Chapter 8

Set Recursion and Higher Types

Primitive recursive set functions and rudimentary set functions have been around for some time, see Jensen-Karp [72], Gandy [40], and also Devlin [19] for their basic properties and further references. The step to a general recursion theory on sets came rather late. There is, perhaps, a reason why: Primitive recursive and rudimentary set functions were introduced to elucidate the rather restricted recursion-theoretic nature of the constructible hierarchy, and in the hands of Jensen [71] have become an important tool in the fine structure theory of L.

Full set recursion was introduced by D. Normann [124], and later rediscovered by Y. Moschovakis, as a tool for developing a companion theory for Kleene-recursion in higher types. The theory has, however, a wider scope and we shall present a general version in the first part of this chapter. In this we follow the exposition in Normann [124]. The approach of Moschovakis uses inductive definability, but the end result is substantially the same.

In Section 3 we work out the detailed connection with Kleene-recursion in higher types. Some of this work has its origin in the theses of Harrington [53], MacQueen [98], and Normann [122]. We believe that the general set-theoretic approach adds both simplicity and insight.

As a testing ground for this belief we turn in Section 4 to the degree theory in higher types. We present a fairly simple priority argument involving ${}^3E$, allowing the reader to explore the full intricacies of the general theory for him- or herself. We just want to make the point that set recursion is a very natural computation theory to use in the study of degrees of functionals.

8.1 Basic Definitions

Set recursion in a relation R on the universe of sets V is generated by the schemes for the functions rudimentary in R augmented with the diagonalization scheme.

8.1.1 Definition. Let $R \subseteq V$ be a relation. The class of partial functions *set-recursive relative to* R is inductively defined by the following clauses

(i) $f(x_1, \ldots, x_n) = x_i$ $\qquad e = \langle 1, n, i\rangle$

(ii) $f(x_1, \ldots, x_n) = x_i - x_j$ $\qquad e = \langle 2, n, i, j\rangle$

(iii) $f(x_1, \ldots, x_n) = \{x_i, x_j\}$ $\qquad e = \langle 3, n, i, j \rangle$

(iv) $f(x_1, \ldots, x_n) \simeq \bigcup_{y \in x_1} h(y, x, \ldots, x_n)$

$e = \langle 4, n, e' \rangle$ where e' is an index for h.

(v) $f(x_1, \ldots, x_n) \simeq h(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))$

$e = \langle 5, n, m, e', e_1, \ldots, e_m \rangle$ where e' is an index for h and $e_1, \ldots, e_m$ are indices for $g_1, \ldots, g_m$, respectively.

(vi) $f(x_1, \ldots, x_n) \simeq x_i \cap R$ $\qquad e = \langle 6, n, j \rangle$

(vii) $f(e_1, x_1, \ldots, x_n, y_1, \ldots, y_m) \simeq \{e_1\}_R(x_1, \ldots, x_n)$

$e = \langle 7, n, m \rangle$.

There are a number of comments to make. The functions generated by clauses (i) to (vi) are the functions rudimentary in R. These functions are all total. In the general case we get partial functions and we then assume in scheme (iv) that the computation is defined only if $h(y, x_2, \ldots, x_n)$ is defined for all $y \in x_1$.

The index e is always a natural number. This is a limiting feature of this version of set-recursion which squares with a characteristic feature of Kleene-recursion in higher types, but makes the theory differ from general admissibility theory. Note that since for each $n \in \omega$ the constant function with value n is rudimentary, these functions will be set-recursive.

The basic theory of the rudimentary set functions is developed in Devlin [19]. We shall use only their most elementary properties: For the more delicate parts of the theory see Jensen [71] (or the exposition in Devlin [19]); the reader will also profit from a study of Simpson's lectures [156].

For emphasis we repeat: Let $R \subseteq V$. A function f on V is called set-recursive in R if there is an index e such that for all $x_1, \ldots, x_n$

$$f(x_1, \ldots, x_n) \simeq \{e\}_R(x_1, \ldots, x_n),$$

where the notation $\{e\}_R$ obtains its meaning through Definition 8.1.1.

The set $\Theta_R = \{(e, \sigma, z); \{e\}_R(\sigma) \simeq z\}$ is a computation theory in the sense of part A of this book. From the schemes in Definition 8.1.1 we derive in a canonical way the notions of

a. *length of a computation*,
b. *subcomputation*,
c. *computation tree*.

These concepts should by now be thoroughly familiar to the reader and we need not repeat the detailed constructions. (A complete exposition exists in Gurrik [52], to which any reader who wishes to learn Norwegian can turn.)

We let $\|a, \sigma, z\|$, for $(a, \sigma, z) \in \Theta_R$, denote the length of the computation $\{e\}_R(\sigma) \simeq z$.

Set-recursion in R is p-normal. This is a consequence of the following simple lemma which combines an application of the schemes (iv) and (vii).

8.1.2 Lemma. *There is an index e such that for arbitrary R, x, e_1, σ*

$$\{e\}_R(x, e_1, \sigma) \simeq \begin{cases} 0 & \textit{if} \quad \forall y \in x \cdot \{e_1\}_R(y, \sigma) \simeq 0 \\ 1 & \textit{if} \quad \forall y \in x \cdot \{e_1\}_R(y, \sigma)\downarrow \quad \textit{and} \quad \exists y \in x \cdot \{e_1\}_R(y, \sigma) \neq 0. \end{cases}$$

The proof is simple. By elementary properties of rudimentary functions we may assume that $\{e_1\}_R$ takes values 0 ($= \varnothing$) and 1 ($= \{\varnothing\}$) only. Let

$$\{e\}_R(x, e_1, \sigma) \simeq \bigcup_{y \in x} \{e_1\}_R(y, \sigma).$$

This is the proof. The lemma shows that we have the crucial property of normality from recursion in higher types built into set-recursion through the scheme (iv) of "bounded" union. If $x = \omega$, bounded union means quantification over ω, i.e. the computability of ${}^2E$. This explains the name "E-recursion" rather than "set-recursion" in Normann [124].

With the usual proof, Lemma 8.1.2 implies p-normality, i.e. stage comparison, and the existence of a selection operator over ω. (See Chapter 3 for details.)

8.1.3 Proposition. *There is a set-recursive function p such that if $\sigma = (e, \sigma_1, z) \in \Theta_R$ or $\sigma' = (e', \sigma'_1, z) \in \Theta_R$ then $p(\sigma, \sigma')\downarrow$, and*

$$\sigma \in \Theta_R \wedge \|\sigma\| \leqslant \|\sigma'\| \Rightarrow p(\sigma, \sigma') \simeq 0$$
$$\|\sigma\| > \|\sigma'\| \Rightarrow p(\sigma, \sigma') \simeq 1.$$

8.1.4 Proposition. *There is an index e such that for arbitrary R, e_1, σ*

$$\{e\}_R(e_1, \sigma)\downarrow \quad \textit{iff} \quad \exists n \in \omega \{e_1\}_R(n, \sigma)\downarrow.$$

And if $\exists n \in \omega \{e_1\}_R(n, \sigma)\downarrow$ then $\{e_1\}_R(\{e\}_R(e_1, \sigma), \sigma)\downarrow$, and

$$\|e, e_1, \sigma\| > \inf\{\|e, n, \sigma\| : \{e\}_R(n, \sigma)\downarrow\}.$$

As our computations are single-valued we abbreviate in the usual fashion, i.e. if $\{e\}_R(\sigma) \simeq z$, we let $\|e, \sigma\|$ denote the length of the computation.

8.1.5 Remark. It is also possible to develop a version of the selection principle of Theorem 4.3.1 in the context of set-recursion. But proving "Grilliot-selection" once is sufficient for us! The reader should, however, consult a forthcoming paper, *A note on reflection*, in Math. Scandinavica by Dag Normann.

8.2 Companion Theory

We aim at a construction generalizing the "next admissible", see the introductory discussion in Section 5.3 as well as the construction of the abstract 1-section corresponding to a Spector theory Θ on ω in Proposition 5.4.20 of the same chapter.

As always there are some preliminary definitions. For completeness we put down the somewhat unexciting

8.2.1 Definition. Let $R \subseteq V$, $\tau \in V^m$. Let φ be a partial function from V^n to V. We say that φ is *set-recursive in τ relative to R* if there is an index e such that for all $\sigma \in V^n$

$$\varphi(\sigma) \simeq \{e\}_R(\sigma, \tau).$$

From this we have the obvious notions of sets recursive and semirecursive in σ relative to R.

The following definition is essential.

8.2.2 Definition. Let $A \subseteq V$, $R \subseteq V$. The *set-recursive closure of A relative to R* is the set

$$M(A; R) = \{\{e\}_R(\sigma) : e \in \omega, \sigma \in A^n, n \in \omega\}.$$

If A is set-recursively closed relative to R, we may split up A as follows

$$\langle M(B; R)\rangle_{B \in {}^f A},$$

where ${}^f A$ is the set of finite subsets of A. This splitting will be of crucial importance in the following theory. As a first result we shall characterize semicomputability relative to R in terms of a special kind of Σ_1-definability over the splitting $\langle M(B; R)\rangle_{B \in {}^f A}$.

But first a linguistic convention. We write *R-recursive in τ* and *R-semirecursive in τ* instead of "set-recursive in τ relative to R" and "set-semirecursive in τ relative to R". Similarly we shall use the phrase *R-recursive closure of A* for the notion introduced in Definition 8.2.2.

8.2.3 Definition. Assume that A is R-recursively closed and that B is a finite subset of A. A set $C \subseteq A$ is called *$\Sigma_B^*(R)$-definable* if for some Δ_0-formula φ with parameters from $M(B; R)$

$$x \in C \quad \text{iff} \quad \exists y \in M(B \cup \{x\}; R) \cdot \varphi(x, y, R).$$

$C \subseteq A$ is called $\Delta_B^*(R)$ if both C and $A - C$ are $\Sigma_B^*(R)$-definable.

If the correct R is clear from the context we write for simplicity Σ_B^*, Δ_B^*, and even $M(B)$.

8.2.4 Proposition. *Let A be R-recursively closed and transitive. Then $\langle M(B; R)\rangle_{B \in {}^f A}$ satisfies Σ^*-collection: Let φ be a Δ_0-formula with parameters from $M(B; R)$ and let $u \in M(B)$. Assume*

$$\forall x \in u \, \exists y \in M(B \cup \{x\}; R) \cdot \varphi(x, y, R),$$

then

$$\exists v \in M(B) \forall x \in u \, \exists y \in v \cdot \varphi(x, y, R).$$

The proof is exactly the same as the proof of Proposition 5.4.20. Let σ_B be a listing of the finite set B. By assumption

$$\forall x \in u \, \exists e \in \omega \cdot \varphi(x, \{e\}_R(\sigma_B, x), R).$$

By Gandy selection, i.e. Proposition 8.1.4, choose one e to each x and use the union scheme (iv) to find the set v. Formally let $\nu(x)$ be the index corresponding to x. Then we can set

$$v = \bigcup_{x \in u} \{\{\nu(x)\}_R(\sigma_B, x, R)\},$$

which is easily seen to belong to $M(B)$.

8.2.5 Proposition. *Well-foundedness is Σ^*-definable.*

We indicate the proof. By the recursion theorem find an index e such that if y is a well-founded relation on x, then $\{e\}_R(y, x)\downarrow$ and $\{e\}_R(y, x)$ is the rank function of y. So, for any x, y, y is a well-founded relation on x iff $\exists f \in M(\{x, y\})$ such that f is a rank function for y. (Of course, this is independent of, hence uniform in, R.)

8.2.6 Theorem. *Let A be R-recursively closed and B a finite subset of A. A set $C \subseteq A$ is R-semirecursive in σ_B iff C is $\Sigma_B^*(R)$-definable.*

This is the promised definability characterization of R-semirecursive sets. We shall return to this result in connection with Kleene-recursion in higher types in the next section.

First, assume that C is R-semirecursive in σ_B, i.e. for some index e, $x \in C$ iff $\{e\}_R(x, \sigma_B)\downarrow$. By a somewhat lengthy analysis using the recursion theorem we may prove that if $\{e\}_R(x, \sigma_B)\downarrow$ then the associated computation tree will be in $M(B \cup \{x\}; R)$. Therefore,

$$x \in C \quad \text{iff} \quad \exists T \in M(B \cup \{x\}; R)[T \text{ is well-founded and } T \text{ is a computation-tree for } \{e\}_R(x, \sigma_B)].$$

By Proposition 8.2.5, this is seen to be a $\Sigma_B^*(R)$-definition of C.

For the converse, assume that C is $\Sigma_B^*(R)$-definable. We may use the same trick as in the proof of Proposition 8.2.4 to find for each $x \in C$ an index $e = \nu(x)$ for the y in the $\Sigma_B^*(R)$-definition of C. $\nu(x)$ will diverge if $x \notin C$, hence $C = \{x : \nu(x)\downarrow\}$, i.e. C is R-semirecursive in σ_B.

We conclude this section by a brief discussion of R-admissibility.

8.2.7 Definition. A family $\langle M_B\rangle_{B\in^f A}$ is called *R-admissible* if it satisfies the following three requirements:

(i) each M_B is rudimentarily closed in R,
(ii) for $B, C \in {}^fA$, $M_B \subseteq M_C$ iff $B \subseteq M_C$,
(iii) the family satisfies $\Sigma^*(R)$-collection.

Note that this is an "arbitrary" splitting, i.e. we have a map from ${}^fA$ into the universe of sets satisfying (i) to (iii) above. We have the following closure result:

8.2.8 Proposition. *Let $\langle M_B\rangle_{B\in^f A}$ be R-admissible. Then each M_B is closed under R-recursion.*

The proof proceeds in the following steps. First by induction on the height of a well-founded relation we prove by Σ^*-collection that if y is a well-founded relation on x, then the rank-function is in $M_{\{x,y\}}$. Thus well-foundedness is Σ^*-definable over $\langle M_B\rangle_{B\in^f A}$. Next, and by the same method, we prove that if $\{e\}_R(\sigma_B)\downarrow$, then the computation tree is in M_B. We then finish the proof by observing that the value of a computation is rudimentary in the computation tree. We remark that this proof also shows that the relation $\{e\}_R(\sigma_B) \simeq z$ is $\Sigma^*(R)$-definable over the family $\langle M_B\rangle_{B\in^f A}$.

Putting Propositions 8.2.4 and 8.2.8 together we see that if A is transitive and R-recursively closed, then $\langle M(B; R)\rangle_{B\in^f A}$ is the *finest splitting* of A into an R-admissible family.

8.3 Set Recursion and Kleene-Recursion in Higher Types

We shall now explain how set-recursion generalizes Kleene-recursion in higher types. Let $I = \mathrm{Tp}(k)$, i.e. I is the set of all total functions of type k. Tp(0) is then the set of natural numbers. We note that I has a natural pairing function, hence we may identify finite subsets of I with elements of I.

In order to get the effect of normality, i.e. the computability of the functional ${}^{k+2}E$ over I, we see from the proof of Lemma 8.1.2 that I must not only be a domain for the computation theory, we must also have recourse to I as an input to computations. This motivates the following definition.

8.3.1 Definition. Let $I = \mathrm{Tp}(k)$ and let $R \subseteq V$ be a relation. By the *spectrum of R over I* is understood the family

$$\mathrm{Spec}(R; I) = \langle M(\{a, I\}; R)\rangle_{a\in I}.$$

For simplicity we often write $M_a(I; R)$ for $M(\{a, I\}; R)$, and occasionally drop the I or the R or both, if their presence is clear from the context.

We let

$$M(I; R) = \bigcup_{a \in I} M_a(I; R).$$

As in Proposition 8.2.4 we see that Spec$(R; I)$ will satisfy $\Sigma^*(R)$-*collection over* I, i.e. if φ is a Δ_0-formula with parameters from $M_a(I; R)$, and if

$$\forall b \in I\, \exists y \in M_{\langle a,b\rangle}(I; R) \cdot \varphi(b, y, R),$$

then

$$\exists v \in M_a(I; R) \forall b \in I\, \exists y \in v \cdot \varphi(b, y, R).$$

We further note that each $M_a(I; R)$ will be rudimentarily closed relative to R. Looking back to Definition 8.2.7 we are led to the following notion of R-admissibility over I.

8.3.2 Definition. A family $\langle M_a \rangle_{a \in I}$ is called *R-admissible over I* if it satisfies the following requirements:

(i) each M_a is rudimentarily closed relative to R,
(ii) $I \in M_a$ for all $a \in I$, and for all $a, b \in I$
$a \in M_b$ iff $M_a \subseteq M_b$,
(iii) the family satisfies $\Sigma^*(R)$-collection over I.

Obviously the family Spec$(R; I)$ is R-admissible over I. We shall prove that it is the minimal family that is R-admissible over I. To do this we need to discuss how to code elements of the spectrum.

8.3.3 Definition. (i) Let $A \subseteq I \times I$ be a transitive, reflexive relation. For $a, b \in I$ let $a \simeq b$ iff $A(a, b)$ and $A(b, a)$. We say that A is a *code for a set* x if $A/\simeq$ is isomorphic to $\langle \mathrm{TC}(\{x\}), \in \rangle$, where TC as usual denotes the transitive closure.

(ii) Let $\langle M_a \rangle_{a \in I}$ be a family over I. $\langle M_a \rangle_{a \in I}$ is called *locally of type I* if for any set x and $a \in I$

$$x \in M_a \quad \text{iff} \quad x \text{ has a code in } M_a.$$

8.3.4 Lemma. Spec$(R; I)$ *is locally of type I.*

The proof is an exercise in the use of the recursion theorem. One first establishes that there is an index e_1 such that if A is a code for x, then $\{e_1\}_R(A, I) = x$. From this, once more by the recursion theorem, one constructs an index e_2 such that given any $e \in \omega$, and any sequence of codes $A_1, \ldots, A_n$ for sets $y_1, \ldots, y_n$, if $\{e\}_R(y_1, \ldots, y_n) \simeq x$, then $\{e_2\}_R(e, A_1, \ldots, A_n)$ is a code for x.

8.3.5 Remark. As the reader will understand from the above hint-of-a-proof we do not wish to go into details of proofs involving codes for sets. Complete expositions can be extracted from D. Normann [124], and also from Gurrik [52].

We can now prove the minimality of the spectrum.

8.3.6 Theorem. Spec$(R; I)$ *is the minimal family R-admissible over I.*

It remains to verify that Spec$(R; I)$ is included in any family $\langle M_a \rangle_{a \in I}$ which is R-admissible over I. The proof is very similar to the proof of Proposition 8.2.8. We prove by induction on the length of the computation that for any $x_1, \ldots, x_n$, if $x_1, \ldots, x_n$ have codes in M_a and $\{e\}_R(x_1, \ldots, x_n)\downarrow$, then both the computation tree and the value will be in M_a. The use of Σ^*-collection over a set x in 8.2.8 is now replaced by the use of a code for x and Σ^*-collection over I.

We come now to the main characterization theorem.

8.3.7 Theorem. *Let F be a functional of type $k + 2$, let $I =$ Tp(k) and $C \subseteq$ Tp$(k + 1)$. The following three statements are equivalent:*

(i) *C is Kleene-semirecursive in ${}^{k+2}E, F$.*
(ii) *C is F-semirecursive in I.*
(iii) *C is $\Sigma_I^*(F)$-definable.*

We recall the notion of $\Sigma_I^*(F)$-definable. C is $\Sigma_I^*(F)$-definable if

$$x \in C \quad \text{iff} \quad \exists y \in M_{\{x\}}(I; F)\varphi(x, y, F).$$

We note that F is here a relation, I is an element. The equivalence of (ii) and (iii) follows from Theorem 8.2.6.

The proof that (i) implies (ii) consists in constructing an index e in F-recursion such that

$$\{e\}_F(e_1, \sigma) \simeq \{e_1\}_{\mathrm{Kleene}}(\sigma, {}^{k+2}E, F).$$

This is tedious but straightforward. For example the scheme

$$\{e_1\}_K(f, \sigma) \simeq f(\lambda t \cdot \{e_2\}_K(f, \sigma, t)),$$

is handled by rewriting in the following form

$$\{e_1\}_K(f, \sigma) \simeq z \quad \text{iff} \quad \exists x \in I\, \forall y \in \mathrm{Tp}(k - 1)[x(y) = \{e_2\}_K(f, \sigma, y) \\ \wedge f(x) = z].$$

A combination of schemes (iv) and (vii) from Definition 8.1.1 will take care of this case, everything else is rudimentary.

For the converse assume that C is F-semirecursive in I, i.e. for some index e

$$f \in C \quad \text{iff} \quad \{e\}_F(f, I)\downarrow.$$

The method is now to simulate the computation $\{e\}_F(f, I)$ as a Kleene-computation in ${}^{k+2}E, F$ on codes. Again the details are not particularly exciting. The reader who for some reason wants to reconstruct the details, will need the following facts.

In Kleene-recursion there is an index e such that if f and g are characteristic functions for codes for x and y, respectively, then

$$\{e_1\}_K({}^{k+2}E, f, g) \simeq \begin{cases} 0 & \text{if} \quad x = y \\ 1 & \text{if} \quad x \neq y. \end{cases}$$

(Use the recursion theorem and induction on min(rank(x), rank(y)).)

From e_1 we construct a Kleene-index e_2 such that if $f_1, \ldots, f_k$ are characteristic functions of codes for $x_1, \ldots, x_k$ and $\{e\}_F(x_1, \ldots, x_n) \simeq y$, then

$$\lambda a \in I \cdot \{e_2\}_K(e, f_1, \ldots, f_k, a, {}^{k+2}E, F),$$

is the characteristic function of a code for y.

As we remarked in the introduction to this chapter, the sources of this theorem, as well as for Theorems 8.2.6 and 8.3.6 are the theses of Harrington [53], MacQueen [98], and Normann [122]. A first version of 8.2.6 can be found in Harrington's thesis, and the first version of Theorem 8.3.6 stems from MacQueen [98]. Further developments can be found in Normann [122] as well as in the (unpublished) lecture notes of Kechris [74].

8.3.8 Examples. We shall separately consider the case $k = 0$ and the case $k > 0$. And we shall first supplement Theorem 8.3.7 with the following observation: *Let F be of type $k + 2$, let $a \in I$ and $A \subseteq I$, then $A \in k + 1 -$ sc$({}^{k+2}E, F, a)$ iff $A \in M_a(I; F)$.* (The proof is implicit in 8.2.6 and 8.3.7.)

First to the case $k = 0$. Then I is the set of natural numbers and we are essentially studying Kleene-recursion in ${}^2E, F$, where F is a total type-2 functional over ω. In this case Spec(F; ω) reduces to one set, viz. the "next admissible" set corresponding to the Spector theory PR[${}^2E, F$]. We have recovered the theory of Section 5.3, and Theorem 8.2.6 is the genuine Gandy-Spector theorem.

In the case $k > 0$, Spec(F; I) $= \langle M_a(I; F)\rangle_{a \in I}$ is non-trivial (since not every $a \in I = $ Tp(k) is recursive in ${}^{k+2}E, F$). Since the spectrum is locally of type I, we see that $M_a(I; F)$ consists of exactly those sets which have a code in k + 1-sc$({}^{k+2}E, F, a)$. (Note, that this was exactly the way we constructed the abstract 1-section of a Spector theory over ω, see Proposition 5.4.20.)

Each set $M_a(I; F)$ is countable, rudimentarily closed in F and satisfies suitable versions of Δ_0-separation and Δ_0-dependent choices. These sets thus have all "good" properties of admissible sets and abstract 1-sections, but they do contain "gaps", i.e. the sets are not transitive. The gaps are necessarily present to reflect gap phenomena in computations in higher types.

The sets in the family Spec(F; I) interact via the $\Sigma^*(F)$-collection principle which leads to the characterization in Theorem 8.3.7. In this case the equivalence of (ii) and (iii) is the best possible version of the original Gandy-Spector theorem, the domain of the existential quantifier cannot be chosen independently of the element we are testing for membership in the given semicomputable set.

Theorem 8.3.7 shows that Kleene-recursion in a normal functional is a special case of set-recursion relative to some relation. We shall now prove a converse.

The following recursive approximation of the spectrum will prove useful.

8.3.9 Definition. Let α be an ordinal and A a set. The *α-approximation* to $M(A; R)$ is the set

$$M^{\alpha}(A; R) = \{\{e\}_B(\sigma_B) : B \in^f A,\ e \in \omega \text{ and } \|e, \sigma_B\| < \alpha\}.$$

From this we derive the correct definitions of $\langle M^{\alpha}(B; R)\rangle_{B \in^f A}$ and $\langle M^{\alpha}(I; R)\rangle_{a \in I}$.

We shall also need the following notion of weak Σ^*-definability.

8.3.10 Definition. Let $\langle M_a\rangle_{a\in I}$ be an R-admissible family over I and let $C \subseteq M$, where $M = \bigcup_{a\in I} M_a$. We say that C is *weakly Σ^*-definable in a relative to R*, in symbols, C is w-$\Sigma^*_a(R)$, if for some Δ_0-formula φ with parameters from M_a,

$$x \in C \quad \text{iff} \quad \forall b \in I(x \in M_{\langle a,b\rangle} \Rightarrow \exists y \in M_{\langle a,b\rangle} \cdot \varphi(x, y, R)).$$

C is w-$\Delta^*_a(R)$ if both C and $M - C$ are w-$\Sigma^*_a(R)$.

We shall comment below on why we need the weak notion. Here we first remark that if $C \subseteq I$, then w-$\Sigma^*_a(R)$ and $\Sigma^*_a(R)$ coincide since $M_{\langle a,x\rangle}$ is the least $M_{\langle a,b\rangle}$ such that $x \in M_{\langle a,b\rangle}$. We also have the following lemma.

8.3.11 Lemma. *Let R be a relation and $\langle M_a\rangle_{a\in I} = \mathrm{Spec}(R; I)$. If $C \subseteq M$ is $\Sigma^*_a(R)$-definable, then C is w-$\Sigma^*_a(R)$-definable.*

Since C is $\Sigma^*_a(R)$-definable, we have

$$x \in C \quad \text{iff} \quad \exists y \in M(\{x, a\}; R)\varphi(x, y, R).$$

Let $x \in M_{\langle a,b\rangle}$. Then

$$x \in C \quad \text{iff} \quad \exists \alpha \in M_{\langle a,b\rangle} \exists y \in M^{\alpha}(\{x, a\}; R)\varphi(x, y, R).$$

It is sufficient to show that the relation $z = M_{\{x,a\}}(R)$ is w-$\Sigma^*(R)$ and that if $\alpha \in M_{\langle a,b\rangle}$ and $x \in M_{\langle a,b\rangle}$, then $M^{\alpha}_{\{x,a\}} \in M_{\langle a,b\rangle}$. This we do by a careful analysis of the inductive definition of $M^{\alpha}_{\{x,a\}}$.

We need one more lemma.

8.3.12 Lemma. *Let R_1 and R_2 be relations. If R_1 is w-$\Delta^*(R_2)$, then* $\mathrm{Spec}(R_1; I) \subseteq \mathrm{Spec}(R_2; I)$ *and if $C \subseteq I$ is $\Sigma^*(R_1)$-definable, then C is also $\Sigma^*(R_2)$-definable.*

For the proof we note that if R_1 is w-$\Delta^*(R_2)$ then $\mathrm{Spec}(R_2; I)$ will be R_1-admissible over I; by Theorem 8.3.6 we conclude that $\mathrm{Spec}(R_1; I) \subseteq \mathrm{Spec}(R_2; I)$.

In connection with Definition 8.3.10 we observe that over I Σ^* and w-Σ^* are the same. So it suffices to prove that w-$\Sigma^*(R_1) \subseteq w$-$\Sigma^*(R_2)$. But this will follow if we can prove that w-$\Delta^*(R_2)$ is closed under bounded quantification.

Since we are not in the admissible case we will give the proof of this closure property. Or rather, we prove this for $\Delta^*(R)$ definability: the w-Δ^* case is similar.

Let θ be $\Delta^*(R)$ definable, i.e. we have formulas ψ_1 and ψ_2 such that

$$\theta(z, x) \quad \text{iff} \quad \exists w \in M_{\{x,z\}} \psi_1(x, z, w)$$
$$\neg\theta(z, x) \quad \text{iff} \quad \exists w \in M_{\{x,z\}} \psi_2(x, z, w).$$

We want to show that

$$\psi(x) \quad \text{iff} \quad \forall z \in y\,\theta(z, x)$$

is $\Delta^*(R)$ definable. But this is an immediate consequence of the following two equivalences

$$\psi(x) \quad \text{iff} \quad \exists f \in M_{\{x\}} \forall z \in y\, \psi_1(x, z, f(z))$$
$$\psi(x) \quad \text{iff} \quad \forall f \in M_{\{x\}} (\forall z \in y[\psi_1(x, z, f(z)) \vee \psi_2(x, z, f(z))] \rightarrow \forall z \in y\, \psi_1(x, z, f(z))).$$

In connection with this argument the reader should appreciate the fact that Σ^* is *not* closed under bounded quantification.

We are now in a position to prove that Kleene-recursion in a normal type $k + 2$ functional over $I = \mathrm{Tp}(k)$ is the same as set-recursion over I relative to some relation.

8.3.13 Theorem. *Let $I = \mathrm{Tp}(k)$ and let R be a relation. Then there is a total type $k + 2$ functional F such that*

(i) $\mathrm{Spec}(R; I) = \mathrm{Spec}(F; I)$.
(ii) *Over I, $\Sigma^*(R)$ and $\Sigma^*(F)$ are the same.*

At this point the reader would do well to recall our discussion of abstract 1-sections in Section 5.4; note that $k = 0$ is permissible in the above theorem.

We proceed to the proof. Let R be given, by Lemma 8.3.12 it is sufficient to construct an F such that F and R are w-Δ^* in each other.

We define approximations F_α to F by induction on α. If $F_\alpha(f)$ is undefined for all ordinals α, we complete the definition of F by setting $F(f) = 1$.

We start off by setting F_0 equal to the empty function. Let $F_{<\alpha} = \bigcup_{\gamma<\alpha} F_\gamma$. At stage α we proceed as follows:

If f is $F_{<\alpha}$-computable by a computation of length $\leqslant \alpha$ and $F_{<\alpha}(f)$ is undefined, let

(i) $F_\alpha(f) = 0$ if f is a pair $\langle f_1, f_2 \rangle$ and f is (the characteristic function of) a code for a set x such that $\mathrm{rank}(x) \leqslant \alpha$ and $x \in R$.

(ii) $F_\alpha(f) = 1$ otherwise.

We separate the proof in four steps.

8.3.14. R is w-$\Delta^*(F)$.

Let $x \in M_a(F)$. Then x will have a code A in $M_a(F)$, see Lemma 8.3.4. We also have that $\alpha = \operatorname{rank}(x) \in M_a(F)$.

There will be a set $B \in M_a(F)$ which has not been computed before stage α. Let f_1, f_2 be the characteristic functions of B, A respectively, and put $f = \langle f_1, f_2 \rangle$. f will then be in $M_a(F)$ and $x \in R$ iff $F(f) = 0$. For any a such that $x \in M_a$ we then see that

$$\begin{aligned} x \in R \quad &\text{iff} \quad \exists f = \langle f_1, f_2\rangle \in M_a(F)[f_2 \text{ is a code for } x \text{ and} \\ &\qquad\qquad f \notin \langle M_b^{\operatorname{rank}(x)}(F)\rangle_{b\in I} \text{ and } F(f) = 0], \\ &\text{iff} \quad \forall f = \langle f_1, f_2\rangle \in M_a(F)[f_2 \text{ is a code for } x \text{ and} \\ &\qquad\qquad f \notin \langle M_b^{\operatorname{rank}(x)}(F)\rangle_{b\in I} \Rightarrow F(f) = 0]. \end{aligned}$$

Thus R is w-$\Delta^*(F)$ and by Lemma 8.3.12 $\operatorname{Spec}(I; R) \subseteq \operatorname{Spec}(I; F)$.

The reader should note that this is a point where weak definability is needed. An arbitrary $x \in R$ need not be an element of I. We are not able to prove for arbitrary x that there is a code for x in $M_{\{x\}}$. Therefore, we started out with a set $M_a(F)$ where $a \in I$. Then, as we pointed out, every $x \in M_a$ will have a code in M_a, and codes are needed for the construction of F.

8.3.15. The relation

$$\text{“}\{e_1\}(\sigma, {}^{k+2}E, F) \simeq n \text{ by a computation of length less than } \alpha\text{”}$$

is R-recursive.

The proof is by an analysis of the definition of F using the recursion theorem for set-recursion relative to R. From this and Σ^*-collection over I we further prove:

8.3.16. If $\{e_1\}(\sigma, {}^{k+2}E, F)\downarrow$, then the length of this computation will be in $M_a(R)$, where a codes the input sequence σ.

We are now ready for the final stage of the argument.

8.3.17. F is w-$\Delta^*(R)$.

Let $f \in M_a(R)$, then by the first part of the proof, $f \in M_a(F)$, hence for some α, $F_\alpha(f)$ is defined. By 8.3.16 we can find such an α in $M_a(R)$. We can then write for $f \in M_a(R)$

$$\begin{aligned} F(f) = 0 \quad &\text{iff} \quad \exists \alpha \in M_a(R) \cdot F_\alpha(f) = 0, \\ &\text{iff} \quad \forall \alpha \in M_a(R)(F_\alpha(f) \text{ is defined} \Rightarrow F_\alpha(f) = 0). \end{aligned}$$

To complete the proof we note that if $\alpha \in M_a(R)$, then F_α is R-recursive in α, I, hence by Lemma 8.3.11 F_α will be w-$\Delta^*(R)$.

8.3.18. Remark. If we are in the case $I = \mathrm{Tp}(k)$, $k > 0$, then well-foundedness is set-recursive relative to I. This fact simplifies the proof of Theorem 8.3.13 considerably, simply define F by

$$F(f) = \begin{cases} 0 & \text{if } f \text{ is a code for a set } x \in R, \\ 1 & \text{otherwise.} \end{cases}$$

8.4 Degrees of Functionals

We shall in this section use the theory of set-recursion to give a priority argument relative to ${}^3E$. This is but an introduction to a vast topic which the reader is invited to investigate for him- or herself. In general the setting will be $I = \mathrm{Tp}(k)$ and we study set-recursion relative to some relation R.

8.4.1 Definition. Let $I = \mathrm{Tp}(k)$ and R be given.

$A \leqslant_R B$ iff A is set-recursive in I and some individual $a \in I$ relative to R and B.

This is a very liberal reducibility notion. Harrington in his thesis [53] used *subindividuals* instead of *individuals* (see Sacks [144] for a very informative exposition). We shall comment on this below.

To simplify the discussion we retreat at once to the case $I = \mathrm{Tp}(1)$. In this case the continuum is assumed given, i.e. we are concerned with the structure $\mathrm{Spec}({}^3E; I) = \langle M_a \rangle_{a \in I}$, where $I = \mathrm{Tp}(1)$. In order to obtain any results at all we shall have to introduce the following

8.4.2 Assumption. We assume for the rest of this section that $V = L$, i.e. there is a wellordering $<$ of I which is recursive in ${}^3E$ and has length $\aleph_1$. We let $\| \ \|$ be the norm associated with $<$.

This assumption is needed if we insist on the liberality of Definition 8.4.1. If we had restricted ourselves to reducibility relative to *sub*individuals there would have been no need of introducing $V = L$. This is tied up with the fact that the set of subindividuals is *strongly* finite, hence we have an associated admissible structure, and it is possible to use the full arsenal of techniques of admissibility theory, such as e.g. the blocking technique (see Chapter 6). On the other hand, Theorem 8.4.6 fails in the presence of strong assumptions of determinacy. We would not be surprised if the same theorem proved independent of ZF; see the similar situation in 6.3.3.

We shall need some technical results derived from the well-ordering $<$ on I.

8.4.3 Definition. For $a \in I$ let a', the *a-jump*, be the $<$-least b such that $b \notin M_a$.

8.4.4 Lemma.

(a) *If* $a < b$, *then* $M_a \subseteq M_b$.
(b) $\|a'\|$ *is the least ordinal not in* M_a.
(c) $M_a \in M_{a'}$.
(d) $M_a <_{\Sigma_1} M_{a'}$, *i.e.* M_a *is a* Σ_1*-substructure of* $M_{a'}$.

For the proof of (a) we note that $\{c \in I : c < b\}$ is countable and thus can be enumerated by an element of I. Using $<$ we may find such an enumeration $\{c_i\}_{i \in \omega}$ in M_b. Then $a = c_i$ for some $i \in \omega$, therefore $a \in M_b$ and thus $M_a \subseteq M_b$.

The proof of (b) follows directly from the definition of the a-jump and the fact that $b \in M_a$ iff $\|b\| \in M_a$.

Part (c) then follows since M_a can be recursively enumerated up to the ordinal $\|a'\|$; thus $M_a \in M_{a'}$.

The last part of the theorem is less trivial, being an application of further reflection, see Theorem 7.1.7. Let c be the characteristic function of a complete $\Sigma_1(M_a)$ subset of ω ($=$ the subindividuals). We see that $c \notin M_a$, so $a' \leqslant c$. On the other hand, $c \in M_{a'}$, since $M_a \in M_{a'}$, therefore $M_{a'} = M_c$. Further reflection is now essentially the statement $M_a <_{\Sigma_1} M_c$, see the Compactness property 7.1.7 and the proof.

We shall need one more technical result. In Section 8.3, see in particular Theorem 8.3.13, we showed how to translate from set-recursion back to Kleene-recursion in higher types. In the proof we had to worry about *weak*-definability, a technical nuisance was the fact that for an arbitrary $x \in M$ there is not necessarily a code for x in $M_{\{x\}}$ (see the remark after 8.3.14). We get around this complication by introducing the set

$$^1M = \{\langle a, x\rangle : x \in M_a\}.$$

^1M is a Σ^*-subset of M and for subsets of ^1M the notions $\mathbf{\Sigma}^*$ and weak-$\mathbf{\Sigma}^*$ coincide. The use of ^1M is unnecessary in the priority argument, but is required for restating the result in terms of Kleene-recursion in higher types. The facts we use are:

(i) If $x \in {}^1M$ then there is an $a \in I$ such that $M_a = M(\{x, I\})$.
(ii) If $x \in {}^1M$, then $M(\{x, I\})$ is locally of type $k + 1$.
(For definitions see in particular 8.3.3.)

We are in the case $I = \mathrm{Tp}(k)$, where $k > 0$. We can now use Remark 8.3.18 to simplify the construction of a functional F_Q corresponding to a given relation Q. In detail: let $Q \subseteq I \times M$ and set

$$A_Q = \{\langle a, f\rangle : f \text{ is the characteristic function of a code for a set } x \text{ and } \langle a, x\rangle \in Q\}.$$

And let

$$F_Q = \text{the characteristic function of } A_Q.$$

By a suitable coding F_Q is a functional of type-2 over the domain I, in our case F_Q is of type 3.

8.4.5 Proposition. *Let* $I = \mathrm{Tp}(k)$, $k > 0$, *and* R *a relation. Let* $\mathrm{Spec}(R; I) = \langle M_a \rangle_{a \in I}$ *and let* $M = \bigcup_{a \in I} M_a$, ${}^1M = \{\langle a, x \rangle : x \in M_a\}$.
Assume that $Q \subseteq M$, *then*

(a) Q, $A_a \cap M$ *and* $F_Q \cap M$ *are* w-$\Delta^*(R)$ *in each other.*
(b) *If* $Q \in w\text{-}\Delta^*_a(R)$, *where* $a \in I$, *then* F_Q *is weakly Kleene-recursive in* F_R, ${}^{k+2}E$, a.
(c) *If* $Q \subseteq {}^1M$ *and* $Q \in \Sigma^*_a(R)$, *then* A_Q *is Kleene-semirecursive in* F_R, ${}^{k+2}E$, a.

For the notion of weakly Kleene-recursive see Definition 4.1.7.
We observe that part (a) is immediate since each M_b in the spectrum is locally of type $k + 1$. To prove part (b) we must find an index e in Kleene-recursion such that

$$F_Q(\lambda b \cdot \{e'\}_K(F_R, {}^{k+2}E, a, b, \sigma)) \simeq \{e\}_K(F_R, {}^{k+2}E, a, \sigma, e').$$

The proof is a bit messy, but the ideas are rather straightforward and by now familiar. Q is w-$\Delta^*_a(R)$. We first observe that Δ_0-formulas can be handled by ${}^{k+2}E$. And the unbounded quantifiers over $M_{a,\sigma}$ needed in the w-Δ^*_a-definition of F_Q from R can be replaced by unbounded quantifiers over $k + 1$-$\mathrm{sc}(F_R, {}^{k+2}E, a, \sigma)$. But objects in the $k + 1$-section have numerical codes, and we may in a familiar way use Gandy-selection over ω to complete the proof. The proof of (c) is similar taking account of technical facts (i) and (ii) above.

Before stating the main theorem we note that Assumption 8.4.2 implies that there is a wellordering of ${}^1M$ recursive in ${}^3E$. Since knowledge of this wellordering is important for the proof of the main theorem, we shall give a fairly detailed description of it. The spectrum $\langle M_a \rangle_{a \in I}$ has an approximation $\langle M^\alpha_a \rangle_{a \in I}$, where

$$M^\alpha_a = \{\{e\}(a, I) : \|e, a, I\| \leqslant \alpha\}.$$

Let ${}^1M^\alpha = \{\langle a, x \rangle : x \in M^\alpha_a\}$.
We note that in a standard way we can introduce a wellordering $<^*$ on ${}^1M^\alpha - \bigcup_{\beta < \alpha} {}^1M^\beta$ of ordertype $\aleph_1$; this wellordering is induced from the given wellordering of I. Let $\alpha(x)$ = least ordinal α such that $x \in {}^1M^\alpha$. We then set

$x <^1 y$ iff $\alpha(x) < \alpha(y)$ or $\alpha(x) = \alpha(y) = \alpha$ and x is less than y in the wellordering $<^*$.

We use $\| \; \|^1$ for the associated norm. Note that $\| \; \|^1$ will be a set-recursive function on ${}^1M$ with a set-recursive inverse.
We can now state the theorem.

8.4.6 Theorem. ($V = L$). *There is a* Σ^**definable subset* Q *of* ${}^1M$ *such that* $M(Q; I) = M$, *but* Q *is not* $\boldsymbol{\Delta}^*$*-definable over* M.

8.4.7 Corollary. ($V = L$). *There is a set $A \subseteq \text{Tp}(2)$ semirecursive in ${}^3E$ such that*

(i) *A is not recursive in ${}^3E$ and a function.*
(ii) *If $B \subseteq I$ is recursive in A, ${}^3E$ and a function, then B is recursive in ${}^3E$ and a function.*
(iii) *No complete semirecursive subset of I is recursive in A, ${}^3E$ and a function.*

The corollary follows immediately from the theorem using the appropriate parts of Proposition 8.4.5:

We have constructed a set $Q \subseteq {}^1M$, let $A = A_Q$. By (c) in 8.4.5, A is semirecursive in ${}^3E$, and since Q is not Δ^* in parameters from I, A is not recursive in ${}^3E$ and a function. Part (ii) of the corollary follows from the equality $M(Q; I) = M$. The same equality also gives (iii), but this needs a small supplementary argument. Let C be a complete semirecursive subset of I, and assume that C is recursive in A, ${}^3E$ and a function. Then there will be an $a \in I$ such that $C \in M_a(Q; I)$. But since C is complete, $C \notin M$, i.e. we have the sought for violation of equality $M(Q; I) = M$, which proves (iii).

We turn now to a proof of the theorem.

Let $W_{e,a}$ be an enumeration of the Σ_a^*-subsets of M and let $W_{e,a}^\sigma$ be recursive approximations, $\sigma \in \text{On} \cap M$. We shall have to worry about two kinds of conditions:

$1 \cdot e \cdot a$ $\quad M - Q \neq W_{e,a}$
$2 \cdot e \cdot a$ $\quad$ Protect the computation $\{e\}_Q(a, I)$.

The first set of conditions will secure that Q is not Δ^*-definable, the second will give us the necessary control over $\text{Spec}(Q; I)$.

Each condition will be coded as a pair $\langle a, n\rangle = \langle a, \langle e, i\rangle\rangle$, $i = 1, 2$, and we order the pairs by the lexicographical ordering on $I \times \omega$. The order type will be $\aleph_1$. We shall let ν denote both a condition and its place in the ordering. This ordering determines our priorities.

We shall also use the following terminology. Let $\nu < \aleph_1$, we say that y is in row ν, or $y \in \text{row } \nu$, if $y \in {}^1M$ and for some ordinal β, $\|y\|^1 = \aleph_1 \cdot \beta + \nu$.

We shall also find the following terminology useful:

a-conditions: $\nu = \langle a, n\rangle$
1-conditions: $\nu = 1 \cdot e \cdot a$
2-conditions: $\nu = 2 \cdot e \cdot a$.

Recall our dual use of ν. Notice that if $a < b$ then all the a-conditions have higher priority than the b-conditions.

Q^σ will be constructed by an induction on the stage $\sigma \in \text{On} \cap M$ such that Q^σ will be uniformly set-recursive in σ. At each state σ we shall worry about one "relevant" 1-condition or all "relevant" 2-conditions.

The 2-conditions will be met by creating requirements freezing certain computations relative to Q. A requirement x for ν is *active* at stage σ if $x \cap Q^\sigma = \varnothing$.

To meet the 1-conditions $\nu = 1 \cdot e \cdot a$ we will see to it that $M - Q$ and $W_{e,a}$ differ on row ν. This will be done by putting an element from row $\nu \cap W_{e,a}$ into Q if possible. In doing this we must be careful not to destroy other parts of the construction. This leads to the following stipulation:

8.4.8. y is a *candidate* for ν at stage σ if

(i) $y \in {}^1M^\sigma$
(ii) $y \in \text{row}\, \nu$
(iii) If x is an active requirement for $\nu_1 < \nu$ at stage σ, then $y \notin x$
(iv) $\sigma \in M_y^{\sigma+1}$
(v) Let $\sigma_0 \leqslant \sigma$ be minimal such that $y \in {}^1M^{\sigma_0}$, then for all σ_1 such that $\sigma_0 \leqslant \sigma_1 \leqslant \sigma$ we require that $y \in M_{\sigma_1}^\sigma$.

The first four conditions on y are quite reasonable: since we want $Q \subseteq {}^1M$ we must insist on (i); (iii) expresses that we do not want to injure active requirements of higher priority, and (iv) expresses our intention to make Q Σ^*, therefore we will not put y into $Q^{\sigma+1}$ unless σ is in M_y. Only (v) needs a comment, it is included purely for technical reasons in order to preserve computations $\{e\}_Q(b, I)$ inside M_b, or $M_{b'}$ at worst.

8.4.9. A condition ν *requires attention* at stage σ if

(i) $\nu = 1 \cdot e \cdot a$, $Q^\sigma \cap \text{row}\, \nu = \varnothing$, and there is a candidate for ν at stage σ in $W_{e,a}^\sigma$, or

(ii) $\nu = 2 \cdot e \cdot a$, there is no active requirement for ν at stage σ, and $\{e\}_{Q^\sigma}(a, I)\downarrow$ by a computation of length $\leqslant \sigma$ which uses elements from M^σ only in the subcomputations.

We can now proceed to the construction of Q. As usual, set $Q^0 = \varnothing$ and $Q^\lambda = \bigcup_{\sigma<\lambda} Q^\sigma$ for λ a limit stage.

8.4.10. Assume that Q^σ is constructed.

If no condition requires attention at stage σ let $Q^{\sigma+1} = Q^\sigma$. Otherwise, let ν be the least condition that requires attention at stage σ. There are two cases

(i) $\nu = 1 \cdot e \cdot a$. Let y be the $<^1$-least candidate for ν in $W_{e,a}^\sigma$ and set $Q^{\sigma+1} = Q^\sigma \cup \{y\}$.

(ii) $\nu = 2 \cdot e \cdot a$. Let $Q^{\sigma+1} = Q^\sigma$ and create the following requirements: For each $\nu_1 = 2 \cdot e_1 \cdot a_1$, if ν_1 requires attention at stage σ, let $M^\sigma - Q^\sigma$ be a requirement for ν_1.

Finally, set $Q = \bigcup_{\sigma \in M} Q^\sigma$.

If x is a requirement and we put a $y \in x$ into Q, then we say that x is *injured*. A rather trivial cardinality argument leads to

8.4.11. If ν is a condition, there will be created at most countably many requirements for conditions $\leqslant \nu$.

A requirement for ν is injured at most countably many times.

This simple observation gives one half of the theorem.

8.4.12. $M - Q \neq W_{e,a}$ for all e, a.

Let $\nu = 1 \cdot e \cdot a$ and let σ be a stage at which all requirements for $\nu_1 < \nu$ ever to be constructed are constructed and all 1-conditions $\nu_1 < \nu$ to be met are met. σ exists by 8.4.11 and we may assume that $\sigma = \kappa_0^b$ for some $b \geqslant a$. (Recall that $\kappa_0^b = \sup(\mathrm{On} \cap M_b)$.)

Assume that $M - Q$ and $W_{e,a}$ coincide on row ν. Let $y \in$ row ν be the element with norm $\langle \sigma, \nu \rangle$ in $\| \; \|^1$. If $y \in Q$ then y must also be an element of $W_{e,a}$, otherwise it would never be put into Q. But this goes against our assumption, therefore $y \in M - Q$ and hence also $y \in W_{e,a}$. Since $W_{e,a}$ is Σ^*, there will be a $\sigma_1 \in M_{a,y}$ such that $y \in W_{e,a}^{\sigma_1}$. We show that we can choose $\sigma_1 \geqslant \sigma$ and $\sigma_1 \in M_y$ by proving the

Claim. $a \in M_y$ and $\sigma \in M_y$.

Let $y \in {}^1M$ be of the form $\langle b', y_1 \rangle$. Then σ is minimal such that $y_1 \in M_{b'}^{\sigma}$, so $\sigma \in M_{b'} = M_y$. Clearly $a \in M_\nu$ and since $y \in$ row ν, we may compute ν from σ and y.

So assume that $\sigma_1 \geqslant \sigma$, $\sigma_1 \in M_y$, and $y \in W^{\sigma_1}$. We shall argue that at stage σ_1 y is a candidate for ν. The troublesome part is (v) of 8.4.8. But by the choice of σ and y we first see that σ is minimal such that $y \in {}^1M^{\sigma}$. It remains to verify that if $\sigma' \geqslant \sigma$ then $y, \sigma \in M_{\sigma'}^{\sigma}$, but this follows immediately from the fact that σ is chosen to be of the form κ_0^b for suitable b. Thus at stage σ_1 y is a candidate for ν, and ν thus requires attention. By choice of σ we do not pay attention to any $\nu_1 < \nu$, so at stage σ_1 something from row ν is put into Q^{σ_1+1}. But then, after all, $M - Q$ and $W_{e,a}$ must differ.

The other half of the theorem requires more detailed information about the construction. (And it is in the proof of the following lemma that the usefulness of (v) in 8.4.8 will become clear.)

8.4.13. (a) Let x be a requirement for ν created at stage $\sigma \in M_a$, where $x, \nu \in M_a$. If x is injured, then x is injured before stage κ_0^a.

(b) If $a = b'$ and $\nu = 2 \cdot e \cdot a_0$ for some $a_0 \leqslant a$, then there is a stage $\sigma \in M_a$ such that if $\sigma \leqslant \sigma_1 < \kappa_0^a$ no 1-condition $\nu_1 < \nu$ is met at stage σ_1.

For the proof of (a) assume that x is injured by putting a y from row ν_1 into Q at stage σ_1. Since $x = M^{\sigma} - Q^{\sigma}$, y will be an element of ${}^1M^{\sigma}$, so, in particular,

$$y \in {}^1M^{\kappa_0^a}.$$

If $\sigma_1 \geqslant \kappa_0^{a'}$, then $y \notin M_{a'}$ by clause (iv) in 8.4.8. By clause (v) of the same definition $y \in M_{\kappa_0^a}^{\sigma_1} \subseteq M_{a'}$. This contradiction shows that $\sigma_1 < \kappa_0^{a'}$. Then

$$M_{a'} \models \exists\sigma_1 \exists y \in \text{row}\, \nu_1 \cap Q^{\sigma_1+1}.$$

By reflection ((d) of 8.4.4) we get

$$M_a \models \exists\sigma_1 \exists y \in \text{row}\, \nu_1 \cap Q^{\sigma_1+1}.$$

This shows that the injury took place before stage κ_0^a, which proves part (a) of the lemma.

To prove (b) let $\nu_1 < \nu$ be a 1-condition. If ν_1 is a c-condition for some $c < a$ and if we meet ν_1 inside M_a, we also meet ν_1 inside M_b since $M_b <_{\Sigma_1} M_a$ and $\nu_1 \in M_b$.

Thus there could be at most finitely many 1-conditions $\nu_1 < \nu$ which we meet between κ_0^b and κ_0^a. This could happen only when $a_0 = a$, i.e. $\nu = \langle a, n\rangle$, and ν_1 is of the form $\nu_1 = \langle a, m\rangle$, where $m < n$. If this were the case, choose $\sigma > \kappa_0^b$ such that all these conditions are met before stage σ.

8.4.14. Let $x \in M$, we say that

$$x \in M_a(Q),$$

is *finally protected* at stage σ if for some $e \in \omega$ the computation $\{e\}_Q(a, I) = x$ is protected by a requirement active at stage σ which is never injured.

8.4.15. Assume that $\{e\}_Q(x_1, \ldots, x_n)\downarrow$. Let $a, c \in I$ and δ be an ordinal in $M_{\langle a,c\rangle}$. Assume that the statements

$$x_1, \ldots, x_n \in M_a(Q),$$

are finally protected at stage δ. Then there is a $\sigma > \delta$ in $M_{\langle a,c\rangle}$ such that

$$\exists x \in M_a^\sigma(Q^\sigma; I)\cdot\{e\}_{Q^\sigma}(x_1, \ldots, x_n) = x.$$

In applications of this lemma $x_1, \ldots, x_n$ will come from the set $I \cup \{I\}$, in which case the assumption is trivially true. The assumption seems, however, necessary in order to make the inductive proof work.

The proof is by induction on the length of the computation $\{e\}_Q(x_1, \ldots, x_n)$. We give case iv and case v of 8.1.1, the other cases are either similar or simpler.

Case iv. Here

$$\{e\}_Q(x_1, \ldots, x_n) \simeq \bigcup_{y \in x_1} \{e_1\}_Q(y, x_2, \ldots, x_n),$$

where $x_1 \in M_a(Q), \ldots, x_n \in M_a(Q)$ all are finally protected at stage δ.

First note that when x_1 is computed from a, I, there will be a function f mapping I onto x_1, uniformly recursive in the computation of x_1. For each $y = f(b) \in x_1$, $y \in M_{a,b}(Q)$ will be finally protected at stage δ.

From the induction hypothesis and Σ^*-collection we can now claim

$$\forall c \forall_\gamma \in M_{a,c} \exists \sigma \in M_{a,c} \forall b \exists \sigma_b [\gamma < \sigma_b < \sigma \wedge \exists x_b \in M^{\sigma_b}_{a,b,c} \cdot \{e_1\}_{Q^{\sigma_b}}(f(b), x_2, \ldots, x_n) \simeq x_b].$$

We may now define a strictly increasing sequence of ordinals $\langle \delta_b \rangle_{b \in I} \in M_{a,c}$ such that if $\|b_2\| = \|b_1\| + 1$, then

$$\forall b \in I \exists \sigma_b [\delta_{b_1} < \sigma_b \leqslant \delta_{b_2} \wedge \exists x_b \in M^{\sigma_b}_{a,b,b_1} \cdot \{e_1\}_{Q^{\sigma_b}}(f(b), x_2, \ldots, x_n) \simeq x_b].$$

Let $\sigma = \sup\{\delta_b : b \in I\}$. Since the cofinality of σ is $\aleph_1$, we may apply the cardinality argument used in 8.4.11 to the construction up to stage σ. The δ_b's are chosen such that for each $b_i \in I$ we will cofinally often below σ try to protect the computation $\{e_1\}_Q(f(b_i), x_2, \ldots, x_n)$. Thus at stage σ they will all be protected by active requirements. But then

$$\exists x \in M^\sigma_{a,c}(Q^\sigma) \cdot x = \bigcup_{b \in I} \{e_1\}_{Q^\sigma}(f(b), x_2, \ldots, x_n),$$

which completes case iv. (At this point the reader will see the usefulness of having $\leqslant$ rather than $<$ in the definition of the approximations M^α_a, x is a computation of length σ.)

Case v. We have composition

$$\{e\}_Q(x_1, \ldots, x_n) = \{e_0\}_Q(\{e_1\}_Q(x_1, \ldots, x_n), \ldots, \{e_m\}_Q(x_1, \ldots, x_n), x_1, \ldots, x_n).$$

To be able to use the induction hypothesis we must find a stage where all statements

$$\{e_i\}_Q(x_1, \ldots, x_n) \in M_a(Q),$$

are finally protected. Since reflection is available we have some freedom to manoeuver.

Let $\delta_0 = \kappa_0^{a,c}$. By the induction hypothesis there will be stages $\delta_1, \ldots, \delta_m$ in $M_{\langle a,c \rangle'}$ such that for $1 \leqslant i \leqslant m$

$$\exists y_i \in M^{\delta_i}_a(Q^{\delta_i}) \cdot \{e_1\}_{Q^{\delta_i}}(x_1, \ldots, x_n) \simeq y_i.$$

The associated conditions will be a-conditions. By 8.4.13 these conditions will be met by requirements which are never injured at a stage $\delta_{m+1} \in M_{\langle a,c \rangle'}$. Thus at stage δ_{m+1} all statements

$$y_i \in M_a(Q),$$

are finally protected, where $y_i = \{e_i\}_Q(x_1, \ldots, x_n)$.

By the induction hypothesis, once more, there is a $\delta_{m+2} \geqslant \delta_{m+1}$ in $M_{\langle a,c\rangle'}$ such that

$$\exists x \in M_a^{\delta_m+2}(Q^{\delta_{m+2}})\cdot\{e_0\}_{Q^{\delta_{m+2}}}y_1, \ldots, y_m, x_1, \ldots, x_n) \simeq x.$$

And such that

$$\exists x \in M_a^{\delta_m+2}(Q^{\delta_{m+2}})\cdot\{e\}_{Q^{\delta_{m+2}}}(x_1, \ldots, x_n) \simeq x.$$

Since $M_{\langle a,c\rangle} <_{\Sigma_1} M_{\langle a,c\rangle'}$ we find an σ in $M_{\langle a,c\rangle}$ with the same property as δ_{m+2} above.

This completes case v.

The following lemma will now complete the proof of the theorem.

8.4.16. If $a = b'$, then $M_a(Q; I) = M_a$.

If $x \in M_a(Q; I)$ there is an index e such that $x = \{e\}_Q(a, I)$. I and a are elements of M_a protected at all stages. Let $\nu = 2\cdot e\cdot a$.

By Lemma 8.4.13 (b) there is a $\sigma_0 \in M_a$ such that between σ_0 and κ_0^a we do not meet any 1-condition $\nu_1 < \nu$. By Lemma 8.4.15 there is a $\sigma_1 > \sigma_0$ in M_a such that

$$\exists x \in M_a^{\sigma_1}\cdot\{e\}_{Q^{\sigma_1}}(a, I) \simeq x.$$

We would then at stage σ_1 create a requirement to protect this computation, and by Lemma 8.4.13 (a) this requirement is never injured.

Thus

$$\{e\}_Q(a, I) = \{e\}_{Q^{\sigma_1}}(a, I) \in M_a,$$

and the lemma is proved.

8.4.17 Remark. As noted in the introduction to this chapter we have concentrated on a fairly simple priority argument to demonstrate the "naturalness" of set-recursion as a computation theory for the study of degrees of functionals.

Theorem 8.4.6 is due to Dag Normann and is not published in this form elsewhere. More advanced results, such as e.g. the splitting theorem, can be found in Normann [125, 127].

Reducibility relative to subindividuals has been studied by Harrington [53], see the exposition in Sacks [144].

There are many open problems, e.g. we would like to have a set-recursive version of the *density theorem*, see 6.3.1.

8.4.18 Remark. G. E. Sacks has in a forthcoming paper *Post's problem, absoluteness and recursion in higher types* (**Kleene symposium**, North-Holland, to appear) shown that in the case $I = \mathrm{Tp}(1)$ the set $M = \bigcup_{a\in I} M_a$ is *not* Σ_b^*-definable for any $b \in I$. It is, however, weakly-Σ^* definable. Thus there was a need for introducing the notion of weak-Σ^* definability and the set ${}^1M$ in our arguments above.

8.5 Epilogue

It is time to take our farewell of the reader. And let us do so by casting a quick glance back on the territory covered, pointing out some open problems, some omissions, and some areas of future research.

8.5.1. *Computing in an algebraic context.* Except for the introductory Pons Asinorum we have concentrated on traditional "hard core" recursion theory. But recursion-theoretic ideas could have wider application, beyond the usual tie-up with definability theory and descriptive set theory. Algebra is one possibility, we gave an example and several references in 0.3.4. There is, too, an interesting study by W. Hodges [63] who uses the Jensen-Karp theory of primitive recursive set functions to study the effectivity of some field constructions.

8.5.2. Part A of the book gave a reasonably thorough *axiomatic analysis of the notion of computation.* Whilst we used many combinatorial tricks first developed in the context of the λ-calculus, we did not look at recursion theory from the point of view of the λ-calculus, so let us make good this omission giving a reference to G. Mitschke, *λ-Kalkül, δ-Konversion und Axiomatische Rekursionstheorie* [104], in which the relationship is explicitly studied; see also Barendregt [9].

We also remarked in Section 2.7 on the difficulties in studying computations relative to a *partial* higher type objects. We gave some examples, but did not have an abstract axiomatic analysis to offer. This deserves further study. The reader should consult the exposition of Platek's thesis in Moldestad [105], and also a recent study by S. C. Kleene [86]. See also part II of [86] and a paper by D. P. Kierstead, *A semantics for Kleene's j-expressions*, to appear in the KLEENE SYMPOSIUM, North-Holland.

8.5.3. In Chapter 3 we gave an outline of the connection between Spector theories and the theory of inductive definability. Spector theories is our general version of hyperarithmetic theory, i.e. of the *effective* theory of Borel sets, and has many applications to *descriptive set theory*, see Moschovakis [118]. Applications beyond the first levels of the projective hierarchy require extra set-theoretic assumptions (e.g. the axiom of constructibility, the existence of measurable cardinals, or the axiom of determinacy). We find these "applications" a bit problematic from a philosophic point of view, and are more fascinated by the recent "sharper" applications of the "absolute" theory, see e.g. Louveau [96, 97]. And see also Fenstad-Normann [32] and references therein, where the interest is how far one can go with certain problems of descriptive set theory within the accepted set-theoretic foundation.

In Chapter 4 we gave a brief introduction to second-order definability theory, and referred to Kechris [76] for a fuller exposition. Here one would like to see applications to "real-life" mathematics.

Our treatment of general hyperarithmetic theory has been both highly selective and rather brief. We can make good one omission by drawing the reader's attention to the "omitting types" paper by Grilliot [51].

8.5.4. In Part C on admissible prewellorderings and degree structure we were on several occasions led to *the borderline between recursion theory and set theory.* We shall not repeat this discussion here. Interesting problems remain in determining the "true" domain for degree theory. The reader is referred back to Section 6.3 and the examples, problems, and references there given.

8.5.5. Our discussion of recursion in higher types in Chapters 4, 7, and 8 was again selective and rather abstract, the reader should consult the book by Hinman [61] for many interesting examples and further developments.

One example we must mention is the *superjump* introduced by Gandy [38]. The study of this object has played an important role in shaping the general theory. The superjump S has a simple definition

$$S(F, \alpha, e) \simeq \begin{cases} 0 & \text{if} \quad \{e\}(\alpha, F)\downarrow \\ 1 & \text{otherwise.} \end{cases}$$

The basic result is that 1-sc(S) is exactly $L_{\rho_0} \cap 2^\omega$, where ρ_0 is the first recursively Mahlo ordinal. There is an extensive literature, Gandy [38], Aczel-Hinman [8], Harrington [53, 54], Normann [126], and Lavori [95].

Another important concrete example is the type three object ${}^3CL$ which is essentially *the diagonalization operator for arbitrary inductive definitions on* ω. This was studied by Harrington (unpublished, but see a brief reference in Kechris [76]). A main result is that 1-sc(${}^2E$, ${}^3CL$) $= L_{\sigma(\pi_0)} \cap 2^\omega$, where π_0 is the least non-projectible ordinal and $\sigma(\pi_0)$ is the least ordinal stable in π_0. There is also an interesting connection to the Kolmogorov R-operator.

To continue our list of omissions: We have said almost nothing about *hierarchies.* Two basic references on the positive aspects of hierarchies in higher recursion theory are Shoenfield [149], and Wainer [170], where "good" hierarchies (in the sense of a genuine building up from below) are given for recursion in an arbitrary total type-2 object; the normal case is due to Shoenfield, the general case to Wainer.

Moving up in types the situation is more problematic. One can always "after the fact" extract a hierarchy, since we have ordinals associated with computations. But a genuine building up from below in the presence of ${}^3E$ does not exist, see Schwichtenberg-Wainer [146] for a good discussion. Notice that for certain type-3 objects, e.g. S and ${}^3CL$ mentioned above, in which ${}^3E$ is not recursive, we do have interesting hierarchies.

Let us conclude this list of omissions by drawing attention to the existence of *gap phenomena* in computations in higher types, see Moldestad [105] for an introduction. This is a fascinating area which invites further study. One should also not neglect the further study of *reflection phenomena* in higher types, see the brief introduction in Section 7.1 and the exposition in Kechris [74]. *Basis results* are discussed in Moldestad-Normann [107].

8.5.6. One growing and important area of general recursion theory has been entirely absent from our discussion, *the theory of countable or continuous functionals.*

By way of an introduction let us discuss the following example of Grilliot [50]. The setting is the total type-2 functionals over ω.

Theorem. *${}^2E$ is recursive in F iff F is effectively discontinuous.*

We indicate the proof of one half of the theorem, viz. that ${}^2E$ is recursive in F if F is effectively discontinuous. So suppose that $\langle g_i \rangle_{i\in\omega}$ and f are recursive in F, that $f = \lim_i g_i$, but $F(f) \neq \lim_i F(g_i)$.

By thinning out the sequence $\langle g_i \rangle_{i\in\omega}$ we may assume that $F(f) \neq F(g_i)$, for all $i \in \omega$ and, further, that $g_i(j) = f(j)$ for $j \leqslant i$.

Introduce an operator J by

$$J(h) = \begin{cases} f & \text{if} \quad \forall x(h(x) \neq 0) \\ g_i & \text{if} \quad h(i) = 0 \ \wedge \ \forall x < i(h(x) \neq 0). \end{cases}$$

J is recursive in F as can be seen from the following equation

$$J(h)(j) = \begin{cases} g_j(j) & \text{if} \quad \forall x \leqslant j(h(x) \neq 0) \\ g_i(j) & \text{if} \quad i \leqslant j \ \wedge \ h(i) = 0 \ \wedge \ \forall x < i(h(x) \neq 0). \end{cases}$$

But then ${}^2E(h) = 1$ iff $F(J(h)) = F(f)$, i.e. ${}^2E$ is recursive in F.

A corollary of this result is that ${}^2E$ is recursive in F and a function iff F is discontinuous.

The topology is here the usual product topology on Baire-space ω^ω. This topology is determined by "finite information", i.e. by a neighborhood basis consisting of sets $N_u = \{\alpha \in \omega^\omega : u \subseteq \alpha\}$, where u is a sequence number and $u \subseteq \alpha$ means that $\alpha(i) = u_i$, $i < \mathrm{lh}(u)$.

We have the following well-known observation

Proposition. *F is a continuous map from Baire-space into ω iff there is a function α_F, called an* associate *of F, which satisfies*

(i) $\forall \alpha \exists n \alpha_F(\bar{\beta}(n)) > 0$,
(ii) $\forall \beta \forall n(\alpha_F(\bar{\beta}(n)) > 0 \Rightarrow \alpha_F(\bar{\beta}(n)) = F(\beta) + 1)$.

We see how ${}^2E$ divides the higher recursion theory: either ${}^2E$ is recursive in F and then F is discontinuous and we are in the case of normal recursion in higher types, i.e. finite theories in the sense of Chapters 3 and 4 or ${}^2E$ is *not* recursive in F which leads to the non-normal case. And a major part of the non-normal case is concerned with the *countable* or *continuous* functionals, where computations are determined by finite information, i.e. by an associate α_F in the sense of the above proposition.

We have mostly been concerned with the full type structure in Part D. But in the countable/continuous case it makes good sense to go to a thinner hierarchy. This is the appropriate notion. A *type structure* is a collection of sets $\{A_\tau : \tau$ a type symbol} such that (i) $A_0 = \omega$, and (ii) if τ is of the form $\tau_1 \times \ldots \times \tau_n \to 0$, then A_τ is a set of maps (but not necessarily all maps) from $A_{\tau_1} \times \ldots \times A_{\tau_n}$ to ω.

One important example of a type hierarchy different from the full type structure is the hierarchy

$$\mathbf{C} = \{C_\tau : \tau \text{ is a type symbol}\},$$

of countable objects where $C_0 = \omega$, $C_1 = \mathrm{Tp}(1)$, but where, from C_2 on, we restrict ourselves by roughly requiring that a map $F\colon C_n \to \omega$ should be allowed in C_{n+1} only if its value at a g in C_n is determined by a finite amount of information about g. In the case of C_2 we can take this to mean that F should have an associate α_F, or, equivalently, that F is continuous.

At higher types the connection with topology is more problematic. Lifting the idea of having an associate presents, however, no difficulties. We let a map $\Phi\colon C_2 \to \omega$ belong to C_3 iff it has an associate α_Φ satisfying

(i) $\forall\beta\forall F \in C_2[\beta$ is an associate for $F \Rightarrow \exists n \alpha_\Phi(\bar{\beta}(n)) > 0]$,

(ii) $\forall\beta\forall n\forall F \in C_2[\beta$ is an associate for $F \wedge \alpha_\Phi(\bar{\beta}(n)) > 0 \Rightarrow \alpha_\Phi(\bar{\beta}(n)) = \Phi(F) + 1]$.

And so we continue.

Associated with the hierarchy $\mathbf{C}$ we have two natural notions of recursiveness. One can be quickly explained:

1. A countable functional F is *recursively countable* if it has a recursive associate α_F.

The other notion makes sense for arbitrary type structures $\{A_\tau : \tau \text{ a type symbol}\}$, provided the structure is sufficiently closed. It is simply the notion of computation obtained by relativizing the standard Kleene schemes S1–S9 (see Kleene [83] or Chapter 4) from the full hierarchy to the thin hierarchy $\{A_\tau\}$. It is easy to convince oneself that $\mathbf{C}$ is sufficiently closed, hence we have a notion of Kleene-computation relative to $\mathbf{C}$.

2. A countable functional F is *Kleene computable* if it has an index, i.e. there is an $e \in \omega$ such that $F(\sigma) = \{e\}(\sigma)$, where the right-hand side is determined by the relativized schemes S1–S9 over $\mathbf{C}$.

Both notions are natural. But they do not coincide. It is easy to show that Kleene computable functionals have recursive associates. The converse is false.

Let $[\gamma] = \{\beta \in C_1 : \forall x \in \omega(\beta(x) \leqslant \gamma(x))\}$. We see that $[\gamma]$ for any $\gamma \in C_1$ is a compact subset of C_1. The *fan functional* $\Phi(F, \gamma)$ computes a uniform *modulus of continuity* for F on $[\gamma]$, i.e.

$$\Phi(F, \gamma) = (\mu n)(\forall\beta, \beta' \in [\gamma])[\bar{\beta}(n) = \bar{\beta}'(n) \to F(\beta) = F(\beta')].$$

Φ is recursively countable (by König's lemma), but the fan functional is not Kleene computable. We indicate the proof.

Let $\gamma \equiv 1$ and $F \equiv 0$, then $\Phi(F, \gamma) = 0$. Assume that Φ is Kleene computable, i.e. there is an index e such that

$$(*) \qquad \{e\}(G, \gamma) = \Phi(G, \gamma)$$

for all G. Select a non-recursive δ in $[\gamma]$. Choose a "restricted associate" α for F, i.e. an α such that: (a) For all recursive β, $\exists n\alpha(\bar{\beta}(n)) > 0$, but (b) $\forall n\alpha(\bar{\delta}(n)) = 0$.

One can now prove that there exist sequence numbers $\sigma_1, \ldots, \sigma_k$ such that

(i) $\alpha(\sigma_i) > 0, \quad i = 1, \ldots, k,$
(ii) if $G = F$ on $N_{\sigma_1} \cup \ldots \cup N_{\sigma_k}$, then $\{e\}(G, \gamma) = \{e\}(F, \gamma)$.

This one proves by reflecting on the meaning of $\{e\}(F, \gamma)\downarrow$, taking into account that α is a restricted associate of F.

We can now choose an n such that $N_{\bar{\delta}(n)} \cap N_{\sigma_i} = \varnothing$ for $i = 1, \ldots, k$. Define a functional G by

$$G(f) = \begin{cases} 1 & \text{if } f \in N_{\bar{\delta}(n)} \\ 0 & \text{ow.} \end{cases}$$

It is immediate from the definition of the fan functional Φ that $\Phi(G, \gamma) = n$. But from (ii) above it equally follows that $\{e\}(G, \gamma) = \{e\}(F, \gamma) = \Phi(F, \gamma) = 0$. This contradicts (*) and shows that Φ is *not* Kleene computable.

A good epilogue is always an introduction to something beyond. And an introduction is an invitation not a complete story. In particular, in the case of countable functionals we urge the reader to go beyond these introductory remarks and here are a few basic references on this topic.

The study of countable functionals was opened up by the papers of Kleene [82] and Kreisel [88] in 1959 which introduced the countable hierarchy, the notion of an associate and the two ways of approaching the notion of recursiveness in C. The non-computability of the fan functional is due to Tait (unpublished), for an exposition see Gandy-Hyland [42] which is an excellent introduction to the field. Hinman [60] taking a lead from the theorem of Grilliot quoted above made a first contribution to the degree theory of continuous functionals.

The theory was further advanced by several contributions of Yu. L. Ershov [22, 23]. The "obvious" axiomatization problem, i.e. how to extend S1–S9 to a set of schemes giving all recursively countable functionals has been discussed by Feferman [25], see also Hyland [66]. Further results here may have an interesting feed-back on the general axiomatics of the notion of *computation*.

Of many more recent contributions we mention the theses of Bergstra [15] and Hyland [64], and the further contributions of Gandy-Hyland [42], Hyland [67], Normann [123, 128] and Normann-Wainer [131]. A systematic introduction to part of the *pure theory* can be found in the Lecture Notes of Normann [129]. There is also an *applied part* of the theory, viz. the application of countable functionals to constructivity and proof theory, see the original paper of G. Kreisel [88]. A survey is given in Troelstra [167]; a recent contribution is Hyland [65].