatest element and $\sup^+ X = \sup X + 1$. If φ is a function from ordinals to ordinals and $X \subseteq \operatorname{Dm} \varphi$, then

$$\inf_{\pi \in X} \varphi(\pi) = \inf\{\varphi(\pi): \pi \in X\}, \quad \text{and}$$

$$\sup^{(+)}_{\pi \in X} \varphi(\pi) = \sup^{(+)}\{\varphi(\pi): \pi \in X\}.$$

An ordinal ρ is a *limit of members of* $X \subseteq \mathrm{Or}$ iff $(\forall \pi < \rho)(\exists \sigma \in X) \,.\, \pi < \sigma < \rho$. Any limit of members of any set X is a limit ordinal. If also $\rho \in X$, then ρ is called a *limit point of* X. We denote by $\operatorname{Lim} X$ the set of limit points of X. A subset $Y \subseteq X$ is *cofinal in* X iff $(\forall \sigma \in X)(\exists \tau \in Y)\sigma \leqslant \tau$.

If $\mathfrak{A}$ is a proposition which may be true $(\mathfrak{A}(\sigma))$ or false $(\neg \mathfrak{A}(\sigma))$ of each ordinal σ, then to prove $\forall \sigma\, \mathfrak{A}(\sigma)$ we may use the method of proof by *transfinite induction*: if $\forall \sigma([(\forall \tau < \sigma)\mathfrak{A}(\tau)] \rightarrow \mathfrak{A}(\sigma))$, then $\forall \sigma\, \mathfrak{A}(\sigma)$. We use frequently also the parallel method of *definition by transfinite recursion*: for any total $k+2$-place function ψ, there exists a $k+1$-place function φ such that for all ρ and $\mathbf{x}$,

$$\varphi(\rho, \mathbf{x}) = \psi(\varphi \restriction_{\mathbf{x}} \rho, \rho, \mathbf{x})$$

where

$$\varphi \restriction_{\mathbf{x}} \rho = \{((\pi, \mathbf{x}), z): \pi < \rho \wedge \varphi(\pi, \mathbf{x}) = z\}.$$

φ is not unique, but any other function φ' which satisfies this equation has $\varphi'(\sigma, \mathbf{x}) = \varphi(\sigma, \mathbf{x})$ for all σ and $\mathbf{x}$.

Since any set X of ordinals is well ordered by the relation $\leqslant$, it is uniquely order-isomorphic to an ordinal which we denote by $\|X\|$, the *order-type* of X. The function φ_X which realizes this isomorphism is recursively defined by:

$$\varphi_X(\rho) = \sup^+\{\varphi_X(\pi): \pi < \rho \wedge \pi \in X\}.$$

We list here some elementary properties of φ_X which will be needed in Chapter VIII:

(1) $\qquad \pi, \rho \in X \wedge \pi < \rho \rightarrow \varphi_X(\pi) < \varphi_X(\rho);$

(2) $\qquad \rho \in X \wedge \sigma < \varphi_X(\rho) \rightarrow \exists \pi[\pi \in X \wedge \pi < \rho \wedge \sigma = \varphi_X(\pi)];$

(3) $\qquad \rho \in X \rightarrow \varphi_X(\rho + 1) = \varphi_X(\rho) + 1;$

(4) $\rho \subseteq X \to \varphi_X(\rho) = \rho$;

(5) X is an ordinal $\leftrightarrow (\forall \rho \in X) . \varphi_X(\rho) = \rho$.

An ordinal κ is an *initial ordinal* or *cardinal* (*number*) iff there is no one-one correspondence between κ and any $\tau < \kappa$. From the axiom of choice it follows that for every set X there is a unique cardinal κ such that there exists a one-one correspondence between X and κ. We denote this κ by Card(X), the *cardinal of* X. Then there is a one-one correspondence between two sets X and Y just in case $\text{Card}(X) = \text{Card}(Y)$. The natural numbers are exactly the finite cardinals, and ω is the least infinite cardinal. A set X is *countable* iff $\text{Card}(X) \leqslant \omega$ and denumerable or countably infinite iff $\text{Card}(X) = \omega$. The infinite cardinals are enumerated by the function $\aleph$ defined by:

$$\aleph_0 = \omega;$$

$$\aleph_{\sigma+1} = \{\rho : \text{Card}(\rho) \leqslant \aleph_\sigma\};$$

$$\aleph_\sigma = \bigcup \{\aleph_\pi : \pi < \sigma\}, \quad \text{for limit } \sigma.$$

In particular, $\aleph_1$ is the set of countable ordinals.

For any X, $\mathbf{P}(X)$ denotes the power set of X, the set of all subsets of X. If $\text{Card}(X) = \kappa$, then $\text{Card}\,\mathbf{P}(X)$ is denoted by 2^κ. If X is infinite, then $\text{Card}({}^X 2) = \text{Card}({}^X \omega) = 2^\kappa$. In particular, $\text{Card}({}^\omega \omega) = \text{Card}({}^{k,l}\omega) = 2^{\aleph_0}$ for all k and all $l > 0$. By *Cantor's Theorem*, $\kappa < 2^\kappa$ for all cardinals κ. The *Continuum Hypothesis* is the statement that $2^{\aleph_0} = \aleph_1$.

1.6 Ordering Relations. For any set X and any $Z \subseteq {}^2X$, the *field* of Z is the set

$$\text{Fld}(Z) = \{x : \exists y\, [(x, y) \in Z \vee (y, x) \in Z]\}.$$

Z is a *pre-partial-ordering* iff

(1) $(\forall x \in \text{Fld}(Z))[(x, x) \in Z]$, ($Z$ is *reflexive*),

and

(2) $\forall x\, \forall y\, \forall z\, [(x, y) \in Z \wedge (y, z) \in Z \to (x, z) \in Z]$ (Z is *transitive*).

Z is a *pre-linear-ordering* iff (1), (2), and

(3) $\forall x\, \forall y\, [x, y \in \text{Fld}(Z) \wedge x \neq y \to (x, y) \in Z \vee (y, x) \in Z]$ (Z is *connected*).

Z is a *pre-wellordering* iff (1), (2), (3), and

(4) $\forall Y (Y \subseteq \mathrm{Fld}(Z) \wedge Y \neq \varnothing \rightarrow (\exists x \in Y)(\forall y \in Y)[(y, x) \in Z \rightarrow (x, y) \in Z])$

(Z is *well founded*).

Z is a *partial-* (*linear-*, *well-*) *ordering* iff Z is a pre-partial- (linear-, well-) ordering and

(5) $\forall x \forall y [(x, y) \in Z \wedge (y, x) \in Z \rightarrow x = y]$ (Z is *antisymmetric*).

From the Axiom of Dependent Choice (DC) it follows that (4) is equivalent to

(4′) $\forall \varphi [\forall m . (\varphi(m+1), \varphi(m)) \in Z \rightarrow \exists m . (\varphi(m), \varphi(m+1)) \in Z]$.

If Z is a pre-wellordering, then there is a unique function $|\cdot|_Z$, the *norm associated with* Z, from $\mathrm{Fld}(Z)$ onto an ordinal such that for all $x, y \in \mathrm{Fld}(Z)$,

(6) $(x, y) \in Z \leftrightarrow |x|_Z \leqslant |y|_Z$.

In fact, for any $y \in \mathrm{Fld}(Z)$,

$$|y|_Z = \sup{}^{+}\{|x|_Z : (x, y) \in Z \wedge (y, x) \notin Z\}.$$

Conversely, if $|\ \ |$ is any function from a set Y into the ordinals, the relation $Z_{|\ |}$ defined by

(7) $(x, y) \in Z_{|\ |} \leftrightarrow |x| \leqslant |y|$

is a pre-wellordering. If the image of $|\ \ |$ is an ordinal, then $|\ \ |$ is the norm associated with $Z_{|\ |}$.

Z is a well-ordering just in case $|\ \ |_Z$ is injective (one-one). The image of $|\ \ |_Z$ is called the *(pre-)order-type* of Z and is denoted by $\|Z\|$. Clearly $\|Z\| < \kappa$, where κ is the least cardinal greater than $\mathrm{Card}(X)$. In the context of of set theory without the Axiom of Choice, a useful measure of the size of a set X in terms of ordinals is $o(X) = \sup{}^{+}\{\|Z\| : Z$ is a pre-wellordering and $\mathrm{Fld}(Z) \subseteq X\}$.

Orderings will generally be denoted by symbols $\leqslant$ or $\preccurlyeq$ with various sub- and superscripts. In any such context, the symbols $<$ or $\prec$ always denote the associated *strict ordering* defined by:

$$x < y \leftrightarrow x \leqslant y \wedge y \nleqslant x.$$

With any $\gamma \in {}^{\omega}\omega$ we associate a binary relation $\leqslant_\gamma$ by

$$m \leqslant_\gamma n \leftrightarrow \gamma(\langle m, n\rangle) = 0.$$

We shall mainly be interested in γ such that $\leqslant_\gamma$ is a partial ordering. We then set

$$\mathrm{Fld}(\gamma) = \mathrm{Fld}(\leqslant_\gamma);$$

$$\mathsf{W} = \{\gamma : \leqslant_\gamma \text{ is a well-ordering}\};$$

$$\|\gamma\| = \begin{cases} \|\leqslant_\gamma\|, & \text{if } \gamma \in \mathsf{W}; \\ \aleph_1, & \text{otherwise}; \end{cases}$$

$$(\gamma \restriction p)(\langle m, n\rangle) = \begin{cases} 0, & \text{if } m \leqslant_\gamma n \wedge n <_\gamma p; \\ 1, & \text{otherwise}; \end{cases}$$

$$|p|_\gamma = \|\gamma \restriction p\|.$$

The following facts are easily verified: for any $\gamma \in \mathsf{W}$ and any p,

(8) $\gamma \restriction p \in \mathsf{W}$, and if $\|\gamma\| > 0$, then $\|\gamma \restriction p\| < \|\gamma\|$;

(9) for any $\sigma < \|\gamma\|$, there is a unique $p \in \mathrm{Fld}(\gamma)$ such that $\|\gamma \restriction p\| = \sigma$;

(10) $\|\gamma \restriction p\| = \sup^+\{\|\gamma \restriction q\| : q <_\gamma p\}$;

(11) $\aleph_1 = \{\|\gamma\| : \gamma \in \mathsf{W}\}$.

1.7 Notes. The idea of coding finite sequences of natural numbers by prime powers goes back (at least) to Gödel [1931]. For readers less familiar with set theory we recommend Lévy [1978], Zuckerman [1974], or the handiest recent text.

2. Topology and Measure

We begin our study of the spaces ${}^{k,l}\omega$ by defining a natural topology and measure theory for them. We define first a topology based on viewing ${}^{k,l}\omega$ as a product of copies of ω, show that with this topology ${}^\omega\omega$ is homeomorphic to the set of binary irrational numbers between 0 and 1 with the topology induced from the reals, and using this homeomorphism, carry Lebesgue measure over to ${}^\omega\omega$.

The set ${}^\omega\omega$ may be viewed as a product $\omega \times \omega \times \cdots \times \omega \times \cdots$ of denumerably many copies of ω. To ω we assign the *discrete topology*: all sets are open (and hence all are also closed). Then to ${}^\omega\omega$ we assign the induced *product topology*: a set $\mathsf{A} \subseteq {}^\omega\omega$ is a *basic open set* iff for some n and some (open) subsets $B_0, \ldots, B_{n-1}$ of ω,

$$\mathsf{A} = B_0 \times B_1 \times \cdots \times B_{n-1} \times \omega \times \cdots \times \omega \times \cdots.$$

In other words, for all α,

$$\alpha \in \mathsf{A} \leftrightarrow (\alpha(0) \in B_0) \wedge (\alpha(1) \in B_1) \wedge \cdots \wedge (\alpha(n-1) \in B_{n-1}).$$

The *open subsets* of ${}^{\omega}\omega$ are then, of course, arbitrary unions of basic open sets. Finally to ${}^{k,l}\omega$ we again assign the product topology: $\mathsf{R} \subseteq {}^{k,l}\omega$ is a *basic open relation* iff for some $A_0, \ldots, A_{k-1} \subseteq \omega$ and some open sets $\mathsf{A}_0, \ldots, \mathsf{A}_{l-1} \subseteq {}^{\omega}\omega$,

$$\mathsf{R} = (A_0 \times \cdots \times A_{k-1}) \times (\mathsf{A}_0 \times \cdots \times \mathsf{A}_{l-1}).$$

For any finite sequence $\mathbf{m} = (m_0, \ldots, m_{n-1})$, the *interval* $[\mathbf{m}]$ is defined by:

$$\alpha \in [\mathbf{m}] \leftrightarrow (\alpha(0) = m_0) \wedge \cdots \wedge (\alpha(n-1) = m_{n-1}).$$

If $\mathbf{m} \subseteq \mathbf{n}$, then $[\mathbf{n}] \subseteq [\mathbf{m}]$ is a *subinterval* of $[\mathbf{m}]$. Clearly each interval is a basic open subset of ${}^{\omega}\omega$. Conversely, if A is the basic open set determined by $B_0, \ldots, B_{n-1}$, then

$$\mathsf{A} = \bigcup \{[\mathbf{m}] \colon m_0 \in B_0 \wedge \cdots \wedge m_{n-1} \in B_{n-1}\}.$$

Thus the set of intervals is also a base for the topology on ${}^{\omega}\omega$.

A partial functional is *partial continuous* iff for all n,

$$\mathsf{F}^{-1}(\{n\}) = \{(\mathbf{m}, \boldsymbol{\alpha}) \colon \mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n\}$$

is open. This is equivalent to the more usual condition that $\mathsf{F}^{-1}(B)$ be open for any open set $B \subseteq \omega$. F is *continuous* iff it is partial continuous and total.

2.1 Lemma. *For any* $\mathsf{R} \subseteq {}^{k,l}\omega$,

(i) R *is open iff* R *is the domain of some partial continuous functional*;

(ii) R *is closed-open iff* K_R *is continuous.*

Proof. If F is partial continuous, then $\mathrm{Dm}\,\mathsf{F} = \bigcup \{\mathsf{F}^{-1}(\{n\}) \colon n \in \omega\}$ is a union of open sets and hence is open. Conversely, if R is open, let

$$\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) \simeq \begin{cases} 0, & \text{if } \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}); \\ \text{undefined, otherwise.} \end{cases}$$

Clearly F is partial continuous and $\mathsf{R} = \mathrm{Dm}\,\mathsf{F}$. (ii) follows immediately from the definitions. $\square$

A set $\mathsf{A} \subseteq {}^{\omega}\omega$ is *dense in* an interval $[\mathbf{m}]$ iff for every subinterval $[\mathbf{m} * \mathbf{n}] \subseteq [\mathbf{m}]$, $\mathsf{A} \cap [\mathbf{m} * \mathbf{n}] \neq \varnothing$. A is *dense* iff it is dense in the interval $[\varnothing] = {}^{\omega}\omega$. A is *nowhere dense* iff it is dense in no interval. A is *meager* (*first category*) iff it is a countable union of nowhere dense sets. A is *non-meager* (*second category*) iff it is not meager. A is *comeager* (*residual*) iff $\sim$ A is meager. An element α of A is *isolated in* A iff there exists a neighborhood $[\alpha \restriction k]$ of α such that $[\alpha \restriction k] \cap A = \{\alpha\}$. A is called *perfect* iff it is closed, non-empty, and has no isolated elements.

We mention first some simple direct consequences of these definitions. A singleton is nowhere dense so any countable set is meager. A countable union of meager sets is meager. A subset of a meager set is meager. A set is nowhere dense iff its closure includes no interval. The complement of an open dense set is nowhere dense. A perfect set has power $2^{\aleph_0}$.

2.2 Baire Category Theorem. *No non-empty open set is meager; no comeager set is meager.*

Proof. Both statements follow from the assertion that no interval is meager. Suppose to the contrary that for some $\mathbf{p}$ and some nowhere dense sets A_n, $[\mathbf{p}] = \bigcup \{A_n : n \in \omega\}$. A_0 is not dense in $[\mathbf{p}]$, so for some sequence $\mathbf{m}^0$, $A_0 \cap [\mathbf{p} * \mathbf{m}^0] = \varnothing$. A_1 is not dense in $[\mathbf{p} * \mathbf{m}^0]$, so for some $\mathbf{m}^1$, $A_1 \cap [\mathbf{p} * \mathbf{m}^0 * \mathbf{m}^1] = \varnothing$. In this way we construct $\mathbf{m}^n$ such that for all n,

$$(A_0 \cup \cdots \cup A_n) \cap [\mathbf{p} * \mathbf{m}^0 * \cdots * \mathbf{m}^n] = \varnothing.$$

There is a function $\alpha \in \bigcap \{[\mathbf{p} * \mathbf{m}^0 * \cdots * \mathbf{m}^n] : n \in \omega\}$ and $\alpha \in [\mathbf{p}]$ but $\alpha \notin A_n$ for all n, a contradiction. $\square$

We shall have occasion to consider the subspace ${}^{\omega}2$ consisting of all α which assume only the values 0 and 1. ${}^{\omega}2$ is just the set of characteristic functions of subsets of ω and thus in a natural one-one correspondence with $\mathbf{P}(\omega)$. The interval $[\mathbf{m}]$ has a non-empty intersection with ${}^{\omega}2$ iff $\mathbf{m}$ is a *binary sequence* — all m_i are either 0 or 1. If X is a set of finite sequences we say X is *closed downward* iff whenever $\mathbf{n} \subseteq \mathbf{m} \in X$, also $\mathbf{n} \in X$.

2.3 Infinity Lemma. *For any set X of binary sequences which is closed downward, if X is infinite, then X contains an infinite branch — that is, for some $\alpha \in {}^{\omega}2$, $\alpha \restriction k \in X$ for all k.*

Proof. Let X satisfy the hypotheses and consider the set

$$Y = \{\mathbf{m} : \mathbf{m} \in X \text{ and } \{\mathbf{n} : \mathbf{m} \subseteq \mathbf{n} \wedge \mathbf{n} \in X\} \text{ is infinite}\}.$$

By hypothesis $\varnothing \in Y$. For any $\mathbf{m}$ and any $\mathbf{n} \neq \mathbf{m}$,

$$\mathbf{m} \subseteq \mathbf{n} \leftrightarrow (\mathbf{m} * (0) \subseteq \mathbf{n})) \vee (\mathbf{m} * (1) \subseteq \mathbf{n}).$$

Hence if $\mathbf{m} \in Y$, then at least one of $\mathbf{m} * (0)$ and $\mathbf{m} * (1)$ also belongs to Y. Thus there exists a unique function α such that for all k,

$$\alpha(k) = \text{least } p.(\alpha \restriction k) * (p) \in Y.$$

Since $Y \subseteq X$, this α satisfies the conclusion of the lemma. $\square$

2.4 Theorem. (i) ${}^{\omega}2$ *is a compact subspace of* ${}^{\omega}\omega$;
(ii) *for any k and l,* $({}^{k}2) \times {}^{l}({}^{\omega}2)$ *is a compact subspace of* ${}^{k,l}\omega$.

Proof. We prove (i) by showing that any open cover $\mathcal{F}$ of ${}^{\omega}2$ has a finite subcover. Let X be the set of all finite binary sequences $\mathbf{m}$ such that $[\mathbf{m}]$ is included in no member of $\mathcal{F}$. Clearly X is closed downward; suppose X is infinite. Then by the Infinity Lemma, X contains an infinite branch α. Since $\mathcal{F}$ is a cover, $\alpha \in \mathsf{A}$ for some $\mathsf{A} \in \mathcal{F}$. As A is open, for some k, $[\alpha \restriction k] \subseteq \mathsf{A}$, a contradiction. Hence X is finite, so for some k all members of X have length less than k. Let $\mathbf{m}^0, \ldots, \mathbf{m}^{2^k-1}$ be a list of all binary sequences of length k. For each $i < 2^k$ we may choose an $\mathsf{A}_i \in \mathcal{F}$ such that $[\mathbf{m}^i] \subseteq \mathsf{A}_i$. Then $\mathcal{F}_0 = \{\mathsf{A}_i : i < 2^k\}$ is the required finite subcover. The proof of (ii) is similar. $\square$

Note that the proof of Theorem 2.4 depends only on the fact that for any $\mathbf{m} \in X$, $\{p : \mathbf{m} * (p) \in X\}$ is finite. Hence, for example, ${}^{\omega}q$ is a compact subspace of ${}^{\omega}\omega$ for any $q \in \omega$.

The original aim of Descriptive Set Theory was the study and classification of sets of real numbers and their properties which are of interest for mathematical analysis. It was early discovered that little is lost and much is gained in simplicity and elegance if one studies sets of irrational numbers. Indeed, for most properties of interest to analysis — measurability, having the power of the continuum, being meager, etc. — the exclusion of a countable set of points (the rationals) has no effect. On the other hand, there are important topological differences between the reals and the irrationals which simplify the theory of sets of irrationals: the irrationals are of topological dimension 0, there is a base for the topology on the irrationals which consists of closed-open sets, and the irrationals are homeomorphic to their own Cartesian powers. Further simplification was obtained by the discovery that the space of irrationals is homeomorphic to ${}^{\omega}\omega$ with the topology described above. Thus many results concerning ${}^{\omega}\omega$ and the product spaces ${}^{k,l}\omega$ have immediate consequences for the spaces of irrational and real numbers (cf. end of § IV.3).

Temporarily, let ${}^{\omega}\omega$ denote (ambiguously) the topological space described above (as well as its underlying set). Let Ir denote similarly the set of irrational numbers x such that $0 < x < 1$ together with the topology induced by the standard topology on the set of real numbers: $Y \subseteq \mathrm{Ir}$ is *open* iff $Y = \mathrm{Ir} \cap Z$ for some open subset Z of the real interval $(0, 1)$. Then the fact we mentioned is: ${}^{\omega}\omega$ and Ir are homeomorphic. We leave the proof of this to Exercise 2.8 and construct here instead a homeomorphism of ${}^{\omega}\omega$ with another subspace of $(0, 1)$, the space BIr of binary irrationals. This correspondence will serve just as well in transferring results from ${}^{\omega}\omega$ to $(0, 1)$ and is somewhat more natural.

A *finite binary decimal* is a representation of a real number in the form:

$$.r_1r_2\ldots r_l = r_1(2^{-1}) + r_2(2^{-2}) + \cdots + r_l(2^{-l})$$

where each $r_i = 0$ or 1. The real numbers which have finite binary representations are exactly those which can be written as a quotient p/q of natural numbers such that q is a power of 2. Clearly such numbers are dense in (0, 1). An *infinite binary decimal* is a representation

$$.r_1r_2\ldots r_l\ldots = \sum_{i=1}^{\infty} r_i(2^{-i})$$

where each $r_i = 0$ or 1. Any such series converges to a real number between 0 and 1 and every such real number has an infinite binary representation. Two infinite binary decimals represent the same real number iff they are of the forms

$$.r_1r_2\ldots r_l\,100\ldots0\ldots,$$

and

$$.r_1r_2\ldots r_l\,011\ldots1\ldots.$$

A *binary irrational* is a real number between 0 and 1 that does not have a finite binary representation. BIr is the topological space consisting of the binary irrationals with the topology induced from (0, 1).

2.5 Theorem. *$^{\omega}\omega$ and* BIr *are homeomorphic.*

Proof. For any $\alpha \in {}^{\omega}\omega$, let $\theta(\alpha)$ be the infinite binary decimal:

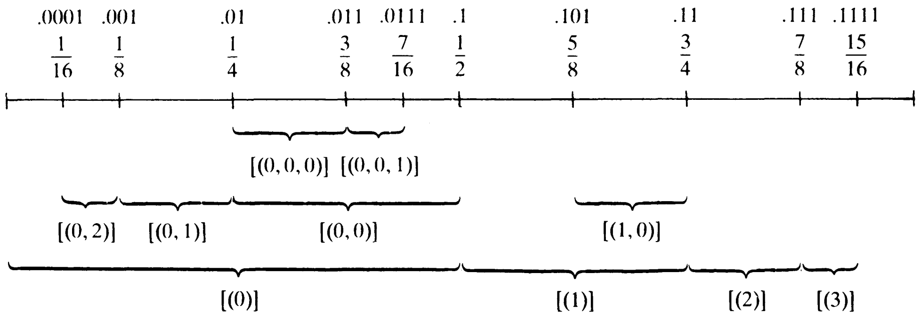
$$\theta(\alpha) = .\underbrace{11\ldots10}_{\alpha(0)+1}\underbrace{00\ldots01}_{\alpha(1)+1}\underbrace{11\ldots10}_{\alpha(2)+1}\underbrace{00\ldots01}_{\alpha(3)+1}\ldots.$$

From the preceding remarks it is obvious that θ maps $^{\omega}\omega$ one-one onto BIr. θ is continuous because if $\alpha\restriction k = \beta\restriction k$, then $|\theta(\alpha) - \theta(\beta)| < 2^{-k}$. For the continuity of θ^{-1}, suppose that $x \in$ BIr and k are given. To insure that $\theta^{-1}(x)\restriction k = \theta^{-1}(y)\restriction k$ it suffices to take $|x - y| < 2^n$, where $n = \theta^{-1}(x)(0) + \cdots + \theta^{-1}(x)(k-1) + k$. Hence θ is a homeomorphism. $\square$

To compute the image of a given interval [**m**] in $^{\omega}\omega$, note that $\theta(\alpha)$ has the following representation:

$$\theta(\alpha) = .\underbrace{1\,1\ldots1\,1}_{\alpha(0)+1}$$
$$-(.\underbrace{0\,0\ldots0\,0}_{\alpha(0)+1}\;\underbrace{1\,1\ldots1\,1}_{\alpha(1)+1})$$
$$+(.\underbrace{0\,0\ldots0\,0}_{\alpha(0)+1}\;\underbrace{0\,0\ldots0\,0}_{\alpha(1)+1}\;\underbrace{1\,1\ldots1\,1}_{\alpha(2)+1})$$
$$-\cdots.$$

Thus θ induces the following correspondence between intervals of ${}^{\omega}\omega$ and of BIr:



To extend these results to the spaces ${}^{k,l}\omega$, it suffices to show that for each k and l, ${}^{k,l}\omega$ is homeomorphic to ${}^{\omega}\omega$. For this we need sequence coding functions which are *onto* ω. Set

$$\langle\langle m, n\rangle\rangle^2 = \tfrac{1}{2}(m^2 + 2mn + n^2 + 3m + n)$$

and, recursively, for $l > 1$,

$$\langle\langle m_0, \ldots, m_l\rangle\rangle^{l+1} = \langle\langle\, \langle\langle m_0, \ldots, m_{l-1}\rangle\rangle^l, m_l\rangle\rangle.$$

We leave it as an exercise (2.9) to check that the map $\theta^{k,l}$ defined by:

$$\theta^{k,l}(\mathbf{m}, \boldsymbol{\alpha}) = (\mathbf{m}) * \lambda p.\langle\langle \alpha_0(p), \ldots, \alpha_{l-1}(p)\rangle\rangle^l$$

is the desired homeomorphism.

The homeomorphism θ induces a natural measure on ${}^{\omega}\omega$. Let $\mathrm{mes}_{\mathrm{Lb}}$ denote Lebesgue measure restricted to BIr. For $\mathsf{A} \subseteq {}^{\omega}\omega$, we set

$$\mathrm{mes}(\mathsf{A}) = \mathrm{mes}_{\mathrm{Lb}}\{\theta(\alpha)\colon \alpha \in \mathsf{A}\}$$

and say A is *measurable* just in case its θ-image is Lebesgue measurable. Because θ is a homeomorphism, all open and closed sets are measurable. The measure is clearly countably additive and has the property that all subsets of a set of measure 0 are measurable (completeness). This measure may also be described as the *product measure* on ${}^{\omega}\omega$ generated by the measure on ω which assigns $\{n\}$ the measure $2^{-(n+1)}$. Thus for any sequence $\mathbf{m} = (m_0, \ldots, m_{k-1})$,

$$\text{mes}([\mathbf{m}]) = 2^{-(m_0+1)} \cdot \cdots \cdot 2^{-(m_{k-1}+1)} = 2^{-(m_0+\cdots+m_{k-1}+k)}.$$

Similarly, we may define a measure on ${}^{k,l}\omega$ either via the homeomorphism $\theta^{k,l}$ or directly by setting

$$\begin{aligned}&\text{mes}(\{m_0\} \times \cdots \times \{m_{k-1}\} \times [p^0] \times \cdots \times [p^{l-1}])\\ &\quad = 2^{-(m_0+1)} \cdot \cdots \cdot 2^{-(m_{k-1}+1)} \cdot \text{mes}([p^0]) \cdot \cdots \cdot \text{mes}([p^{l-1}]).\end{aligned}$$

2.6–2.13 Exercises

2.6. Let α^* range over ${}^{\omega}2$. Show that for any set X of finite sequences of 0's and 1's,

$$\forall \alpha^* \exists p \,.\, \alpha^* \restriction p \in X \leftrightarrow \exists n \forall \alpha^* (\exists p < n) \,.\, \alpha^* \restriction p \in X.$$

2.7. For any $\mathsf{A} \subseteq {}^{\omega}\omega$, let $s * \mathsf{A} = \{s * \alpha : \alpha \in \mathsf{A}\}$ and $\mathsf{A}^{(s)} = \{\alpha : s * \alpha \in \mathsf{A}\}$. The class Ka of *Kalmar sets* is the smallest class X of subsets of ${}^{\omega}\omega$ such that $\varnothing, {}^{\omega}\omega \in X$ and if for all $n, \mathsf{A}_n \in X$, then $\bigcup \{\langle n \rangle * \mathsf{A}_n : n \in \omega\} \in X$. Show for all A and s,

(i) $\mathsf{A} \in \text{Ka} \to \mathsf{A}^{(s)} \in \text{Ka}$;
(ii) $\mathsf{A} \in \text{Ka} \leftrightarrow s * \mathsf{A} \in \text{Ka}$;
(iii) $\forall n \,.\, \mathsf{A}^{(\langle n \rangle)} \in \text{Ka} \to \mathsf{A} \in \text{Ka}$;
(iv) $\forall \beta \exists n \,.\, \mathsf{A}^{(\bar{\beta}(n))} \in \text{Ka} \to \mathsf{A} \in \text{Ka}$;
(v) $\mathsf{A} \in \text{Ka} \leftrightarrow \mathsf{A}$ is closed-open.

2.8. Prove that the topological spaces ${}^{\omega}\omega$ and Ir are homeomorphic. (Since the sets of rationals and binary rationals are each countable and dense in (0, 1), there is a one-one order-preserving correspondence between them. This may be extended in a unique way to a homeomorphism of (0, 1) with itself. The restriction of this homeomorphism to Ir is a homeomorphism of Ir with BIr.)

2.9. Show that $\theta^{k,l}$ is a homeomorphism of ${}^{k,l}\omega$ onto ${}^{\omega}\omega$.

2.10. (The Zero-One Law). Show that for any measurable set $\mathsf{A} \subseteq {}^{\omega}\omega$, if for all s,

$$\text{mes}(\mathsf{A} \cap [s]) = \text{mes}(\mathsf{A}) \cdot \text{mes}([s]),$$

then mes(A) is either 0 or 1. (Show that this equation holds with $[s]$ replaced by any measurable set.)

2.11. Show that in the usual topology on the real interval (0, 1), Ir and BIr are G_δ sets (countable intersection of open sets) but not F_σ sets (countable union of closed sets).

2.12. Show that a relation $\mathrm{R} \subseteq {}^{k,l}\omega$ is open iff for some $S \subseteq {}^{k+1}\omega$,

$$\mathrm{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists p\, S(\mathbf{m}, \bar{\alpha}_0(p), \ldots, \bar{\alpha}_{l-1}(p)).$$

2.13. Show that a homeomorphism between ${}^\omega\omega$ and Ir may be constructed directly. (To each finite sequence $\mathbf{m}$ assign a rational number $\theta(\mathbf{m})$ recursively by the rules:

$$\theta(\varnothing) = 0 \quad \text{and} \quad \theta((p, \mathbf{m})) = \frac{1}{p + 1 + \theta(\mathbf{m})}\,.$$

For all α, $\theta(\alpha \restriction k)$ converges to a limit $\theta^*(\alpha)$.)

3. *Inductive Definitions*

Let X be any fixed set. A function Γ from the power set of X into itself is called an *operator over* X. Γ is said to be *inclusive* iff for all $Y \subseteq X$, $Y \subseteq \Gamma(Y)$, *monotone* iff for all $Y \subseteq Z \subseteq X$, $\Gamma(Y) \subseteq \Gamma(Z)$, and *inductive* iff Γ is either inclusive or monotone. An operator Γ *defines inductively* a subset $\bar{\Gamma}$ of X as follows. We define by transfinite recursion the sequence Γ^σ by $\Gamma^\sigma = \Gamma(\bigcup\{\Gamma^\tau : \tau < \sigma\})$ and set $\bar{\Gamma} = \bigcup\{\Gamma^\sigma : \sigma \in \mathrm{Or}\}$. We write $\Gamma^{(\sigma)}$ for $\bigcup\{\Gamma^\tau : \tau < \sigma\}$ so that $\Gamma^\sigma = \Gamma(\Gamma^{(\sigma)})$.

We think of the set $\bar{\Gamma}$ as being "built up" in stages. Starting from the empty set we get successively $\Gamma(\varnothing), \Gamma(\Gamma(\varnothing)), \ldots$. Γ^σ is called the σ-th *stage* or *level.*

3.1. Lemma. *For any inductive operator Γ and any ordinal σ,*

(i) $\Gamma^{(\sigma)} \subseteq \Gamma^\sigma$;

(ii) $\Gamma^{\sigma+1} = \Gamma(\Gamma^\sigma)$;

(iii) $\Gamma^{(\sigma)} = \Gamma^\sigma \rightarrow \Gamma^\tau = \Gamma^\sigma = \bar{\Gamma}$ *for all* $\tau \geqslant \sigma$;

(iv) $\Gamma^{(\sigma)} = \Gamma^\sigma$ *for some σ such that* $\mathrm{Card}(\sigma) \leqslant \mathrm{Card}(X)$.

Proof. For inclusive Γ, (i) is immediate from the definitions; for monotone Γ it follows from the obvious fact that for $\tau \leqslant \sigma$, $\Gamma^{(\tau)} \subseteq \Gamma^{(\sigma)}$. (ii) is immediate from the observation that by (i), $\Gamma^{(\sigma+1)} = \Gamma^\sigma$. (iii) is proved by induction on τ: for $\tau = \sigma$, clearly $\Gamma^\tau = \Gamma^\sigma$; for $\tau > \sigma$, the induction hypothesis yields $\Gamma^{(\tau)} = \Gamma^\sigma$ and we have $\Gamma^\tau = \Gamma(\Gamma^{(\tau)}) = \Gamma(\Gamma^\sigma) = \Gamma(\Gamma^{(\sigma)}) = \Gamma^\sigma$. For (iv), suppose that for each σ with $\mathrm{Card}(\sigma) \leqslant \mathrm{Card}\, X$, $\Gamma^{(\sigma)} \subsetneqq \Gamma^\sigma$, and let x_σ be an element of $\Gamma^\sigma \sim \Gamma^{(\sigma)}$. If $\tau \neq \sigma$, also

$x_\tau \neq x_\sigma$, and this defines an injection of $\{\sigma\colon \mathrm{Card}(\sigma) \leq \mathrm{Card}(X)\}$ into X. But this set is exactly the least cardinal *larger* than $\mathrm{Card}\, X$, so this is impossible. □

We denote by $|\Gamma|$ the least σ such that $\Gamma^{(\sigma)} = \Gamma^\sigma$, the *closure ordinal* of Γ. Then

3.2 Corollary. *For any inductive operator Γ over X,* $\mathrm{Card}(|\Gamma|) \leq \mathrm{Card}(X)$ *and* $\bar{\Gamma} = \Gamma^{(|\Gamma|)}$. □

Thus we need not think of the sequence Γ^σ as extended over all ordinals but only over those less than $|\Gamma|$. In particular, if $X = \omega$ we need only consider countable ordinals.

Note that for any inductive Γ, $\Gamma(\bar{\Gamma}) = \bar{\Gamma}$; but for $\sigma < |\Gamma|$, $\Gamma(\Gamma^{(\sigma)}) \neq \Gamma^{(\sigma)}$. In other words, $\bar{\Gamma}$ is the first *fixed point* of Γ in the sequence Γ^σ.

3.3 Theorem. *For any monotone operator Γ over a set X, $\bar{\Gamma}$ is the smallest fixed point of Γ — that is,*

$$\bar{\Gamma} = \bigcap \{Z\colon Z \subseteq X \wedge \Gamma(Z) = Z\}.$$

Proof. By the preceding remark, $\bar{\Gamma} \in \{Z\colon Z \subseteq X \wedge \Gamma(Z) = Z\}$ so that the intersection of this set is included in $\bar{\Gamma}$. Conversely, let Z be any subset of X such that $\Gamma(Z) = Z$; we prove by induction on σ that for all σ, $\Gamma^\sigma \subseteq Z$. Assume as induction hypothesis that this holds for all $\tau < \sigma$ so $\Gamma^{(\sigma)} \subseteq Z$. Then by monotonicity, $\Gamma^\sigma = \Gamma(\Gamma^{(\sigma)}) \subseteq \Gamma(Z) = Z$. □

Note that the proof yields also that for monotone Γ,

$$\bar{\Gamma} = \bigcap \{Z\colon Z \subseteq X \wedge \Gamma(Z) \subseteq Z\}.$$

These results have two distinct aspects. First, they give a characterization of $\bar{\Gamma}$ which does not involve ordinals. Second, they provide a very convenient way of proving that all $x \in \bar{\Gamma}$ have some property: one shows that the set Z of all $x \in X$ which have the property satisfies $\Gamma(Z) \subseteq Z$. In applying this method we say that the proof is by *Γ-induction* or by *induction over $\bar{\Gamma}$*.

In many contexts where we are defining inductively a particular set Y it will be convenient to avoid direct reference to the inductive operator involved. Thus if Y is defined as $\bar{\Gamma}$, we may write Y^σ and $Y^{(\sigma)}$ instead of Γ^σ and $\Gamma^{(\sigma)}$ and describe proofs by Γ-induction as proofs by *induction over Y*.

In the remainder of this section we consider the properties of two special classes of inductive definitions. Let Y be a subset of X and $\mathscr{F}$ a family of finitary functions on X — that is, for each $\varphi \in \mathscr{F}$, there is a natural number $k(\varphi)$ such that $\mathrm{Dm}\,\varphi = {}^{k(\varphi)}X$ and $\mathrm{Im}\,\varphi \subseteq X$. For each such pair $(Y, \mathscr{F})$, we define an inductive operator $\Gamma_{Y,\mathscr{F}}$ by:

$$\Gamma_{Y,\mathscr{F}}(Z) = Y \cup \{\varphi(\mathbf{z}) \colon \varphi \in \mathscr{F} \wedge \mathbf{z} \in {}^{k(\varphi)}Z\}.$$

The resulting set $\bar{\Gamma}_{Y,\mathscr{F}}$ is called the *closure of* Y *under* $\mathscr{F}$. Since $\Gamma_{Y,\mathscr{F}}$ is clearly monotone, $\bar{\Gamma}_{Y,\mathscr{F}}$ is also the *smallest set including* Y *and closed under* $\mathscr{F}$.

3.4 Lemma. *For any* Y *and* $\mathscr{F}$ *as above,* $|\bar{\Gamma}_{Y,\mathscr{F}}| \leq \omega$.

Proof. Let Y and $\mathscr{F}$ be fixed and write Γ for $\Gamma_{Y,\mathscr{F}}$. By 3.1 (i) it suffices to show $\Gamma^{\omega} \subseteq \Gamma^{(\omega)}$. Let x be any element of $\Gamma^{\omega} = \Gamma(\Gamma^{(\omega)})$. If $x \in Y$, then $x \in \Gamma^0 \subseteq \Gamma^{(\omega)}$. Otherwise, for some $\varphi \in \mathscr{F}$ and some $\mathbf{z} \in {}^{k(\varphi)}(\Gamma^{(\omega)})$, $x = \varphi(\mathbf{z})$. For each $i < k(\varphi)$, let r_i be the least natural number such that $z_i \in \Gamma^{r_i}$, and set $r = \max\{r_i \colon i < k(\varphi)\}$. Then $\mathbf{z} \in {}^{k(\varphi)}\Gamma^r$ so $x \in \Gamma(\Gamma^r) = \Gamma^{r+1} \subseteq \Gamma^{(\omega)}$. □

The method of inductive definition is a generalization of the definition of the set ω of natural numbers in set theory: ω is the smallest set including $\{0\}$ and closed under the successor function, $\mathrm{Sc}(x) = x \cup \{x\}$. Many of the fundamental notions of elementary logic are most naturally defined inductively, often by operators of the form $\Gamma_{Y,\mathscr{F}}$. For example, the set of formulas of a (finitary) first-order formal language is the closure of the set of atomic formulas under functions corresponding to the propositional connectives and quantifiers (cf. § III.5). The set of formal theorems of an axiomatic theory is the closure of the set of axioms under functions corresponding to the rules of inference. An example which is not a closure under finitary functions is the class of formulas of the infinitary language $L_{\omega_1\omega}$ (cf. Keisler [1971]).

We shall also need a generalization of the method of definition by recursion. Roughly speaking, for any set X^* we may define a function $\theta \colon \omega \to X^*$ by specifying a value $\theta(0)$ and a method for calculating $\theta(m+1)$ from $\theta(m)$. The corresponding generalization will allow us to define a function $\theta \colon \bar{\Gamma}_{Y,\mathscr{F}} \to X^*$ by specifying the values $\theta(y)$ for $y \in Y$ and methods for calculating $\theta(\varphi(\mathbf{x}))$ from $\theta(x_0), \dots, \theta(x_{k(\varphi)-1})$ for all $\varphi \in \mathscr{F}$ (the x_i should be thought of as the immediate predecessors of $\varphi(\mathbf{x})$). In the case of ω, m is uniquely determined by $m+1$, but for arbitrary Y and $\mathscr{F}$ it may happen that $\varphi(\mathbf{x}) = \varphi'(\mathbf{x}')$ or $\varphi(\mathbf{x}) \in Y$ so that the rules would not determine a unique value for $\theta(\varphi(\mathbf{x}))$.

We call the pair $(Y, \mathscr{F})$ *monomorphic* iff all $\varphi \in \mathscr{F}$ are one-one and the sets Y and $\{\mathrm{Im}\, \varphi \colon \varphi \in \mathscr{F}\}$ are pairwise disjoint. The inductive definitions of ω and the class of formulas of a first-order language are monomorphic whereas that of the class of formal theorems is not.

3.5 Theorem (Definition by Recursion). *For any monomorphic pair* $(Y, \mathscr{F})$ *and any set* X^*, *suppose that* $\psi \colon X^* \to X^*$ *and for each* $\varphi \in \mathscr{F}$, $\varphi^* \colon {}^{k(\varphi)}X^* \to X^*$. *Then there exists a unique function* $\theta \colon \bar{\Gamma}_{Y,\mathscr{F}} \to X^*$ *such that*

(i) *for all* $y \in Y$, $\theta(y) = \psi(y)$;

(ii) *for all* $\varphi \in \mathscr{F}$ *and all* $\mathbf{x} \in {}^{k(\varphi)}\bar{\Gamma}_{Y,\mathscr{F}}$,

$$\theta(\varphi(\mathbf{x})) = \varphi^*(\theta(x_0), \ldots, \theta(x_{k(\varphi)-1})).$$

Proof. Let $\Gamma = \Gamma_{Y,\mathcal{F}}$, X^*, ψ, and φ^* be given as in the hypothesis. We define functions $\theta_r: \Gamma^r \to X^*$ by ordinary recursion as follows. $\theta_0 = \psi$. Suppose θ_r is defined and $x \in \Gamma^{r+1}$. If $x \in \Gamma^r$ we set $\theta_{r+1}(x) = \theta_r(x)$. If $x \in \Gamma^{r+1} \sim \Gamma^r$, then by the assumption that $(Y, \mathcal{F})$ is monomorphic there exist unique $\varphi \in \mathcal{F}$ and $\mathbf{z} \in {}^{k(\varphi)}\Gamma^r$ such that $x = \varphi(\mathbf{z})$. We then set $\theta_{r+1}(x) = \varphi^*(\theta_r(z_0), \ldots, \theta_r(z_{k(\varphi)-1}))$. Finally, $\theta = \bigcup\{\theta_r : r \in \omega\}$. We leave to the reader the easy verification that θ satisfies conditions (i) and (ii) (Exercise 3.11). □

Our second special class consists of operators over a product space $X \times Y$. Operators of this type will be used in defining subsets of ${}^{k,l}\omega$.

3.6 Definition. An operator Γ over $X \times Y$ is *decomposable* iff there exists a family of operators Γ_y over X, indexed by $y \in Y$, such that for any $Z \subseteq X \times Y$,

$$\Gamma(Z) = \{(x, y) : x \in \Gamma_y(\{z : (z, y) \in Z\})\}.$$

3.7 Lemma. *For any decomposable operator Γ over $X \times Y$,*
(i) $\bar{\Gamma} = \{(x, y) : y \in Y \wedge x \in \bar{\Gamma}_y\}$;
(ii) $|\Gamma| = \sup\{|\Gamma_y| : y \in Y\}$.

Proof. Both parts follows easily from the assertion that for all σ

$$\Gamma^\sigma = \{(x, y) : y \in Y \wedge x \in \Gamma_y^\sigma\}.$$

To establish this by induction, suppose that it holds for all $\tau < \sigma$. Then

$$\Gamma^{(\sigma)} = \bigcup_{\tau<\sigma}\{(x, y) : y \in Y \wedge x \in \Gamma_y^\tau\} = \{(x, y) : y \in Y \wedge x \in \Gamma_y^{(\sigma)}\},$$

and

$$\Gamma^\sigma = \Gamma(\Gamma^{(\sigma)}) = \{(x, y) : y \in Y \wedge x \in \Gamma_y(\Gamma_y^{(\sigma)})\} = \{(x, y) : y \in Y \wedge x \in \Gamma_y^\sigma\}. \quad \square$$

Decomposable inductive operators over ${}^{k,l}\omega$ are given by families of operators Γ_α over ${}^k\omega$. By Corollary 3.2 each $|\Gamma_\alpha|$ is countable and thus $|\Gamma| \leq \aleph_1$, whereas the closure ordinal of an arbitrary operator over ${}^{k,l}\omega$ is bounded only by the least cardinal greater than $2^{\aleph_0}$. This fact will play an important role in § III.3.

3.8–3.13 Exercises

3.8. Show that any monotone operator over a set X has a largest fixed point $\bar{\Gamma}$. In fact, $\bar{\Gamma} = \sim\Gamma^\circ$, where Γ° is a monotone operator defined by $\Gamma^\circ(Y) = \sim\Gamma(\sim Y)$.

3.9. Construct an example of a non-monotone inductive operator which has no smallest fixed point.

3.10. An operator Γ is called κ-*compact* (for any cardinal κ) iff whenever $x \in \Gamma(Y)$, also $x \in \Gamma(Z)$ for some $Z \subseteq Y$ with $\mathrm{Card}(Z) < \kappa$. Note that any $\Gamma_{Y,\mathscr{F}}$ is ω-compact. What can be said in general about the closure ordinal of a κ-compact inductive operator?

3.11. Complete the proof of Theorem 3.5. Sketch an alternative proof in which θ is defined inductively as the smallest set of pairs (x, x^*) such that ... .

3.12. Suppose $Y \subseteq X$ and $\mathscr{F}$ is a family of finitary functions on X such that $(Y, \mathscr{F})$ is monomorphic. For each $x \in \bar{\Gamma}_{Y,\mathscr{F}}$, define $\mathrm{Sp}(x)$, the *support* of x, recursively by:

$$\mathrm{Sp}(y) = \varnothing, \quad \text{for} \quad y \in Y;$$

$$\mathrm{Sp}(\varphi(\mathbf{x})) = \bigcup \{\mathrm{Sp}(x_i): i < k(\varphi)\} \cup \{x_i: i < k(\varphi)\}.$$

Establish the following *principle of proof by course-of-values induction* over $\bar{\Gamma}_{Y,\mathscr{F}}$: for any $Z \subseteq \bar{\Gamma}_{Y,\mathscr{F}}$, if $(\forall x \in \bar{\Gamma}_{Y,\mathscr{F}})[\mathrm{Sp}(x) \subseteq Z \rightarrow x \in Z]$, then $Z = \bar{\Gamma}_{Y,\mathscr{F}}$.

3.13. There is also a natural notion of definition by course-of-values recursion that says roughly that we may define a function θ: $\bar{\Gamma}_{Y,\mathscr{F}} \rightarrow X^*$ by specifying the values $\theta(y)$ for $y \in Y$ and methods for calculating $\theta(x)$ from values $\theta(z)$ for $z \in \mathrm{Sp}(x)$. Formulate precisely a principle of this kind as general as possible and prove that it is valid.

3.14 Notes. Inductive definitions have long played a fundamental role in many areas of mathematics but have been studied as objects only much more recently. Definitions in Algebra of the subgroup, subring, etc. generated by certain elements are all inductive. The class of Borel sets of a topological space is inductively defined as is (the complement of) the perfect kernel of a set of reals (cf. Exercise 3.8). The principal objects of study of Logic and Recursion Theory are all inductively defined. The general study of inductive definability begins explicitly with Spector [1961] but it is close to the surface in many earlier papers of Kleene, especially [1955] and [1955a]. Moschovakis [1974, pp. 3–4] gives a more extended history of the subject.