Part C

Initial Segments of $\mathscr{D}$ and the Jump Operator

Chapter IX
Minimal Degrees and High/Low Hierarchies

The material presented in Part B dealt with constructions of initial segments of $\mathscr{D}$ which were controlled by an oracle of degree $\mathbf{0}^{(2)}$. We now introduce more powerful techniques which permit constructions of initial segments of $\mathscr{D}$ which are controlled by an oracle of degree $\mathbf{0}'$. The forcing approach is dropped, and partial trees are used in order to carry out constructions more effectively.

The main results of this chapter are the construction of a minimal degree below $\mathbf{0}'$ and the construction of a minimal degree below an arbitrary degree $\mathbf{d} \in \mathbf{GH}_1$. A stronger version of the latter theorem is used to show that the minimal degrees below $\mathbf{0}'$ form an automorphism base for $\mathscr{D}[\mathbf{0}, \mathbf{0}']$.

1. Partial Recursive Trees

The constructions of this chapter will use partial trees in place of total trees. Most definitions dealing with these new trees remain unchanged from their counterparts for total trees. We will not repeat definitions unless there are changes to be made.

1.1 Definition. A *partial f-tree* T is a partial function $T\colon \mathscr{S}_f \to \mathscr{S}_f$ which satisfies the following conditions:

(i) $\quad \forall \sigma, \tau \in \mathscr{S}_f(\sigma \subseteq \tau \;\&\; T(\tau)\downarrow \;\to T(\sigma)\downarrow \;\&\; T(\sigma) \subseteq T(\tau)).$

(ii) $\quad \forall \sigma, \tau \in \mathscr{S}_f(\sigma \mid \tau \;\&\; T(\sigma)\downarrow \;\&\; T(\tau)\downarrow \;\to T(\sigma) \mid T(\tau)).$

(iii) $\quad \forall \sigma \in \mathscr{S}_f \forall i < f(\mathrm{lh}(\sigma))(T(\sigma * i)\downarrow \;\to \forall j < f(\mathrm{lh}(\sigma))(T(\sigma * j)\downarrow)).$

Thus T is defined on an initial segment of $\mathscr{S}_f$ under the $\subseteq$ ordering, and is defined on all possible extensions of σ of length $1 + \mathrm{lh}(\sigma)$ or no such extensions. If the function T is partial recursive, then T is called a *partial recursive f-tree*.

Henceforth, the word *tree* will be used to denote a partial f-tree. We wish to distinguish those $\sigma \in \mathscr{S}_f$ for which $\sigma \subset T$ but σ has no extensions on T.

1.2 Definition. Let T be a tree, and let $\sigma \in \mathscr{S}_f$ be given. Then σ *is terminal on* T if $\sigma \subset T$ but for all $\tau \subset \mathscr{S}_f$, if $\tau \supset \sigma$ then $\tau \not\subset T$.

Every total f-tree is a partial f-tree. Hence all trees used in Part B of this book can still be used. In particular Id_f will be used, and we note again that it is a recursive tree.

The next three types of trees will be built as subtrees T^* of a given tree T. All the subtrees will have the following properties in common.

(1) If T is partial recursive, then T^* is partial recursive.

(2) An oracle of degree $\mathbf{0}'$ can determine whether or not T^* exists uniformly from an index for T as a partial recursive function, and if T^* exists, this oracle can find an index for T^* as a partial recursive function.

1.3 Definition. Let T be a tree and let $\sigma \in \mathcal{S}_f$ be given such that $T(\sigma)\downarrow$. Define the tree $\mathrm{PExt}_f(T, \sigma)$, the subtree of T extending $T(\sigma)$ by:

$$\mathrm{PExt}_f(T, \sigma)(\tau) = \begin{cases} T(\sigma * \tau) & \text{if} \quad T(\sigma * \tau)\downarrow \\ \uparrow & \text{otherwise.} \end{cases}$$

1.4 Remark. $\mathrm{PExt}_f(T, \sigma)$ is defined if and only if $T(\sigma)\downarrow$, a fact which can be determined by an oracle of degree $\mathbf{0}'$ uniformly from an index for T as a partial recursive function. Hence (1) and (2) hold. Furthermore, if $\mathrm{PExt}_f(T, \sigma)$ exists, then $\mathrm{PExt}_f(T, \sigma) \subseteq T$.

The construction of a minimal degree is equivalent to embedding the lattice with two elements $u_0 < u_1$ as an initial segment of $\mathcal{D}$. To this end, we define $\langle e, 1, 0\rangle$-differentiating trees to insure that the degree corresponding to u_1 is not recursive.

1.5 Definition. Let T be a tree, and let $e \in N$ be given. Define the tree T^* by

$$T^* = \begin{cases} \mathrm{PExt}_f(T, 0) & \text{if} \quad \exists x < \mathrm{lh}(T(0))(T(0)(x)\downarrow \neq \Phi_e(x)\downarrow) \\ \mathrm{PExt}_f(T, 1) & \text{otherwise.} \end{cases}$$

1.6 Remark. Let T^* be the subtree of T specified in Definition 1.5. Then T^* satisfies (1) and (2) and $T^* \subseteq T$. Furthermore, T^* is $\langle e, 1, 0\rangle$-differentiating.

1.7 Definition. Let $\mathrm{PDiff}_f(T, e)$ be the subtree T^* of T constructed in Definition 1.5.

The last type of tree which we consider is an e-splitting subtree. The property of being e-splitting, in isolation, is not sufficient. Rather, e-splitting subtrees must be maximal with respect to this property within the given trees.

1.8 Definition. Let T and T^* be binary trees such that $T^* \subseteq T$, and fix $e \in N$. We say that T^* is an *e-splitting subtree* of T if for all $\sigma, \eta \in \mathcal{S}_2$ such that $T^*(\sigma)\downarrow = T(\eta)$, either $T(\eta)$ is not terminal on T^* and $\langle T^*(\sigma * 0), T^*(\sigma * 1)\rangle$ is an e-splitting, or $T(\eta)$ is terminal on T^* and there are no e-splittings on $\mathrm{PExt}_2(T, \eta)$.

With the above definition of e-splitting subtree, the proof of the Computation Lemma (V.2.6) is easily modified to yield the following:

1.9 Computation Lemma. *Let T be a tree and let $e \in N$ be given. Let T^* be an e-splitting subtree of T. Then for all branches g of T^*, if Φ_e^g is total then $g \leqslant_T \Phi_e^g$. (Note* *that for every $\sigma \in \mathcal{S}_2$ such that $T^*(\sigma) \subset g$, there is an e-splitting of $T^*(\sigma)$ on T^*.) And for all $\sigma \in \mathcal{S}_2$ and $g \subset T$ such that $T^*(\sigma) \subset g$ and $T^*(\sigma)$ is terminal on T^*, if Φ_e^g is total, then Φ_e^g is recursive.*

We now construct e-splitting subtrees.

1.10 Lemma. *Let T be a binary tree and let $e \in N$ be given. Then there is an e-splitting subtree T^* of T which satisfies* (1) *and* (2).

Proof. We proceed by induction on $\mathrm{lh}(\sigma)$ for $\sigma \in \mathcal{S}_2$. We begin by setting $T^*(\emptyset) = T(\emptyset)$. Assume that $T^*(\sigma)$ has already been defined. Search for the least $\langle \sigma_0, \sigma_1, x \rangle \in \mathcal{S}_2^2 \times N$ (under some fixed recursive one-one correspondence of $\mathcal{S}_2^2 \times N$ with N) such that $\langle T(\sigma_0), T(\sigma_1) \rangle$ is an e-splitting of $T^*(\sigma)$ on x. For $j \leqslant 1$, define

$$T^*(\sigma * j) = \begin{cases} T(\sigma_j) & \text{if } \langle \sigma_0, \sigma_1, x \rangle \text{ exists} \\ \uparrow & \text{otherwise.} \end{cases}$$

We note that T^* has the desired properties. ▯

1.11 Definition. Let T be a binary tree and let $e \in N$ be given. Define $\mathrm{PSp}_2(T, e)$ to be the e-splitting subtree T^* of T constructed in Lemma 1.10.

The trees introduced in this section are put to use in the next section to construct a minimal degree below **0′**.

1.12 Remark. Partial recursive trees were first used by Sacks [1961] to construct a minimal degree below **0′**.

1.13–1.14 Exercises

1.13 Let T be a tree and let $e \in N$ be given. Show that there is a subtree $T^* = \mathrm{PTot}_f(T, e)$ of T which satisfies (1) and (2) and: (i) For every branch g of T^*, Φ_e^g is total. (ii) For every $\sigma \in \mathcal{S}_f$ such that $T(\sigma)$ is terminal on T^*, there is an $x \in N$ such that for all $\tau \in \mathcal{S}_f$ which extend σ, if $T(\tau)\downarrow$ then $\Phi_e^{T(\tau)}(x)\uparrow$.

1.14 Let T be a tree and let $e \in N$ and $B \subseteq N$ be given. Show that there is a subtree T^* of T which satisfies (1) and (2) and: (i) For all $\tau \in \mathcal{S}_f$, if τ is terminal on T^* then there is no $g \subset T$ such that $\tau \subset g$. (ii) For every branch g of T^*, if $\Phi_e^{g \oplus B}$ is total, then for some $x \in N$, $\Phi_e^{g \oplus B}(x) \neq g'(x)$. ($g'$ is $\{n\colon \Phi_n^g(n)\downarrow\}$.)

2. *Minimal Degrees Below* **0′**

We continue with local existence theorems for minimal degrees by showing that there is a minimal degree $\mathbf{a} < \mathbf{0}'$. We then show that if $\mathbf{c} < \mathbf{0}'$, then we can construct such a minimal degree $\mathbf{a}$ with $\mathbf{a} \not< \mathbf{c}$.

We will construct a minimal degree $< \mathbf{0}'$ through the use of a priority argument. Priorities are needed to compensate for the fact that we are allowed to ask questions only of an oracle of degree $\mathbf{0}'$, as opposed to the oracle of degree $\mathbf{0}^{(2)}$ which was available to us in the proof of Theorem V.2.11. The non-availability of an oracle of degree $\mathbf{0}^{(2)}$ will prevent us from determining whether a given tree T has a total e-splitting subtree. Thus we will first try to build our set A of minimal degree on an e-splitting subtree T^* of T, but if we reach some $\sigma \subset A$ such that σ is terminal on T^* (and so find that we cannot build A on T^*), then we will leave T^* and build A on a subtree $T^\#$ of T, all of whose branches extend σ. In the latter case, there will be no e-splittings on $T^\#$, so Φ_e^A will be recursive. Priorities are used to keep track of whether we are trying to build A on T^* or on $T^\#$, with T^* having higher priority than $T^\#$.

The change of trees mentioned above must be carried out without interfering with the construction of A recursively in $\emptyset'$. Thus we construct $A = \bigcup\{\alpha_s\colon s \in N\}$, where $\{\alpha_s\colon s \in N\}$ is a sequence of strings which is defined during the construction through the use of an oracle of degree $\mathbf{0}'$. Any new tree T which is specified during stage $t > s$ of the construction will have the property that α_s is on T; in fact, it will be the case that $\alpha_s \subset T(\emptyset)$. Thus A will be on all trees in the final sequence which is to be used to satisfy all requirements.

2.1 Theorem. *There is a minimal degree* $\mathbf{a} < \mathbf{0}'$.

Proof. By Theorem III.3.3, $\mathbf{0}'$ is not a minimal degree, so it suffices to construct $\mathbf{a} \leqslant \mathbf{0}'$. We will construct a set A of degree $\mathbf{a}$ which satisfies the following requirements:

P_e: $A \neq \Phi_e$.

Q_e: If Φ_e^A is total, then either Φ_e^A is recursive or $A \leqslant_T \Phi_e^A$.

We will use an oracle of degree $\mathbf{0}'$ to construct a sequence of strings $\{\alpha_s\colon s \in N\}$, a function $k\colon N \to N$, and an array $\{T_i^s\colon s \in N \,\&\, i \leqslant k(s)\}$ by induction on $\{s\colon s \in N\}$. The function k will be used to specify the last tree being used at stage s to satisfy a requirement. This tree, $T_{k(s)}^s$ will also be the first tree to change from the sequence of trees used at stage $s - 1$. The following induction hypotheses will be satisfied at the end of stage s.

(1) $\quad s \geqslant 1 \to \alpha_{s-1} \subset \alpha_s$.

(2) $\quad \alpha_s \subset T_{k(s)}^s$.

(3) $\quad \forall s \in N(T_0^s = \mathrm{Id}_2)$.

(4) $\quad \forall i < k(s)(T_{i+1}^s \subseteq T_i^s)$.

(5) $\quad s > 1 \to \forall i < k(s)(T_i^s = T_i^{s-1})$.

$A = \bigcup\{\alpha_s\colon s \in N\}$ will be the set of minimal degree.

The construction proceeds as follows:

Stage 0. Set $k(0) = 0$, $T_0^0 = \mathrm{Id}_2$, and $\alpha_0 = \emptyset$.

Stage $s + 1$. Let $r(s + 1)$ be the greatest $r \leqslant k(s)$ such that α_s is not terminal on T_r^s. (By (3), $T_0^s = \mathrm{Id}_2$, so $r(s + 1)$ must exist.) Define $k(s + 1) = r(s + 1) + 1$, and $T_i^{s+1} = T_i^s$ for all $i < k(s + 1)$. By (2) and (4), we can fix $\eta_s \in \mathscr{S}_2$ such that

$T^s_{r(s+1)}(\eta_s)=\alpha_s$. Let $T^{s+1}_* = \mathrm{PExt}_2(T^{s+1}_{r(s+1)}, \eta_s)$ and let $T^{s+1}_+ = \mathrm{PDiff}_2(T^{s+1}_*, r(s+1))$. Set $\alpha_{s+1} = T^{s+1}_+(\emptyset)$ and define

$$T^{s+1}_{k(s+1)} = \begin{cases} T^{s+1}_+ & \text{if } r(s+1) < k(s) \\ \mathrm{PSp}_2(T^{s+1}_+, r(s+1)) & \text{if } r(s+1) = k(s). \end{cases}$$

It is easily verified by choice of $r(s+1)$ that $T^{s+1}_{k(s+1)}$ is well-defined.

This completes the construction. We leave it to the reader to verify the induction hypotheses.

By (1), A is a subset of N. Furthermore, since the question "is β terminal on T?" can be answered by an oracle of degree $\mathbf{0}'$ uniformly in β and an index for the partial recursive tree T, it follows from Remarks 1.4 and 1.6 and Lemma 1.10 that A has degree $\leqslant \mathbf{0}'$. The following lemma is the heart of the proof of the theorem.

2.2 Lemma. *Let $e, s \in N$ be given such that $k(s) = e$ and $k(t) > e$ for all $t > s$. Then*

(i) $\quad \forall t \geqslant s(\alpha_t \subset T^s_e)$.

Furthermore, $k(s+1) = e+1$, and there is a $\beta \in \mathcal{S}_2$ such that

(ii) $\quad T^{s+1}_e(\beta) = \alpha_{s+1}, \quad T = \mathrm{PExt}_2(T^{s+1}_e, \beta), \quad$ *and* $\quad T^{s+1}_{e+1} = \mathrm{PSp}_2(T, e)$.

If there is a $u > s+1$ such that $k(u) = e+1$, then

(iii) $\quad \alpha_{u-1}$ *is terminal on* T^{s+1}_{e+1}

and

(iv) $\quad \forall t > u(k(t) > e+1)$.

Proof. (It will follow from this lemma that $\lim_s k(s) = \infty$.) Fix e and s as in the hypothesis of the lemma. Since $k(s) = e$ and $k(t) > e$ for all $t > s$, (2), (4) and (5) combine to show that α_t is on $T^t_e = T^s_e$ for all $t \geqslant s$, so (i) holds. By choice of s, $k(s+1) > k(s) = e$. But $k(t+1) \leqslant k(t) + 1$ for all $t \in N$. Hence $k(s+1) = e+1$. Since $k(s+1) > k(s)$, $r(s+1) = k(s)$ so (ii) follows from the definition of T^{s+1}_{e+1}. Suppose that for some $u > s+1$, $k(u) = e+1$. Fix the least such u. By choice of s and u, $k(t) > e+1$ for all $t \in N$ such that $s+1 < t < u$, so by the construction, α_{u-1} must be terminal on $T^{u-1}_{e+1} = T^{s+1}_{e+1}$. Hence (iii) holds. Furthermore, by the definition of T^u_{e+1} (note that $r(u) < k(u-1)$), $T^u_{e+1} = \mathrm{PExt}_2(T^u_e, \beta)$ where $T^u_e(\beta) = \alpha_u$. It thus follows from (1) and (5) that for all $t \geqslant u$, α_t is terminal on T^t_e if and only if α_t is terminal on T^t_{e+1}. By choice of s, $r(t) \geqslant e$ for all $t > s$, so for such t, $k(t) = r(t) + 1 > e+1$. Thus (iv) holds. ▯

We now claim that

(6) $\quad \forall e \in N \exists s \in N \forall t \geqslant s(k(t) \geqslant k(s) = e)$.

The claim is proved by induction on e and is clearly true for $e = 0$. Assume that (6) holds for $e = n$. Since $k(t+1) \leqslant k(t) + 1$ for all $t \in N$, there must be a greatest stage s such that $k(s) = n - 1$. Applying Lemma 2.2 for $n - 1$ in place of e, we see that there is a stage $u > s$ such that $k(u) = n$ and $k(t) > n$ for all $t > u$; hence $k(t) \geqslant n + 1$ for all $t > u$ and $k(u + 1) = n + 1$, completing the induction.

We next verify that A is not recursive by showing that A satisfies P_e for all $e \in N$. Fix $e \in N$. By (6), we can fix a stage s so that $k(s) = e + 1$ and for all $t > s$, $k(t) > e + 1$. By the construction, $\alpha_s \subset T^s_{e+1} \subset \mathrm{PDiff}_2(T^s_e, \beta)$ for some $\beta \in \mathscr{S}_2$. By (5) and the choice of s, $T^t_{e+1} = T^s_{e+1}$ for all $t \geqslant s$, hence A is a branch of $\mathrm{PDiff}_2(T^s_e, \beta)$. By Remark 1.6, $A \neq \Phi_e$.

We now complete the proof of the theorem by showing that Q_e is satisfied for all $e \in N$. Fix $e \in N$ and suppose that Φ^A_e is total. By (6), fix the stage s such that $k(s) = e$ and for all $t > s$, $k(t) > e$. By Lemma 2.2, $k(s + 1) = e + 1$. Furthermore, by Lemma 2.2(ii), $T^{s+1}_{e+1} = \mathrm{PSp}_2(T, e)$ where $T = \mathrm{PExt}_2(T^{s+1}_e, \beta)$ and $T^{s+1}_e(\beta) = \alpha_{s+1}$. If there is no $u > s + 1$ such that $k(u) = e + 1$, then by (5) and the choice of s, A is on T^{s+1}_{e+1}, so by the Computation Lemma, $A \leqslant_T \Phi^A_e$. If such a u exists, then it is unique by Lemma 2.2(iv). By Lemma 2.2(iii), α_{u-1} must then be terminal on $T^{s+1}_{e+1} = \mathrm{PSp}_2(T, e)$. Since $\alpha_{s+1} \supseteq T^{s+1}_e(\beta)$, $A \subset T$. Hence again by the Computation Lemma, we see that Φ^A_e is recursive. ▯

Sacks' [1961] original construction of a minimal degree below $\mathbf{0}'$ did not make use of a $\mathbf{0}'$ oracle. Rather, Sacks recursively approximated to A as a function lying on each tree in a path through a tree of trees. We outline Sacks' proof in Exercise 2.7.

The proof of Theorem 2.1 can be modified so that if a set C of degree $< \mathbf{0}'$ is given, then the set A of minimal degree which is constructed also satisfies $A \nleqslant_T C$. To accomplish this, we must satisfy the requirements $\{R_e \colon e \in N\}$ where

$$R_e\colon \Phi^C_e \neq A.$$

We cannot hope to satisfy R_e by fixing a witness x beforehand and forcing $\Phi^C_e(x) \neq A(x)$. For if $\Phi^C_e(x)\downarrow$, it may take a long time to discover this fact, and by the time we discover that $\Phi^C_e(x)\downarrow$, we may already have been forced to define $A(x)$ in order to insure that the degree of A is $\leqslant \mathbf{0}'$. Instead, we fix a set D of degree $\mathbf{0}'$, and whenever Φ^C_e and A agree on a large enough interval $[0, i]$, we try to code D into A by setting $T(\sigma * j) \subset A$ where $D(i) = j$, T is a tree fixed for e, and σ is chosen recursively in C. For this strategy to succeed, the sequence $\{\alpha_s \colon s \in N\}$ must be defined so that the function $h\colon N \to N$ defined by $h(s) = \mathrm{lh}(\alpha_s)$ is recursive in C. (We will, in fact, let h be the identity function.) Thus we will be using the *slowdown* procedure introduced in III.5.6, and we will appoint *targets* $\{\beta_s \colon s \in N\}$ such that α_s must be extended in the direction of β_s. We will then be able to argue that if $\Phi^C_e = A$, then $D \leqslant_T C \oplus A \equiv_T C$, contradicting the choice of C.

2.3 Theorem. *Let* $\mathbf{c} \in \mathbf{D}$ *be given such that* $\mathbf{c} < \mathbf{0}'$. *Then there is a minimal degree* $\mathbf{a} < \mathbf{0}'$ *such that* $\mathbf{a} \nleqslant \mathbf{c}$.

Proof. We indicate how to modify the proof of Theorem 2.1. Fix sets C and D of degree $\mathbf{c}$ and $\mathbf{0}'$ respectively. Let $S = \{\langle e, i\rangle \in N^2 \colon \Phi^C_e(j)\downarrow \text{ for all } j \leqslant i\}$. Since S is recursively enumerable in C, we can fix a one-one function f which is recursive in C and enumerates S. Without loss of generality, we may assume that f has the following property:

(7) $\quad \forall s, t, e, i, j \in N(f(s) = \langle e, i\rangle \,\&\, f(t) = \langle e, j\rangle \,\&\, i < j \to s < t).$

We will use an oracle of degree $\mathbf{0}'$ to construct sequences of strings $\{\alpha_s : s \in N\}$ and $\{\beta_s : s \in N\}$ (the latter being the targets for the former), a function $k: N \to N$, and an array of trees $\{T_i^s : s \in N \,\&\, i \leqslant k(s)\}$ by induction on $\{s : s \in N\}$. We will still have (1), (3), (4) and (5) as induction hypotheses, together with:

(8) $\quad \mathrm{lh}(\alpha_s) = s.$

(9) $\quad \alpha_s \subseteq \beta_s \subset T_{k(s)}^s.$

$A = \bigcup\{\alpha_s : s \in N\}$ will be the set of minimal degree.

The use of targets will introduce more cases into the construction. The choice of $k(s)$ will be determined as follows: We say that k *requires attention at stage* $s+1$ if there are $i \in N$ and $j \leqslant 1$ such that one of the following conditions holds:

(10) $\quad \alpha_s$ is terminal on T_k^s.

(11) $\quad k < k(s)$, $f(s) = \langle k-1, i\rangle$, $D(i) = j$, $\Phi_{k-1}^C \upharpoonright i+1 = \alpha_s \upharpoonright i+1$, and there is a $\tau \in \mathcal{S}_2$ such that $\mathrm{lh}(\tau) = s$ and $\alpha_s \subset T_k^s(\tau * j)$. (In this case, we want to define a new target in order to satisfy R_{k-1}.)

(12) $\quad k = k(s)$, $f(s) = \langle k-1, i\rangle$, $D(i) = j$, $\Phi_{k-1}^C \upharpoonright i+1 = \alpha_s \upharpoonright i+1$, and there is a $\tau \in \mathcal{S}_2$ such that $\mathrm{lh}(\tau) = s$ and $\beta_s \subset T_k^s(\tau * j)$. (In this case, we already have a target for the partial satisfaction of R_{k-1}, and we must continue with our attempt to code D into A in order to satisfy R_{k-1}. We therefore must extend the old target to a longer one.)

(13) $\quad k = k(s)$ and $\alpha_s \neq \beta_s$. (In this case, we still have a target which has not yet been reached, and we want to continue to head towards that target.)

(14) $\quad k = k(s) + 1$. (In this case, we want to begin to tackle a new requirement.)

The construction proceeds as follows:

Stage 0. Set $k(0) = 0$, $T_0^0 = \mathrm{Id}_2$, and $\alpha_0 = \beta_0 = \emptyset$.

Stage $s+1$. Fix the least k which requires attention at stage $s+1$. (By (14), such a k will exist.) Set $k(s+1) = k$ and $T_j^{s+1} = T_j^s$ for all $j < k(s+1)$. Adopt the appropriate case below, according to the first of (10)–(14) which is true for k.

Case 1. (10) or (14) holds: By (3), $T_0^s = \mathrm{Id}_2$, so $k(s+1) > 0$. By (4) and (9), we can fix $\eta_s \in \mathcal{S}_2$ such that $T_{k-1}^s(\eta_s) \supseteq \alpha_s$. Let $T_*^{s+1} = \mathrm{PExt}_2(T_{k-1}^{s+1}, \eta_s * 0)$. Let α_{s+1} be the string of length s such that $\alpha_{s+1} \subseteq T_*^{s+1}(\emptyset)$, and let $\beta_{s+1} = T_*^{s+1}(\emptyset)$. Define

$$T_k^{s+1} = \begin{cases} T_*^{s+1} & \text{if (10) holds} \\ \mathrm{PSp}_2(T_*^{s+1}, k-1) & \text{if (10) does not hold (so (14) holds).} \end{cases}$$

It is easily verified by choice of $k = k(s+1)$ that T_k^{s+1} is well-defined.

Case 2. (11) or (12) holds: Set $T_k^{s+1} = T_k^s$. Fix $i \in N$, $j \leqslant 1$ and $\tau \in \mathcal{S}_2$ as in (11) or (12). Let $\beta_{s+1} = T_k^s(\tau * j)$ and $\alpha_{s+1} = \beta_{s+1} \upharpoonright s+1$. By (8), α_{s+1} is well-defined. (Note that if (11) holds, then $k < k(s)$ and β_{s+1} may be incomparable with β_s. And if (12) holds, then $\beta_{s+1} \subset \beta_s$ and $k = k(s)$.)

Case 3. (13) holds: Set $T_k^{s+1} = T_k^s$ and $\beta_{s+1} = \beta_s$. Let $\alpha_{s+1} = \beta_{s+1} \upharpoonright s + 1$. By (8), α_{s+1} is well-defined.

This completes the construction. We leave it to the reader to verify the induction hypotheses. In addition, we note the following fact:

(15) If Case 2 or Case 3 is followed at stage $s + 1$, then $T_{k(s+1)}^{s+1} = T_{k(s+1)}^{s}$.

By (1), A is a subset of N. Furthermore, since f and D have degree $\leqslant \mathbf{0}'$ and since the question "is β terminal on T?" can be answered by an oracle of degree $\mathbf{0}'$ uniformly in β and an index for T as a partial recursive tree, it follows from Remark 1.4 and Lemma 1.10 that A has degree $\leqslant \mathbf{0}'$. The following lemma replaces Lemma 2.2.

2.4 Lemma. *Let $e, s \in N$ be given such that $k(s) = e$ and $k(t) > e$ for all $t > s$. Then*

(i) $\forall t \geqslant s(\beta_t \subset T_e^s)$.

Furthermore, $k(s + 1) = e + 1$ and there is a $\gamma \in \mathscr{S}_2$ such that

(ii) $T_e^{s+1}(\gamma) = \beta_{s+1}, \quad T = \mathrm{PExt}_2(T_e^{s+1}, \gamma), \quad$ *and* $\quad T_{e+1}^{s+1} = \mathrm{PSp}_2(T, e)$.

If there is a $u > s + 1$ such that $k(u) = e + 1$, then there is at most one such u at which Case 1 is followed, and for this u,

(iii) α_{u-1} *is terminal on* T_{e+1}^{s+1}.

We leave the proof of this lemma to the reader, since it is substantially the same as the proof of Lemma 2.2. (9) must be used in place of (2), (15) must sometimes be used together with (5), and $r(s + 1) = k(s + 1) - 1$ must be defined.

Claim (6) in the proof of Theorem 2.1 is replaced with the following lemma.

2.5 Lemma. $\forall e \in N \exists s \in N \forall t \geqslant s(k(t) \geqslant k(s) > e)$. *Furthermore, if* $e > 0$, *then* $\Phi_{e-1}^C \neq A$.

Proof. The lemma is proved by induction on e and is clearly true for $e = 0$. Assume that the lemma holds for $e = n - 1$. Since $k(t + 1) \leqslant k(t) + 1$ for all $t \in N$, it follows by induction that there must be a greatest stage s such that $k(s) = n - 1$. Applying Lemma 2.4 for $n - 1$ in place of e, we see that there is a stage $u > s$ such that $k(u) = n$ and for all $t > u$, if $k(t) = n$ then Case 2 or Case 3 is followed at stage t. Fix such a stage u.

Assume that $U = \{t \geqslant u\colon k(t) = n\}$ is infinite, for the sake of obtaining a contradiction. By (5) and (15), we note that $T_n^t = T_n^u$ for all $t \geqslant u$. If the complement of U is infinite, then there are infinitely many $t > u$ such that $t \in U$ and $t + 1 \notin U$. For such t, $k(t) > n$ and so, by (11), we must follow Case 2 at stage $t + 1$. And if U is cofinite, then there is a stage w such that for all $t \geqslant w$, $t \in U$. Case 3 can only be followed at finitely many successive stages of U (since $\beta_{t+1} = \beta_t$ if Case 3 is followed at stage $t + 1$). Hence in either case, since U is infinite, Case 2 must be followed at infinitely many stages $t \in U$. Hence $\Phi_{n-1}^C = A$. If $t \in U$ and $\alpha_t \neq \beta_t$, then by (12), there will be a least stage $t^* > t$ such that $\alpha_{t^*} = \beta_t$ and for all v such that $t \leqslant v \leqslant t^*$, $v \in U$ and Case 2 or Case 3 is followed at stage v. Thus for every $t \in U$, $\beta_t \subset A$. We obtain the desired contradiction by showing how to compute $D(x)$ for all sufficiently large x recursively from an $A \oplus C$ oracle: Since $A \oplus C \equiv_T C$, we conclude that $\mathbf{0}' \leqslant \mathbf{c}$. To compute $D(x)$, find a stage t such that $f(t) = \langle n-1, x\rangle$. Such a stage must exist since $\Phi^C_{n-1} = A$. (We neglect the finite number of integers x for which the corresponding t is $< u$.) Note that t is obtained from x through the use of a C oracle. We now search for $\tau \in \mathcal{S}_2$ such that $\mathrm{lh}(\tau) = t$ and $T^u_n(\tau * j) \subset A$ for some $j \in \{0, 1\}$. By (9) and the choice of u, such a τ must exist. Then $T^u_n(\tau * j) = \beta_t \subset A$ so $x \in D \leftrightarrow j = 1$. Note that τ and j are found through the use of an A oracle, so $D \leqslant_T A \oplus C \equiv_T C$, yielding the desired contradiction.

We conclude that $\{t: k(t) = n\}$ is finite. Hence the set of stages t at which (11) or (12) holds for $t-1$ in place of s and n in place of k is finite. There are three possibilities. Either $\{i \in N: \exists v \in N(f(v) = \langle n, i\rangle)\}$ is finite, in which case Φ^C_{n-1} is not total; or Φ^C_{n-1} is total and $\{i \in N: \Phi^C_{n-1} \upharpoonright i+1 = A \upharpoonright i+1\}$ is finite, in which case there is an $x \in N$ such that $\Phi^C_{n-1}(x)\downarrow \neq A(x)$; or there is an $m \in N$ such that for all $i \geqslant m$ and $v \in N$, if $f(v) = \langle n, i\rangle$ and $D(i) = j$ and $\Phi^C_{n-1} \upharpoonright i+1 = \alpha_v \upharpoonright i+1$, then there is no $\tau \in \mathcal{S}_2$ such that $\mathrm{lh}(\tau) = v$ and $\alpha_v \subset T^s_n(\tau * j)$. This last case is impossible, since by (8), $\mathrm{lh}(\alpha_v) = v$ and $\alpha_v \subset A \subset T^s_n$, so such τ and j must exist. We therefore conclude that $\Phi^C_{n-1} \neq A$. ▯

It follows from Lemma 2.5 that $A \nleqslant_T C$, so A is not recursive. The fact that A has minimal degree follows substantially as in the proof of Theorem 2.1. We leave the details to the reader. ▯

The constructions given in this section can be modified to prove similar results below degrees in $\mathbf{H}_1$. We discuss such generalizations in the next section.

2.6 Remarks. Theorem 2.1 was proved by Sacks [1961] using a recursive approximation construction instead of the $\mathbf{0}'$ oracle construction which we gave. The $\mathbf{0}'$ oracle construction is much simpler, and was introduced by Shoenfield [1966] where Theorem 2.3 was first proved. Our proof follows along the lines of Shoenfield [1971].

2.7–2.11 Exercises

2.7 Show that there is a minimal degree $\mathbf{a} < \mathbf{0}'$ such that $\mathbf{a} \notin \mathbf{GL}_1$.

2.8 Give a proof of Theorem 2.1 which does not make use of an oracle of degree $\mathbf{0}'$ in the construction. (*Hint*: Note that each tree T used in the proof of Theorem 2.1 can be expressed as the union of a recursive sequence of finite trees $\{T^s: s \in N\}$ and that this recursive sequence can be defined in a uniform way from an index for the tree. Construct $A = \lim_s \alpha_s$ and note that by the Limit Lemma, the degree of A is $\leqslant \mathbf{0}'$. We index trees by elements of $\mathcal{S}_2$, $T_0 = \mathrm{Id}_2$, $T_{\sigma*0} = \mathrm{PSp}_2(T_\sigma, \mathrm{lh}(\sigma))$, and whenever $\alpha_s = T_\sigma(\eta_s)$ is terminal on $T^{s+1}_{\sigma*0}$ but not on T^{s+1}_σ, we define $T_{\sigma*1} = \mathrm{PExt}_2(T_\sigma, \eta_s)$. Priorities of trees are determined by the lexicographical ordering of the strings indexing the trees. At stage $s+1$, we choose a path through this tree of trees by following the highest priority path which allows us to make progress towards the satisfaction of the requirements, in order. Thus we choose to follow $T_{\sigma*1}$ instead of $T_{\sigma*0}$ only if we are forced to do so by having chosen a string which is terminal on $T^{s+1}_{\sigma*0}$. We later modify the path when some α_t previously chosen which looked terminal on T^t_σ is no longer terminal on T^{s+1}_σ. α_{s+1} is chosen along this path. We now either use Posner's Lemma to satisfy P_e, or weave trees to satisfy these requirements into the construction. Posner's Lemma is discussed in the next exercise.)

2.9 (Posner's Lemma). Show that if we just satisfy the requirements $\{Q_e: e \in N\}$, then $\{P_e: e \in N\}$ will also be satisfied. (See the hint to Exercise V.2.15.)

2.10 Let D be a set of degree $\mathbf{0}'$, and let f be a function of degree $\leqslant \mathbf{0}'$ such that for all $n \in N$, $\Phi^D_{f(n)}$ is total and has degree $\mathbf{c_n}$. Show that $\{\mathbf{c_n}: n \in N\}$ has a minimal upper bound $\leqslant \mathbf{0}'$.

2.11 (Shore). Give an oracle proof of Theorem 2.3 which does not employ the slowdown technique. (*Hint*: Use partial narrow trees in the following way to satisfy R_e. We assume that T_0 is the final tree in the sequence used to satisfy all requirements of higher priority than R_e. We begin by letting $T^0_0 = \text{Nar}(T)$ and $T^0_1 = \text{Nar}(T^0_0)$, and building A on T^0_1 until (and if) at stage s_0, $\langle e, 0\rangle$ appears in $\text{rng}(f)$ and $\Phi^C_e \upharpoonright 1 = \alpha_{s_0-1} \upharpoonright 1$. When this happens, we say that R_e requires attention at stage s_0, and we define $\alpha_{s_0} \subset T_1$ where $T_1 = T_0 - T^0_0$ if $0 \in D$ and $T_1 = T^0_0 - T^0_1$ if $0 \notin D$. We continue this process with $\langle e, 1\rangle$ etc. Show that R_e requires attention at only finitely many stages, else $D \leqslant_T A \oplus C \equiv_T C$. Thus all requirements can be satisfied.)

3. *Minimal Degrees Below Degrees in* $\mathbf{GH_1}$

Let $\mathscr{C}$ be a class determined by the generalized high/low hierarchy. We place $\mathscr{C} \in \mathscr{M}$ if every degree in $\mathscr{C}$ bounds a minimal degree. We consider the question "is $\mathscr{C} \in \mathscr{M}$?" for various classes $\mathscr{C}$. If $\mathscr{C}$ lies near the bottom of the hierarchy, then $\mathscr{C} \notin \mathscr{M}$. For example, if $\mathscr{C} = \mathbf{GL_2} - \{\mathbf{0}\}$, then by Theorem VIII.1.9, there is a degree $\mathbf{c} \in \mathscr{C}$ such that $\mathscr{D}[\mathbf{0}, \mathbf{c}] \simeq Q$ (the ordering of the rational numbers in the interval $[0, 1]$), so $\mathbf{c}$ cannot bound a minimal degree. By Exercise VIII.1.13, $\mathbf{GL_2} - \mathbf{GL_1} \notin \mathscr{M}$. By IV.2.10(vi) and (ix), $\mathbf{GL_1} - \{\mathbf{0}\}$, $\mathbf{GL_3} - \mathbf{GL_2} \notin \mathscr{M}$. (The techniques introduced in Chaps. XI and XII will also show that $\mathbf{L_1} - \{\mathbf{0}\} \notin \mathscr{M}$.) The main result of this section is a positive result, namely, $\mathbf{GH_1} \in \mathscr{M}$. An extension of this result along the lines of Theorem 2.3 can be used to show that the minimal degrees $< \mathbf{0}'$ form an automorphism base for $\mathscr{D}[\mathbf{0}, \mathbf{0}']$.

The proofs of this section will combine the use of the Recursion Theorem with the proofs of Sect. 2. Suppose that $\mathbf{b} \in \mathbf{GH_1}$ is given, and fix a set B of degree $\mathbf{b}$. We will construct $\mathbf{A} = \cup\{\alpha_s: s \in S\}$ for some set $S \in \{N\} \cup \{[0, n]: n \in N\}$, using Φ^B_i to define A. Thus A will be obtained as $\Phi^B_{g(i)}$ for some recursive function g. By the Recursion Theorem, we will be able to assume that for some $i \in N$, $\Phi^B_i = \Phi^B_{g(i)}$, and the minimal degree will be constructed using Φ^B_i for such an i.

It will be necessary for us to specify trees through particular recursive approximations to the trees. Fix a recursive enumeration $\{Z_k: k \in N\}$ of all partial recursive binary trees. Thus for each $k \in N$, Z_k is a partial recursive binary tree, the relation $Z_k(\xi) = \sigma$ is a partial recursive relation on $N \times \mathscr{S}_2^2$, and for every partial recursive tree T, $T = Z_k$ for some $k \in N$. Furthermore, we assume that any algorithm for computing a partial recursive tree can be recursively identified with a tree Z_k which it computes.

The non-availability of an oracle of degree $\mathbf{0}'$ prevents us from determining, given $\sigma \in \mathscr{S}_2$ and a partial recursive tree T^*, whether σ is terminal on T^*. This question arises when we are building an e-splitting subtree T^* of T, and are trying to determine whether we can continue to build the set A of minimal degree on T^*. We thus replace this question with one which asks whether we will ever find a $\sigma \subset A$ such that $\sigma \subset T$ and there are no e-splittings of σ on T. This question can be answered by an oracle of degree $(\mathbf{b} \cup \mathbf{0}')' = \mathbf{b}'$ since $\mathbf{b} \in \mathbf{GH_1}$, so we can approximate to this answer recursively in B. This recursive approximation is given by what we call a *predictor function* and determines whether or not we will try to extend σ on T^*. If the predictor function later changes its mind, then we may have to begin a new attempt to build an e-splitting subtree of T. But since such changes of mind can occur only finitely often, we will eventually determine a tree which can be used to satisfy Q_e.

3.1 Definition. Given a partial recursive function h, define $h^* = \{\langle e, k\rangle: \exists \xi, \sigma \in \mathscr{S}_2(\sigma \subset h \,\&\, Z_k(\xi) = \sigma \,\&$ there is no e-splitting of σ on $Z_k)\}$.

3.2 Remark. $h^* \in \Sigma_1(h \oplus \emptyset')$ uniformly in h. Hence if $h = \Phi_e^B$ and $\mathbf{B} \in \mathbf{GH_1}$, then $\mathbf{h}^* \leqslant (\mathbf{b} \cup \mathbf{0}')' = \mathbf{b}'$. As B is fixed, an index for h^* as a set recursive in B' can be found as a recursive function of e. By the Limit Lemma, there is a function $h^+ : N^3 \to \{0, 1\}$ which is recursive in B such that for all $e, k \in N$, $h^*(e, k) = \lim_s h^*(e, k, s)$, and an index for $h^+ = \Phi_{g(e)}^B$ can be found uniformly from e.

We use Remark 3.2 to prove the following theorem.

3.3 Theorem. *Let* $\mathbf{b} \in \mathbf{GH_1}$ *be given. Then there is a minimal degree* $\mathbf{a}$ *such that* $\mathbf{a} \leqslant \mathbf{b}$.

Proof. Fix $\mathbf{b} \in \mathbf{GH_1}$ and let $B \in \mathbf{b}$ be given. Fix $e \in N$ and let $h = \Phi_e^B$. Let h^* and h^+ be as in Definition 3.1 and Remark 3.2 respectively.

We will use a B oracle to construct a sequence of strings $\{\alpha_s\colon s \in S\}$ where $S \in \{N\} \cup \{[0, n]\colon n \in N\}$, a function $k\colon S \to N$, and an array of trees $\{T_j^s\colon s \in S \,\&\, j \leqslant k(s)\}$ by induction on $\{s\colon s \in N\}$. The construction will produce a recursive function $g\colon N \to N$ such that $\Phi_{g(e)}^B = A_e = \bigcup\{\alpha_s\colon s \in S\}$. The recursion theorem will then provide us with an $e \in N$ such that $\Phi_e^B = \Phi_{g(e)}^B$ and for such an e, it will have to be the case that $S = N$. Thus for this e, A_e will be a set of minimal degree.

The subtree operations which will be used, taking e-splitting subtrees and extension subtrees, have the property that given an index for T as a partial recursive tree, we can recursively find an index for the subtree so defined. Hence without explicitly defining the function, we will assume during the construction that we have a function $t\colon \{\langle j, s\rangle\colon s \in S \,\&\, j \leqslant k(s)\} \to N$ which is partial recursive in B such that $t(j, s)$ provides an index for T_j^s as a partial recursive function.

We will also define an *indicator function*

$$i\colon \{\langle j, s\rangle\colon s \in S \,\&\, j < k(s)\} \to \{0, 1\}.$$

$i(j,s)$ will be 0 exactly when T^s_{j+1} is defined to be a splitting subtree of T^s_j. The function i is closely related to h^+ in that h^+ plays the role of a *predictor function*; $h^+(j,t(j,s),s)$ predicts whether or not to expect T_{j+1} to be a splitting subtree of T_j on which A_e is an infinite branch. The use of the recursion theorem will provide us with an $e \in N$ such that the predictor function and the indicator function agree for all sufficiently large s.

The following induction hypotheses will be satisfied at the end of stage s.

(1) $s \geqslant 1 \to \alpha_{s-1} \subset \alpha_s$.

(2) $\alpha_s \subset T^s_{k(s)}$.

(3) $T^s_0 = \mathrm{Id}_2$.

(4) $\forall j < k(s)(T^s_{j+1} \subseteq T^s_j)$.

(5) $s \geqslant 1 \to \forall j < k(s)(T^s_j = T^{s-1}_j \,\&\, i(j-1,s) = i(j-1,s-1))$.

(6) $\forall j < k(s)(i(j,s) = 0 \leftrightarrow \exists \eta \in \mathscr{S}_2(T^s_{j+1} \subseteq \mathrm{PSp}_2(\mathrm{PExt}_2(T^s_j,\eta),j)))$.

The construction proceeds as follows.

Stage 0. Set $k(0) = 0$, $T^0_0 = \mathrm{Id}_2$, and $\alpha_0 = \emptyset$.

Stage $s+1$. For all $j \leqslant k(s)$, let η^s_j be defined by $T^s_j(\eta^s_j) = \alpha_s$ and let $Y^s_j = \mathrm{PSp}_2(\mathrm{PExt}_2(T^s_j,\eta^s_j),j)$. By (2) and (4), η^s_j must exist for all $j \leqslant k(s)$. (Note that $Y^s_j = Z_m$ for some $m \in N$ and that such an m can be found recursively from $t(j,s)$.)

Use the B oracle to execute a search, uniformly in e, for $k \leqslant k(s)$ and $r > s$ such that

(7) α_s is not terminal on T^s_k;

(8) $\forall j < k(i(j,s) = h^+(j,t(j,s),r))$;

and either

(9) $h^+(k,t(k,s),r) = 0 \,\&\, \alpha_s$ is not terminal on $Y^s_k \,\&$ if $k < k(s)$ then $i(k,s) = 1$,

or

(10) $h^+(k,t(k,s),r) = 1 \,\&$ if $k < k(s)$ then $i(k,s) = 0$.

((8)–(10) require that the predictor function h^+ and the indicator function i agree on arguments $< k$ and either disagree on k or the indicator function is not defined on k. The clause of (9) which requires that α_s not be terminal on Y^s_k is there to rule out a blatant inaccuracy in the prediction made by h^+.) If no such k and r are found, then *the construction terminates at stage* $s+1$. Otherwise, let $\langle k,r\rangle$ be the first such pair found under some fixed recursive one-one correspondence of N with N^2. Let $k(s+1) = k+1$, let $T^{s+1}_j = T^s_j$ for all $j \leqslant k$, and let $i(j,s+1) = i(j,s)$ for all $j < k$. Let $T^{s+1}_* = \mathrm{PExt}_2(T^{s+1}_k,\eta^s_k)$,

$$T^{s+1}_{k(s+1)} = \begin{cases} \mathrm{PDiff}_2(Y^s_k,k) & \text{if (9) holds} \\ \mathrm{PDiff}_2(T^{s+1}_*,k) & \text{if (10) holds,} \end{cases}$$

and

$$i(k, s+1) = \begin{cases} 0 & \text{if (9) holds} \\ 1 & \text{if (10) holds.} \end{cases}$$

Set $\alpha_{s+1} = T^{s+1}_{k(s+1)}(\emptyset)$. We leave it to the reader to verify that (1)–(6) hold for $s+1$ in place of s if the construction does not terminate at stage $s+1$.

This completes the construction. Since the construction is recursive in B uniformly in e, it produces a recursive function g such that for all $e \in N$, $A_e = \Phi^B_{g(e)}$. Hence by the Recursion Theorem, there is an $e \in N$ such that $A_e = \Phi^B_{g(e)} = \Phi^B_e = h$. Fix such an e for the remainder of the proof, and let $A = A_e$. The following two lemmas allow us to conclude that the construction never terminates.

3.4 Lemma. *Let $s \in N$ be given such that the construction does not terminate at any stage $u \leqslant s$. Let $k \leqslant k(s)$ be given such that both $h^*(j, t(j,s)) = i(j,s)$ for all $j < k$ and if $k > 0$, then α_s is not terminal on T^s_{k-1}. Then α_s is not terminal on T^s_k. Furthermore, if $h^*(k, t(k,s)) = 0$ then α_s is not terminal on Y^s_k.*

Proof. If $k = 0$, then by (3), $T^s_0 = \mathrm{Id}^s_2$, so by (2) and (4), α_s is not terminal on T^s_k. Assume that $k > 0$. If $i(k-1, s) = 1$, then $T^s_k = \mathrm{PExt}_2(T^s_{k-1}, \eta)$ for some $\eta \in \mathscr{S}_2$. By (2) and (4), $\alpha_s \subset T^s_k \subseteq T^s_{k-1}$ so by Definition 1.3, α_s cannot be terminal on T^s_k. If $i(k-1, s) = 0$, then by hypothesis, $h^*(k-1, t(k-1, s)) = i(k-1, s) = 0$. Since $\alpha_s \subset T^s_{k-1}$, if follows from Definition 3.1 that there must be a $k-1$-splitting of α_s on T^s_{k-1}. Hence α_s cannot be terminal on T^s_k. If $h^*(k, t(k,s)) = 0$, then since $\alpha_s \subset T^s_k$, it follows from Definition 3.1 that there must be a k-splitting of α_s on T^s_k. Hence α_s cannot be terminal on Y^s_k. ▯

3.5 Lemma. *For all $s \in N$, the construction does not terminate at stage s.*

Proof. Note that the construction cannot terminate at stage 0. Given $s \in N$, fix the greatest $k \leqslant k(s)$ such that $h^*(j, t(j,s)) = i(j,s)$ for all $j < k$. Fix $r \in N$ such that for all $u \geqslant r$ and $j < k$, $h^*(j, t(j,s)) = h^+(j, t(j,s), u)$. By Lemma 3.4, (7) and (8) hold. Thus by choice of k and r, if $h^+(k, t(k,s), r) = 0$ then (9) holds and if $h^+(k, t(k,s), r) = 1$ then (10) holds. Hence the construction will not terminate at stage $s+1$. ▯

The next lemma is used to show that all requirements are satisfied.

3.6 Lemma. *For all $n \in N$ there is an $s \in N$ such that for all $u \geqslant s$, $k(u) > n$, $T^u_n = T^s_n$, and if $n > 0$ then $i(n-1, u) = h^*(n-1, t(n-1, s))$.*

Proof. By Lemma 3.5, the construction never terminates. Hence at any stage $s+1$, if k is given as in the construction then $k(s+1) > k \geqslant 0$. Thus the lemma follows for $n = 0$.

Fix $n > 0$ and assume by induction that the lemma holds for all $m < n$. Fix $r \in N$ so that for all $u \geqslant r$ and $j < n$, $T^u_j = T^r_j$, $h^+(j, t(j,u), u) = h^*(j, t(j,u))$, $k(u) > n-1$, and $i(j-1, u) = h^*(j-1, t(j-1, r))$ if $j > 0$. If there is no $u > r$ such that $k(u) = n$, then by (5), (8), and Lemma 3.5, the lemma is true for n. Hence we may assume that we have fixed $s > r$ such that $k(s) = n$. Since $k(s-1) \geqslant n$, the k of stage s is $k(s) - 1 = n - 1 < k(s-1)$. Hence by (9) and (10), $h^*(n-1, t(n-1, r)) \neq i(n-1, s-1)$. At stage s, we set $i(n-1, s) = h^*(n-1, t(n-1, r))$. By (5) and the choice of r, $i(n-1,u) = h^*(n-1, t(n-1,r)) = h^+(n-1, t(n-1,u), v)$ for all $u, v \geqslant s$, hence by (9) and (10), $k(u) > n$ for all $u > s$. By (5), $T_n^u = T_n^s$ and $i(n-1,u) = i(n-1,s)$ for all $u \geqslant s$. By (8) and since $k(u) > n$ for all $u > s$, we must have $i(n-1,u) = h^+(n-1, t(n-1,u), v) = h^*(n-1, t(n-1,u)) = h^*(n-1, t(n-1,s))$ for all $u, v > s$. ▯

We conclude from Lemma 3.6 that $T_n = \lim_s T_n^s$, $i(n) = \lim_s i(n,s)$, and $t(n) = \lim_s t(n,s)$ are well-defined for all $n \in N$.

The construction guarantees that $A \leqslant_T B$. We now show that A has minimal degree. Fix $n \in N$. Since A is a branch of $T_{n+1} = \mathrm{PDiff}_2(T, n)$ for some partial recursive tree T, we conclude from Remark 1.6 that $A \neq \Phi_n$ for all $n \in N$, so A is not recursive. First assume that $i(n) = 1$. By Lemma 3.6, $h^*(n, t(n)) = 1$. Since $h = A$, it follows from Definition 3.1 that there is a $\sigma \subset A$ such that $\sigma \subset T_n$ and σ has no n-splittings on T_n. By (2), (4) and Lemma 3.6, $A \subset T_n$. It thus follows from the Computation Lemma that Φ_n^A is recursive. Finally, assume that $i(n) = 0$. By Lemma 3.6, $h^*(n, t(n)) = 0$. Since $h = A$, it follows from Definition 3.1 that T_{n+1} n-splits along A, i.e., for every $\sigma \in \mathscr{S}_2$ such that $T_{n+1}(\sigma) \subset A$, $\langle T_{n+1}(\sigma * 0), T_{n+1}(\sigma * 1)\rangle$ form an n-splitting of $T_{n+1}(\sigma)$. Again by the Computation Lemma (Lemma 1.9), $A \leqslant_T \Phi_n^A$. Thus A is a set of minimal degree. ▯

By Corollary IV.3.6, if $\mathbf{a}$ is a minimal degree then $\mathbf{a} \in \mathbf{GL_2}$. By Theorem IV.1.10(ii), $\mathbf{GH_1} \cap \mathbf{GL_2} = \emptyset$. We can thus strengthen the conclusion of Theorem 3.3 as follows.

3.7 Corollary. *Let* $\mathbf{b} \in \mathbf{GH_1}$ *be given. Then there is a minimal degree* $\mathbf{a} < \mathbf{b}$.

Shore has noted that there is a version of Theorem 2.3 which can be proved below any degree in $\mathbf{GH_1}$. Thus we can construct the minimal degree $\mathbf{a}$ of Theorem 3.3 with $\mathbf{a} \nleqslant \mathbf{c}$ if $\mathbf{c}' \leqslant \mathbf{b}'$ and $\mathbf{c} \ngeqslant \mathbf{b}$. This result can then be used to show that the minimal degrees below $\mathbf{0}'$ form an automorphism base for $\mathscr{D}[\mathbf{0}, \mathbf{0}']$. This result was first proved by Posner. Posner, in fact, showed that the minimal degrees below $\mathbf{0}'$ generate $\mathscr{D}[\mathbf{0}, \mathbf{0}']$. These results are covered in the exercises.

3.8 Remarks. Theorem 3.3 was proved by Jockusch [1977]. Cooper [1973] had earlier proved a weaker result, showing that every $\mathbf{b} \in \mathbf{H}_1$ bounds a minimal degree. Cooper's methods were more difficult than those introduced by Jockusch.

3.9–3.11 Exercises

3.9 Let $\mathbf{b} \in \mathbf{GH_1}$ and $\mathbf{c} < \mathbf{b}$ be given. Show that there is a minimal degree $\mathbf{a} < \mathbf{b}$ such that $\mathbf{a} \not< \mathbf{c}$. (*Hint*: Combine the proofs of Theorems 2.3 and 3.3.)

3.10 Let $\mathbf{b} \in \mathbf{GH_1}$ and $\mathbf{c} \in \mathbf{D}$ be given such that $\mathbf{b} \nleqslant \mathbf{c}$ and $\mathbf{c}' \leqslant \mathbf{b}'$. Show that there is a minimal degree $\mathbf{a} < \mathbf{b}$ such that $\mathbf{a} \nleqslant \mathbf{c}$. (*Hint*: If $\mathbf{c} \nleqslant \mathbf{b}$, then we cannot determine whether $\Phi_e^C(x)\downarrow$ by appealing to the B oracle, where C is a set of degree $\mathbf{c}$. We use the fact that $\mathbf{c}' \leqslant \mathbf{b}'$ to get B-recursive approximations to both C and $W_e^C = \{x \in N: \forall y \leqslant x\, \exists \sigma \subset C(\Phi_e^\sigma(y)\downarrow)\}$, uniformly in e. We use these approximations to code B into A as in the proof of Theorem 2.3 (or alternatively, as in Exercise 2.11). This coding is performed when the approximation to C provides a suitable computation on index e for argument x, and the approximation to W_e^C tells us that $x \in W_e^C$. Thus the approximation to W_e^C allows us to code for e infinitely often only if Φ_e^C

is total. In this case, we show that $\Phi_e^C(x) \neq A(x)$ for some x else we can compute B from $A \oplus C \equiv_T C$.)

3.11 Use Exercise 3.10 to show that the minimal degrees form an automorphism base for $\mathscr{D}[\mathbf{0}, \mathbf{0}']$. (*Hint*: Use the fact that $\mathbf{H}_1$ is an automorphism base for $\mathscr{D}[\mathbf{0}, \mathbf{0}']$.)