Chapter VII
Finite Lattices

We completely characterize the finite ideals of $\mathscr{D}$ in this chapter as the set of all finite lattices. It is not known whether all finite lattices have finite homogeneous lattice tables, so we replace these tables with *weakly homogeneous sequential lattice tables* which are possessed by all finite lattices. We extend the methods of Chap. VI, using such tables to embed finite lattices as ideals of $\mathscr{D}$. This embedding theorem is used to locate decidable fragments of Th($\mathscr{D}$); the $\forall_2$-theory of $\mathscr{D}$ is decidable, but the $\forall_3$-theory of $\mathscr{D}$ is undecidable. Results from Appendices A.2 and B.2 are used in this chapter.

1. Weakly Homogeneous Sequential Lattice Tables

We define the tables needed to characterize the finite ideals of $\mathscr{D}$, motivating the definition by discussing the way in which the properties specified by the tables relate to the proofs of various lemmas in Chap. VI. Throughout this chapter, f will denote a non-decreasing recursive function such that $f(x) \geqslant 2$ for all $x \in N$. Recall that for such an f, $\mathscr{S}_f = \{\sigma \in \mathscr{S}\colon \forall x \in N(\sigma(x) < f(x))\}$. Our forcing conditions will be f-branching trees, i.e., trees $T\colon \mathscr{S}_f \to \mathscr{S}_f$, and will be referred to as trees, dropping the word *f-branching*.

The key lattice theoretic fact used in Chapter VI to embed finite distributive lattices as ideals of $\mathscr{D}$ was that all such lattices have finite homogeneous lattice tables. Since it is unknown whether all finite lattices have finite homogeneous lattice tables, we use a weaker type of table to embed all finite lattices as ideals of $\mathscr{D}$. All finite lattices have countable lattice tables, so we will have to find conditions to replace finiteness and homogeneity for the tables. Lemma VI.1.4, which asserts that the coding obtained from the table produces a usl homomorphism, will then have the same proof in this new setting.

It is crucial to the proof of a computation lemma that each tree T used in the construction have the property that for all strings $\sigma \in \operatorname{dom}(T)$, $S_\sigma = \{n \in N\colon T(\sigma * n)\downarrow\}$ is finite. This set S_σ however, must be generated from a lattice table which will not be finite, in order to always be able to find interpolants for the greatest lower bound preservation property. This apparent conflict is resolved by building the table as the union of an increasing sequence of finite usl tables, with the interpolants needed for a given member of the union appearing in the next member of the union. Specifically, we build a table $\Theta = \cup\{\Theta_i : i \in N\}$ such that for each $\alpha, \beta \in \Theta_i$ the interpolants $\gamma_0, \gamma_1, \ldots, \gamma_m$ of Definition VI.3.4 appear in Θ_{i+1}. This condition will be sufficient to give a new proof of Proposition VI.3.6 which asserts that whenever $u_i \wedge u_j = u_k$ and there is an e-splitting mod k on T (where T is sufficiently nice) then there is an e-splitting mod i or an e-splitting mod j on T. Thus for each tree T, we will be able to build an e-splitting subtree of T for some i or find a subtree $T^* \subseteq T$ such that for all $g \subset T^*$, Φ_e^g is not total.

The homogeneity of lattice tables is important for the construction of e-splitting trees, but it is not known whether all finite lattices have homogeneous lattice tables. We thus must replace homogeneity with the notion of *weak homogeneity*, which is a condition requiring the existence of an interpolant γ_1 between $\beta_0 = \gamma_0$ and $\beta_1 = \gamma_2$ in the definition of homogeneity (VI.3.8) and maps ψ_i for $i \leqslant 1$ replacing ψ with the same properties as ψ but applied to γ_i and γ_{i+1} instead of β_0 and β_1. As in the preceding paragraph, the interpolant for $\beta_0, \beta_1 \in \Theta_i$ must appear in Θ_{i+1}.

The above strategy, combined with a more delicate construction of e-splitting trees, enables us to construct the desired initial segments. We now present the properties which we require of our tables.

1.1 Definition. Let $\mathscr{L} = \langle L, \leqslant, \vee, \wedge \rangle$ be a finite lattice, and let $L = \{u_i : i \leqslant n\}$. Let $\{\Theta_i : i \in N\}$ be a sequence of sets of $n + 1$-tuples of integers. Then $\{\Theta_i : i \in N\}$ is a *sequential lattice table for* $\mathscr{L}$ if:

(i) $\forall i \in N(\Theta_i$ is a finite usl table for $\mathscr{L})$.

(ii) $\forall i \in N(\Theta_i \subseteq \Theta_{i+1})$.

(iii) $\forall i, j, k \leqslant n(u_i \wedge u_j = u_k \leftrightarrow \forall r \in N \forall \alpha, \beta \in \Theta_r(\alpha \equiv_k \beta \leftrightarrow$

$\exists \gamma_0, \ldots, \gamma_m \in \Theta_{r+1}(\alpha = \gamma_0 \equiv_i \gamma_1 \equiv_j \gamma_2 \equiv_i \cdots \equiv_j \gamma_m = \beta)))$.

Since we are working with usl tables, Lemma VI.1.4 will hold. We restate this lemma.

1.2 Lemma. *Let $\Theta \subseteq N^{n+1}$ be a recursive usl table for the usl $\langle L, \leqslant, \vee \rangle$ where $L = \{u_0, \ldots, u_n\}$. Let $g: N \to N$ be given such that for all $x \in N$, there is exactly one $\alpha_x \in \Theta$ for which $g(x) = \alpha_x^{[n]}$. For all $i \leqslant n$, define $g_i(x) = \alpha_x^{[i]}$. Let $G = \{g_i : i \leqslant n\}$ and $\mathbf{G} = \{\mathbf{g_i} : i \leqslant n\}$. Then the map $\psi: \langle L, \leqslant, \vee \rangle \to \langle \mathbf{G}, \leqslant, \cup \rangle$ defined by $\psi(u_i) = \mathbf{g_i}$ for all $i \leqslant n$* is a usl homomorphism.

As mentioned previously, we will require a *weak homogeneity* property of our tables.

1.3 Definition. Let $\{\Theta_i : i \in N\}$ be a sequential lattice table for the lattice $\mathscr{L} = \langle L, \leqslant, \vee, \wedge \rangle$ where $L = \{u_0, \ldots, u_n\}$. Then $\{\Theta_i : i \in N\}$ is *weakly homogeneous* if for all $r \in N$ and $\alpha_0, \alpha_1, \beta_0, \beta_2 \in \Theta_r$, if

(i) $\forall i \leqslant n(\alpha_0 \equiv_i \alpha_1 \to \beta_0 \equiv_i \beta_2)$

then there is a $\beta_1 \in \Theta_{r+1}$ and functions $\psi_s: \Theta_r \to \Theta_{r+1}$ for $s = 0, 1$ such that for all

$s \leqslant 1$ and $\alpha, \beta \in \Theta_r$, the following conditions hold:

(ii) $\psi_s(\alpha_0) = \beta_s \,\&\, \psi_s(\alpha_1) = \beta_{s+1}$.

(iii) $\forall i \leqslant n(\alpha \equiv_i \beta \rightarrow \psi_s(\alpha) \equiv_i \psi_s(\beta))$.

In Appendix B.2.9, we prove that every finite lattice has a weakly homogeneous sequential lattice table $\{\Theta_i : i \in N\}$ for which the set of pairs $\langle \alpha, i \rangle \in N^{n+1} \times N$ such that $\alpha \in \Theta_i$ is recursive, as is the function $f: N \rightarrow N$ where $f(i) = |\Theta_i|$ for all $i \in N$. We now fix a finite lattice $\mathscr{L} = \langle L, \leqslant, \vee, \wedge \rangle$ and such a table $\{\Theta_i : i \in N\}$ for $\mathscr{L}$ with the corresponding function f, and let $L = \{u_0, \ldots, u_n\}$. These remain fixed through the end of Sect. 3. We assume, without loss of generality, that for all $i \in N$, $\{\alpha^{[n]} : \alpha \in \Theta_i\} = [0, f(i))$.

1.4 Remark. Weakly homogeneous sequential representations were introduced by Lerman [1971].

2. *Uniform Trees*

All the trees needed for the constructions of this chapter, with the exception of e-splitting trees, are introduced in this section. Since the differences with Chap. VI.2 are slight, we leave most of the details to the reader. All trees are f-branching trees, where f was previously defined by $f(i) = |\Theta_i|$ for all $i \in N$. We begin with the identity tree.

2.1 Definition. The *identity* f-branching tree $\mathrm{Id}_f : \mathscr{S}_f \rightarrow \mathscr{S}_f$ is defined by $\mathrm{Id}_f(\sigma) = \sigma$ for all $\sigma \in \mathscr{S}_f$.

2.2 Remark. Id_f is a recursive uniform tree.

Although the trees extending σ are defined in the same way as before, they do not contain all branches of the original tree T which are compatible with σ. For since f is nondecreasing, f-branching trees tend to have a larger number of branches at higher levels than at lower levels. Thus extension trees must eliminate some branches. It is exactly this property which will allow us to prove the interpolation lemmas which will be needed.

2.3 Definition. Let T be a tree and let $\sigma \in \mathscr{S}_f$ be given. Define $\mathrm{Ext}_f(T, \sigma)$, the subtree of T *extending* $T(\sigma)$, by $\mathrm{Ext}_f(T, \sigma)(\tau) = T(\sigma * \tau)$ for all $\tau \in \mathscr{S}_f$.

2.4 Remark. $\mathrm{Ext}_f(T, \sigma) \subseteq T$ and if T is uniform then $\mathrm{Ext}_f(T, \sigma) \subseteq_u T$. Furthermore, for all $h: N \rightarrow N$, if T is recursive in h then $\mathrm{Ext}_f(T, \sigma)$ is recursive in h.

The construction of $\langle e, i, j \rangle$-differentiating subtrees whenever $u_i \nleqslant u_j$ remains virtually unchanged from the one given in Lemma VI.2.13. We merely replace p-branching trees with f-branching trees. We restate the relevant lemma, leaving its proof to the reader.

2.5 Lemma. *Let T be a uniform tree and let $e \in N$ and $i, j \leqslant n$ be given. Assume that $u_i \nleqslant u_j$. Then there is a tree $T^* \subseteq_u T$ such that T^* is $\langle e, i, j \rangle$-differentiating. Furthermore, for all $h\colon N \to N$, if T is recursive in h then T^* is recursive in h.*

2.6 Definition. Let $\mathrm{Diff}_f(T, e, i, j)$ be the $\langle e, i, j \rangle$-differentiating subtree of T defined in Lemma 2.5.

There are f-branching counterparts of other types of trees defined in earlier chapters. We leave the constructions of such trees to the reader.

2.7–2.8 Exercises

2.7 Let T be a uniform f-branching tree and let $e \in N$ be given. Assume that for all $\sigma \in \mathscr{S}_f$ and $x \in N$ there is a $\tau \in \mathscr{S}_f$ such that $\tau \supseteq \sigma$ and $\Phi_e^{T(\tau)}(x)\downarrow$. Show that T has a recursive uniform e-total subtree $\mathrm{Tot}_f(T, e)$.

2.8 Let T be a uniform f-branching tree and let $C \subseteq N$ and $h\colon N \to N$ be given such that T is recursive in h and h is recursive in C. Show that T has a C-pointed subtree $\mathrm{Pt}_f(T, C)$ which is recursive in C.

3. Splitting Trees

The trees introduced in the last section were constructed almost exactly as were their counterparts in Chap. VI. The splitting trees which are introduced in this section must be constructed more carefully than were their counterparts in Chap. VI. It is here that the failure of each Θ_i to be a homogeneous lattice table needs to be overcome.

Most of the definitions of Chap. VI carry over virtually unchanged to this section. Obvious changes need to be made, e.g., changing $\mathscr{S}_p$ to $\mathscr{S}_f$ and replacing bounds of p with bounds of $f(k)$ for some $k \in N$, usually $k = \mathrm{lh}(\sigma)$ for the properly chosen σ. We leave the reformulation of these definitions to the reader.

The Computation Lemma for f-branching trees is the counterpart to VI.3.3. Its proof is left to the reader, as it is a straightforward modification of the proof of the Computation Lemma for p-branching trees (VI.3.3).

3.1 Computation Lemma. *Let T be a uniform tree and let $e \in N$ and $i \leqslant n$ be given such that T is an e-splitting tree for i. Let $h\colon N \to N$ be given such that T is recursive in h. Then for every branch g of T, if Φ_e^g is total, then $\Phi_e^g \leqslant_T g_i \oplus h$ and $g_i \leqslant_T \Phi_e^g \oplus h$, where g_i is defined as in Lemma 1.2.*

The proof that, given a sufficiently well-behaved tree T and $e \in N$, we can find some $i \leqslant n$ such that T has a recursive uniform e-splitting subtree for i depended on the GLB Interpolation Lemma. This lemma was needed to show that if $u_i \wedge u_j = u_k$ and T is sufficiently well-behaved and has an e-splitting mod k, then T has an e-splitting mod i or an e-splitting mod j. We now prove these results for f-branching trees. It will be crucial to keep track of where the interpolants are.

3.2 GLB Interpolation Lemma. *Let $i, j, k \leqslant n$ and $\sigma, \tau, \rho \in \mathscr{S}_f$ be given such that $u_i \wedge u_j = u_k$, $\mathrm{lh}(\sigma) > 0$, $\mathrm{lh}(\tau) = \mathrm{lh}(\rho)$, and $\tau \equiv_k \rho$. Then there is a sequence*

$\tau = \tau_0, \tau_1, \ldots, \tau_m = \rho$ *such that for all* $p \leqslant m$, $\mathrm{lh}(\tau_p) = \mathrm{lh}(\tau_0)$ *and* $\sigma * \tau_p \in \mathscr{S}_f$, *and* $\tau_0 \equiv_i \tau_1 \equiv_j \tau_2 \equiv_i \cdots \equiv_j \tau_m$.

Proof. We proceed by induction on $\mathrm{lh}(\tau)$. The lemma is trivial for $\tau = \emptyset$. Assume that the lemma holds whenever $\mathrm{lh}(\tau) = s$. Let $i, j, k \leqslant n$ and $\sigma, \tau, \rho \in \mathscr{S}_f$ be given satisfying the hypothesis of the lemma with $\mathrm{lh}(\tau) = s + 1$. Fix $q, r < f(s)$ such that $\tau = \tau^- * q$ and $\rho = \rho^- * r$. By induction, there are interpolants $\tau^- = \rho_0, \rho_1, \ldots, \rho_v = \rho^-$, all of length s, such that $\rho_0 \equiv_i \rho_1 \equiv_j \rho_2 \equiv_i \cdots \equiv_j \rho_v$ and $\sigma * \rho_p \in \mathscr{S}_f$ for all $p \leqslant v$. By 1.1(iii), there are $q = q_0, \ldots, q_w = r$ such that for all $p \leqslant w$, $q_p < f(s + 1)$ and $q_0 \equiv_i q_1 \equiv_j q_2 \equiv_i \cdots \equiv_j q_w$. Since $\mathrm{lh}(\sigma) > 0$, the sequence $\rho_0 * q_0, \rho_1 * q_0, \ldots, \rho_v * q_0, \rho_v * q_1, \ldots, \rho_v * q_w$ has the desired properties. ▯

We note that the condition that $\mathrm{lh}(\sigma) > 0$ is necessary in the GLB Interpolation Lemma. For q_p need not be $< f(s)$, so the interpolants $\rho_t * q_r$ need not lie in $\mathscr{S}_f$. However, since $q_p < f(s + 1)$, $\sigma * \rho_t * q_r \in \mathscr{S}_f$ as long as $\mathrm{lh}(\sigma) > 0$.

The GLB Interpolation Lemma is used to prove the following important proposition. We note that its hypothesis differs from that of Proposition VI.3.6 in that we place some restrictions on the location of the e-splitting mod k.

3.3 Proposition. *Let T be a uniform tree, and let $e \in N$, $i, j, k \leqslant n$ and $\sigma \in \mathscr{S}_f$ be given such that $u_i \wedge u_j = u_k$ and* $\mathrm{lh}(\sigma) > 0$. *Assume that there is an e-splitting* mod k *on* $\mathrm{Ext}_f(T, \sigma) = T^*$, *and that*

$$\forall \sigma \in \mathscr{S}_f \forall x \in N \exists \tau \in \mathscr{S}_f (\Phi_e^{T^*(\tau)}(x)\downarrow).$$

Then either T has an e-splitting mod i *or T has an e-splitting* mod j.

Proof. Let $\langle T^*(\tau), T^*(\rho)\rangle$ be an e-splitting mod k on x. Without loss of generality, we may assume that $\mathrm{lh}(\tau) = \mathrm{lh}(\rho)$. By the GLB Interpolation Lemma, we can fix a sequence $\tau = \rho_0, \rho_1, \ldots, \rho_m = \rho$ of strings, all of the same length, such that for all $c < m$, $\sigma * \rho_c \in \mathscr{S}_f$ and there is a $d \in \{i, j\}$ for which $\rho_c \equiv_d \rho_{c+1}$. Define $\tau_0 = \sigma * \rho_0$, and assuming that $c < m$ and τ_c has been defined, let τ_{c+1} be the least ξ (under some fixed recursive one-one correspondence of $\mathscr{S}_f$ with N) such that $\xi \supseteq \mathrm{tr}(\sigma * \rho_c \to \sigma * \rho_{c+1}; \tau_c)$ and $\Phi_e^{T(\xi)}(x)\downarrow$. Then there is a least $c < m$ such that $\langle T(\tau_c), T(\tau_{c+1})\rangle$ e-splits on x. For some $d \in \{i, j\}$, $\tau_c \equiv_d \tau_{c+1}$. Since T is uniform, we have produced an e-splitting mod i or an e-splitting mod j on T. ▯

We note that we needed to have the e-splitting mod k in Proposition 3.3 lie on $\mathrm{Ext}_f(T, \sigma)$ for some $\sigma \in \mathscr{S}_f$ such that $\mathrm{lh}(\sigma) > 0$ in order to apply the GLB Interpolation Lemma. This restriction will cause no problems in building an e-splitting subtree of T for some $i \leqslant n$ because of the way that i is chosen.

Let T be a uniform tree, and let $e \in N$ and $i \leqslant n$ be given so that the following conditions hold:

(1) $\forall \sigma \in \mathscr{S}_f \forall j \leqslant n (u_j \ngeqslant u_i \to \exists \tau, \rho \in \mathscr{S}_f (\langle \mathrm{Ext}_f(T, \sigma)(\tau), \mathrm{Ext}_f(T, \sigma)(\rho)\rangle$ is an e-splitting mod j)).

(2) $\forall \sigma, \tau \in \mathscr{S}_f (\sigma \equiv_i \tau \to \langle T(\sigma), T(\tau)\rangle$ is not an e-splitting).

(3) $\forall \sigma \in \mathscr{S}_f \forall x \in N \exists \tau \subset \mathrm{Ext}_f(T, \sigma)(\Phi_e^{\tau}(x)\downarrow)$.

Under these circumstances, we will want to build an e-splitting subtree $T^* \subseteq_u T$ for i (see VI.3.2). The construction of such a tree proceeds level by level. At each level, we iterate a certain basic procedure which, when completed, will guarantee that $\langle T^*(\sigma), T^*(\tau)\rangle$ is an e-splitting for the particular σ and τ chosen with $\sigma \not\equiv_i \tau$. Conditions (1) and (3) allow us to carry out this procedure, as we show in the next lemma.

3.4 Lemma. *Let T be a uniform tree, and let $e \in N$ and $i \leqslant n$ be given so that* (1) *and* (3) *hold. Let $\sigma_0, \sigma_1 \in \mathscr{S}_f$ and $j \leqslant n$ be given such that* $\mathrm{lh}(\sigma_0) = \mathrm{lh}(\sigma_1)$, $\sigma_0 \equiv_j \sigma_1$, *and $u_j \not\geqslant u_i$. Then there are $\tau_0, \tau_1 \in \mathscr{S}_f$ such that* $\mathrm{lh}(\tau_0) = \mathrm{lh}(\tau_1)$, $\sigma_0 \subseteq \tau_0$, $\sigma_1 \subseteq \tau_1$, *and $\langle T(\tau_0), T(\tau_1)\rangle$ is an e-splitting mod j.*

Proof. Exactly as in Lemma VI.3.7. ▯

We continue the discussion preceding Lemma 3.4. Lemma 3.4 allows us to find an e-splitting mod j, $\langle T(\tau_0), T(\tau_1)\rangle$ which, if we were to extend the definition of T^* by specifying that $T(\tau_0) \subseteq T^*(\sigma)$ and $T(\tau_1) \subseteq T^*(\tau)$, would not cause outright damage to the uniformity of T^*. However, since the table which is being used need not be homogeneous, we cannot always extend the definition of T^* as above and still preserve its uniformity. Hence it is no longer sufficient to find an e-splitting mod j; we must find such an e-splitting which can be placed on T^* while allowing the extension of T^* to a uniform tree. If T satisfies (3), then we can convert our original e-splitting into an *extendible* one through the use of the interpolant provided by the weak homogeneity property of tables. The extendibility property which is needed is presented in the following definition.

3.5 Definition. Let $m \in N, p, q < f(m)$, and $\tau, \rho \in \mathscr{S}$ be given such that $\mathrm{lh}(\tau) = \mathrm{lh}(\rho)$ and for all $i \leqslant n$, if $p \equiv_i q$ then $\tau \equiv_i \rho$. We say that $\langle \tau, \rho\rangle$ *is m-extendible for* $\langle p, q\rangle$ if there is a map ψ: $[0, f(m)) \to \{\xi : \mathrm{lh}(\xi) = \mathrm{lh}(\tau) \,\&\, 0_{m+1} * \xi \in \mathscr{S}_f\}$ such that the following conditions hold:

(i) $\quad \psi(p) = \tau \quad$ and $\quad \psi(q) = \rho$.

(ii) $\quad \forall k \leqslant n \forall r, t < f(m)(r \equiv_k t \to \psi(r) \equiv_k \psi(t))$.

3.6 Remark. Let $m \in N$, $u, v < f(m)$ and $\tau_0, \tau_1, \tau_0 * \xi \in \mathscr{S}_f$ be given such that $\langle \tau_0, \tau_1\rangle$ is m-extendible for $\langle u, v\rangle$. Then $\langle \tau_0 * \xi, \tau_1 * \xi\rangle$ is m-extendible for $\langle u, v\rangle$. For if ψ is the original extension map, and we define $\psi^*(p) = \psi(p) * \xi$ for $p < f(m)$, then ψ^* witnesses that $\langle \tau_0 * \xi, \tau_1 * \xi\rangle$ is m-extendible for $\langle u, v\rangle$.

The next lemma is used to pass from the point in the construction of e-splitting trees where we have found an extendible e-splitting, to the definition of T^* on the part of the level on which we are working.

3.7 Extension Lemma. *Let $m \in N$ and $\{\sigma_q : q < f(m)\}$ be given such that for all $q, r < f(m)$, $\sigma_q \in \mathscr{S}_f$ and* $\mathrm{lh}(\sigma_q) = \mathrm{lh}(\sigma_r) > m$, *and for all $k \leqslant n$, if $q \equiv_k r$ then $\sigma_q \equiv_k \sigma_r$. Let $u, v < f(m)$ and $\tau_u, \tau_v \in \mathscr{S}_f$ be given such that* $\mathrm{lh}(\tau_u) = \mathrm{lh}(\tau_v)$ *and $\langle \tau_u, \tau_v\rangle$ is m-extendible for $\langle u, v\rangle$. Then there exist $\{\rho_q : q < f(m)\}$ such that for all $q, r < f(m)$, $\rho_q \in \mathscr{S}_f$,* $\mathrm{lh}(\rho_q) = \mathrm{lh}(\rho_r)$, $\rho_u = \sigma_u * \tau_u$, $\rho_v = \sigma_v * \tau_v$, $\sigma_q \subseteq \rho_q$, *and for all $k \leqslant n$, if $q \equiv_k r$ then $\rho_q \equiv_k \rho_r$.*

Proof. Let ψ: $[0, f(m)) \to \{\xi\colon \mathrm{lh}(\xi) = \mathrm{lh}(\tau_u) \,\&\, 0_{m+1} * \xi \in \mathscr{S}_f\}$ be the map whose existence is guaranteed by Definition 3.5. Let $\tau_q = \psi(q)$ and $\rho_q = \sigma_q * \tau_q$ for all $q < f(m)$. Since $\mathrm{lh}(\sigma_q) > m$ for all $q \leqslant f(m)$, $\rho_q \in \mathscr{S}_f$ for all $q < f(m)$. The lemma now follows from 3.5(i) and (ii). ▯

The remaining problem which we must solve is the problem of passing from an e-splitting mod j to one which is appropriately extendible. The next lemma will allow us to carry out that step.

3.8 Extendibility Interpolation Lemma. *Let $m \in N$, $u, v < f(m)$, and $\tau_0, \tau_2 \in \mathscr{S}_f$ be given such that* $\mathrm{lh}(\tau_0) = \mathrm{lh}(\tau_2)$ *and for all $k \leqslant n$, if $u \equiv_k v$ then $\tau_0 \equiv_k \tau_2$. Then there is a τ_1 such that $0_{m+1} * \tau_1 \in \mathscr{S}_f$,* $\mathrm{lh}(\tau_1) = \mathrm{lh}(\tau_0)$ *and for each $j \leqslant 1$, $\langle \tau_j, \tau_{j+1} \rangle$ is m-extendible for $\langle u, v \rangle$.*

Proof. We identify $p < f(m)$ with the unique $\alpha \in \Theta_m$ such that $\alpha^{[n]} = p$. Since $\{\Theta_k\colon k \in N\}$ is weakly homogeneous, for each $x < \mathrm{lh}(\tau_0)$ there is a $\tau_1(x) < f(m+1+x)$ and maps $\psi_{s,x}\colon \Theta_m \to \Theta_{m+1+x}$ for $s \leqslant 1$ such that for all $s \leqslant 1$ and $p, q < f(m)$, $\psi_{s,x}(u) = \tau_s(x)$, $\psi_{s,x}(v) = \tau_{s+1}(x)$, and for all $k \leqslant n$, if $p \equiv_k q$ then $\psi_{s,x}(p) \equiv_k \psi_{s,x}(q)$. τ_1 is now seen to have the desired properties. ▯

We can now convert arbitrary e-splittings mod j to e-splittings mod j which can be used to build uniform trees.

3.9 Lemma. *Let T be a uniform tree for which* (3) *holds. Let $\sigma \in \mathscr{S}_f$, $m \in N$ and $u, v < f(m)$ be given such that* $\mathrm{lh}(\sigma) = m$. *Let $j \leqslant n$ be chosen so that $u_j = \bigvee \{u_k\colon u \equiv_k v\}$. Assume that there is an e-splitting $\langle T^*(\tau_0), T^*(\tau_2) \rangle$* mod j *on x, where $T^* =$* $\mathrm{Ext}_f(T, \sigma * u)$. *Then there is an e-splitting $\langle T(\sigma * u * \xi), T(\sigma * v * \eta) \rangle$* mod j *on x such that $\langle \xi, \eta \rangle$ is m-extendible for $\langle u, v \rangle$.*

Proof. The proof can be followed in Fig. 3.1. By the Extendibility Interpolation Lemma, we can find τ_1 such that $0_{m+1} * \tau_1 \in \mathscr{S}_f$, $\mathrm{lh}(\tau_1) = \mathrm{lh}(\tau_0)$, and for all $s \leqslant 1$, $\langle \tau_s, \tau_{s+1} \rangle$ is m-extendible for $\langle u, v \rangle$. Since $\mathrm{lh}(\sigma * u) = m + 1$ and $\sigma * u \in \mathscr{S}_f$, $\sigma * u * \tau_1 \in \mathscr{S}_f$. Since (3) holds for T, there is a $\delta \in \mathscr{S}_f$ such that $\Phi_e^{T(\sigma * u * \tau_1 * \delta)}(x)\downarrow$. Hence for some $s \leqslant 1$ which we now fix, $\langle T(\sigma * u * \tau_s * \delta), T(\sigma * u * \tau_{s+1} * \delta) \rangle$ is

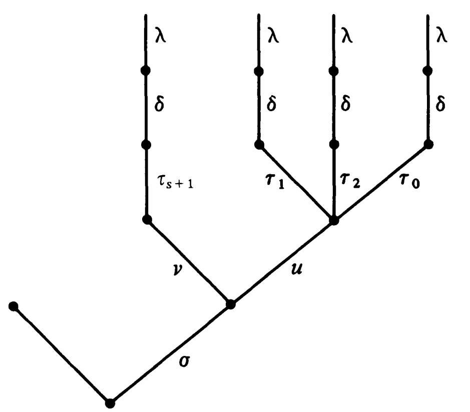

Fig. 3.1

an e-splitting mod j on x. Again since (3) holds for T, there is a $\lambda \in \mathcal{S}_f$ such that $\Phi_e^{T(\sigma * v * \tau_{s+1} * \delta * \lambda)}(x)\downarrow$. Hence either $\langle T(\sigma * u * \tau_s * \delta * \lambda),\ T(\sigma * v * \tau_{s+1} * \delta * \lambda)\rangle$ or $\langle T(\sigma * u * \tau_{s+1} * \delta * \lambda),\ T(\sigma * v * \tau_{s+1} * \delta * \lambda)\rangle$ is an e-splitting mod j on x, and by Remark 3.6, both $\langle \tau_s * \delta * \lambda, \tau_{s+1} * \delta * \lambda\rangle$ and $\langle \tau_{s+1} * \delta * \lambda, \tau_{s+1} * \delta * \lambda\rangle$ are m-extendible for $\langle u, v\rangle$. Hence the lemma will be satisfied for some choice of $\xi \in \{\tau_s * \delta * \lambda, \tau_{s+1} * \delta * \lambda\}$ and $\eta = \tau_{s+1} * \delta * \lambda$. ▯

We now have all the pieces needed to construct an e-splitting subtree for i. We construct such subtrees of uniform trees.

3.10 Lemma. *Let T be a uniform tree, and let $e \in N$ and $i \leqslant n$ be given so that* (1)–(3) *hold. Then there is a tree $T^* \subseteq_u T$ which is e-splitting for i. Furthermore, for all $h\colon N \to N$, if T is recursive in h then T^* is recursive in h.*

Proof. We proceed by induction on the levels of T^*. At stage s of the induction, we define $T^*(\sigma)$ for all $\sigma \in \mathcal{S}_f$ such that $\mathrm{lh}(\sigma) = s$. We begin, at stage 0, by setting $T^*(\emptyset) = T(\emptyset)$.

Stage $s+1$. Let $\{\langle \eta_m, q_m, r_m\rangle : m < m_0\}$ list all $\langle \eta, q, r\rangle \in \mathcal{S}_{f(s)} \times [0, f(s))^2$ such that $\mathrm{lh}(\eta) = s$ and $q \not\equiv_i r$. For $m < m_0$, let $\xi_m \in \mathcal{S}_{f(s)}$ be the string such that $T(\xi_m) = T^*(\eta_m)$. We perform a subinduction on $\{m : m \leqslant m_0\}$, defining, at step m, a sequence $\{\rho_j^m : j < f(s)\}$ such that $0_{s+1} * \rho_j^m \in \mathcal{S}_{f(s)}$ for all $j < f(s)$ and which satisfies:

(4) $\quad \mathrm{lh}(\rho_0^m) = \mathrm{lh}(\rho_1^m) = \cdots = \mathrm{lh}(\rho_{f(s)-1}^m).$

(5) $\quad \forall j < f(s)(m > 0 \to \rho_j^m \supseteq \rho_j^{m-1}).$

(6) $\quad \forall q, r < f(s)\, \forall k \leqslant n(q \equiv_k r \to \rho_q^m \equiv_k \rho_r^m).$

(7) $\quad m > 0 \to \langle T(\xi_{m-1} * \rho_{q_{m-1}}^m), T(\xi_{m-1} * \rho_{r_{m-1}}^m)\rangle$ is an e-splitting.

We begin, for $m = 0$, setting $\rho_j^0 = j$ for all $j < f(s)$.

Substage $m+1$. Let u_{k_m} be the greatest element of L such that $q_m \equiv_{k_m} r_m$. Such a k_m must exist, as is seen immediately from the Least Upper Bound Property for usl tables (Definition VI.1.2(iv)). By (1), there is an e-splitting mod k_m on $\mathrm{Ext}_f(T, \rho_{q_m}^m) \subseteq \mathrm{Ext}_f(T, \xi_m * q_m)$. Hence by Lemma 3.9, there are τ_{q_m}, τ_{r_m} such that $0_{s+1} * \tau_{q_m}, 0_{s+1} * \tau_{r_m} \in \mathcal{S}_f$, $\tau_{q_m} \supseteq \rho_{q_m}^m$, $\tau_{r_m} \supseteq \rho_{r_m}^m$, and $\langle T(\xi_m * \tau_{q_m}), T(\xi_m * \tau_{r_m})\rangle$ is an e-splitting mod k_m for which $\langle \tau_{q_m}, \tau_{r_m}\rangle$ is s-extendible for $\langle q_m, r_m\rangle$. By (4) and (6), the hypotheses for the Extension Lemma are satisfied, so the Extension Lemma produces $\{\rho_j^{m+1} : j < f(s)\}$ satisfying (4)–(6) with $m+1$ in place of m. Since $\rho_{q_m}^{m+1} = \tau_{q_m}$ and $\rho_{r_m}^{m+1} = \tau_{r_m}$, (7) is also satisfied.

Once the subinduction is completed, we define $T^*(\eta * q) = T(\xi * \rho_q^{m_0})$ for all $\eta \in \mathcal{S}_f$ such that $\mathrm{lh}(\eta) = s$ and all $q < f(s)$, where ξ is defined by $T(\xi) = T^*(\eta)$.

Since the subinduction satisfies (4)–(6) and by the definition of T^*, we note that T^* is a uniform tree. Since the subinduction satisfies (5) and (7), T^* is an e-splitting tree for i. (Note that this is the only place where (2) is used.) Furthermore, if T is recursive in h, then the construction of T^* can be carried out recursively in h. ▯

3.11 Definition. Let T be a uniform tree and let $e \in N$ and $i \leqslant n$ be given so that (1)–(3) are satisfied. Then Lemma 3.10 constructs $T^* \subseteq_u T$ such that T^* is an e-splitting tree for i. We give that tree T^* a name, $\mathrm{Sp}_f(T, e, i)$.

The final result of this section shows that if T is a uniform tree which satisfies (3) and $e \in N$, then there is some $i \leqslant n$ such that T has an e-splitting subtree for i.

3.12 Lemma. *Let T be a uniform tree for which* (3) *holds, and let $e \in N$ be given. Then there is an $i \leqslant n$ and a tree $T^* \subseteq_u T$ such that T^* is e-splitting for i. Furthermore, for all h: $N \to N$, if T is recursive in h then T^* is recursive in h.*

Proof. For each $\sigma \in \mathcal{S}_f$, let $\mathcal{U}_\sigma = \{u_k \in L$: there are no e-splittings mod k on $\mathrm{Ext}_f(T, \sigma)\}$. Note that if $\sigma \subseteq \tau$ then $\mathcal{U}_\sigma \subseteq \mathcal{U}_\tau$. Since L is finite, we can fix $\sigma \in \mathcal{S}_f$ such that $\mathcal{U}_\sigma = \mathcal{U}_\tau$ for all $\tau \supseteq \sigma$. Let $u_i = \bigwedge \mathcal{U}_\sigma$. By Proposition 3.3, $u_i \in \mathcal{U}_\sigma$, hence (1) and (2) hold for $\mathrm{Ext}_f(T, \sigma)$. Hence by Lemma 3.10, $\mathrm{Sp}_f(\mathrm{Ext}_f(T, \sigma), e, i)$ is the desired tree. ▯

3.13 Remark. The results of this section are due to Lerman [1971].

4. Finite Ideals of $\mathcal{D}$

We will now characterize the finite ideals of $\mathcal{D}$. This characterization will be used to locate natural decidable and undecidable classes of sentences of Th($\mathcal{D}$).

The development of trees used to force certain requirements in this chapter parallels the development of similar trees in Chap. VI; we replaced the p-branching trees of Chap. VI with f-branching trees, and proved counterparts of all the lemmas used in the proof of Theorem VI.4.2. Thus a proof similar to that given in Chap. VI will produce the following result.

4.1 Theorem. *Let $\mathcal{L}$ be a finite lattice. Then $\mathcal{L} \hookrightarrow^* \mathcal{D}$.*

Theorem 4.1 allows us to characterize the finite ideals of $\mathcal{D}$. For an ideal of $\mathcal{D}$ must be a usl with least element, and since every finite usl with least element is a lattice (the greatest lower bound of two elements is defined to be the least upper bound of all elements of the usl which are less that or equal to both of the given elements), all finite ideals of $\mathcal{D}$ are finite lattices. We thus conclude from Theorem 4.1 that:

4.2 Corollary. *The class of isomorphism types of finite ideals of $\mathcal{D}$ is the class of all finite lattices.*

Let $\mathcal{P} = \langle P, \leqslant \rangle$ be any finite poset, and let $\mathcal{P}^* = \langle P^*, \leqslant \rangle$ be a poset extending $\mathcal{P}$ such that $P^* = P \cup \{d\}$ where $d > p$ for all $p \in P$. If $\mathcal{P}$ is an initial segment of a usl, then $\mathcal{P}^*$ is a usl. Hence the finite initial segments of $\mathcal{D}$ are characterized as follows:

4.3 Corollary. *The class of isomorphism types of finite posets which are initial segments of $\mathcal{D}$ is exactly the class of posets which are initial segments of finite lattices.*

Other trees can be mixed into the construction of initial segments described in this chapter, thus producing initial segments of $\mathcal{D}$ of a given finite isomorphism type which possess various other properties. Some such properties are discussed in the exercises, as are relativizations of Theorem 4.1.

We now use Theorem 4.1 to identify decidable and undecidable classes of sentences of Th($\mathscr{D}$). We begin with a decidability result.

4.4 Theorem. *The* $\forall_2$*-theory of* $\mathscr{D}$ *is decidable.*

Proof. Let $\mathscr{L}$ be the language of the predicate calculus with one binary symbol, $\leqslant$. A *specification* is a conjunction of atomic formulas and negations of atomic formulas in this language. Let $\psi(z_0, \ldots, z_k)$ be a specification. Set $S_\psi = \{a_0, \ldots, a_k\}$ and define a binary relation $\leqslant$ on S_ψ by $a_i \leqslant a_j$ if $z_i \leqslant z_j$ is a conjunct of ψ, and $a_i \nleqslant a_j$ if $z_i \nleqslant z_j$ is a conjunct of ψ. The specification ψ is said to be *consistent* if there is a poset $\langle S_\psi, \leqslant^* \rangle$ such that the structure $\langle S_\psi, \leqslant \rangle$ is embeddable into $\langle S_\psi, \leqslant^* \rangle$, i.e., for all $a_0, a_1 \in S_\psi$, if $a_0 \leqslant a_1$ then $a_0 \leqslant^* a_1$ and if $a_0 \nleqslant a_1$ then $a_0 \nleqslant^* a_1$. A consistent specification is called a *partial diagram.* The specification ψ is said to be *complete* if for all $a_0, a_1 \in S_\psi$, either $a_0 \leqslant a_1$ or $a_0 \nleqslant a_1$. A consistent complete specification is called a *diagram.* The diagram ψ is said to be a *usl diagram* if every pair of elements of S_ψ has a least upper bound. The usl diagram $\psi(z_0, \ldots, z_k)$ is said to be *generated* by the diagram $\theta(z_0, \ldots, z_r)$ if $r \leqslant k$, $\langle S_\theta, \leqslant \rangle$ is embeddable into $\langle S_\psi, \leqslant \rangle$, and for all $i \in N$ such that $r < i \leqslant k$ there is a subset $I \subseteq [0, r]$ such that $a_i = \bigvee \{a_j : j \in I\}$ where the $\vee$ operation is defined in the usl $\langle S_\psi, \leqslant \rangle$.

The following facts are easily verified.

(1) Given variables $z_0, \ldots, z_n$, we can uniformly and effectively list all diagrams whose variables are contained in $\{z_0, \ldots, z_n\}$.

(2) Given a diagram $\psi(z_0, \ldots, z_n)$, we can uniformly and effectively list all usl diagrams $\theta(z_0, \ldots, z_{n+k})$ which are generated by ψ. (Each such usl diagram is a subusl of the free usl with $n + 1$ generators.)

(3) Given a partial diagram $\psi(z_0, \ldots, z_n)$, we can uniformly and effectively list all diagrams $\theta(z_0, \ldots, z_n)$ which extend ψ.

Fix an $\forall_2$-sentence $\sigma \equiv \forall\bar{x}\exists\bar{y}(\xi(\bar{x}, \bar{y}))$ of $\mathscr{L}$ where $\bar{x} = \langle x_0, \ldots, x_n \rangle$ and $\bar{y} = \langle y_0, \ldots, y_m \rangle$. Let $\{\psi_i : i \leqslant r\}$ be a list of all diagrams in variables among $\{x_0, \ldots, x_n\}$. Then σ is true if and only if the following sentence is true:

$$\bigwedge_{i=0}^{r} (\forall\bar{x}\exists\bar{y}(\psi_i(\bar{x}) \to \xi(\bar{x}, \bar{y}))).$$

Hence by (1), it suffices to decide the truth of sentences of the form $\forall\bar{x}\exists\bar{y}(\psi(\bar{x}) \to \xi(\bar{x}, \bar{y}))$ where $\psi = \psi_i$ for some $i \leqslant r$. Fix such a sentence, σ_1.

Let $\{\theta_i : i \leqslant s\}$ be a list of all usl diagrams which are generated by ψ, and let θ_i be such a diagram whose variables lie in the set $\{x_0, \ldots, x_n, z_0, \ldots, z_k\}$. Then σ_1 is true if and only if the following sentence is true:

$$\bigwedge_{i=0}^{s} (\forall\bar{x}\forall\bar{z}\exists\bar{y}(\theta_i(\bar{x}, \bar{z}) \to \xi(\bar{x}, \bar{y})))$$

where $\bar{z} = \langle z_0, \ldots, z_k \rangle$. Hence by (2), is suffices to decide the truth of all sentences of the form $\forall\bar{x}\forall\bar{z}\exists\bar{y}(\theta(\bar{x}, \bar{z}) \to \xi(\bar{x}, \bar{y}))$ where $\theta = \theta_i$ for some $i \leqslant s$. Fix such a sentence, σ_2.

Write $\xi(\bar{x}, \bar{y})$ in disjunctive normal form. Then

$$\xi(\bar{x}, \bar{y}) \equiv \bigvee_{i=0}^{t} \xi_i(\bar{x}, \bar{y})$$

where each ξ_i is a specification. Since we can uniformly and effectively decide, given ξ_i, whether or not ξ_i is consistent, we may assume without loss of generality that each ξ_i is consistent. Let $\{\eta_i\colon i \leqslant u\}$ be a list of all diagrams in the variables $\{x_0, \ldots, x_n, y_0, \ldots, y_m, z_0, \ldots, z_k\}$ which extend ξ_i for some $i \leqslant u$. Then σ_2 is true if and only if the following sentence is true:

$$\forall \bar{x}\, \forall \bar{z}\, \exists \bar{y}(\theta(\bar{x}, \bar{z}) \to \bigvee_{i=0}^{u} \eta_i(\bar{x}, \bar{y}, \bar{z}))).$$

Hence by (3) and (1), it suffices to decide the truth over $\mathscr{D}$ of all sentences of the form

$$\forall \bar{x}\, \exists \bar{y}(\theta(\bar{x}) \to \bigvee_{i=0}^{u} (\eta_i(\bar{x}, \bar{y}))$$

where θ is a usl diagram and each η_i is a diagram extending θ. Fix such a sentence, σ^*.

We now digress to consider a related question. Suppose that we are given finite posets $\mathscr{P} = \langle P, \leqslant_P \rangle \subseteq \langle M, \leqslant_M \rangle = \mathscr{M}$ and an isomorphic copy $\mathscr{T}$ of $\mathscr{P}$ which is a subposet of $\mathscr{D}$, and suppose that $\leqslant_P$ defines usl structure on P. We ask when it is possible to extend $\mathscr{T}$ to $\mathscr{V} = \langle \mathbf{V}, \leqslant_V \rangle \subseteq \mathscr{D}$ so that the following diagram commutes:

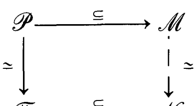

Fig. 4.1

Since every usl can be extended to a usl with least element by adjoining such a least element if it is not present, it follows from Theorem 4.1 that the following condition is necessary:

(4) $\forall a \in M - P\, \forall b \in P(a \nleqslant_M b)$.

Since $\mathscr{P}$ has usl structure, the following condition is also necessary:

(5) $\forall m \in M\, \forall p, q \in P(p \leqslant_M m \;\&\; q \leqslant_M m \to p \vee_P q \leqslant_M m)$.

By Theorem II.4.11, conditions (4) and (5) are also sufficient conditions for the existence of $\mathscr{V}$ such that Fig. 4.1 commutes.

We now conclude that the sentence σ^* is true if and only if there is an $i \leqslant u$ such that (4) and (5) hold for the diagram θ replacing $\mathscr{P}$ and the diagram η_i replacing $\mathscr{M}$. ▯

When we pass from the $\forall_2$-theory of $\mathscr{D}$ to the $\forall_3$-theory of $\mathscr{D}$, we pass from decidability to undecidability. This fact is a corollary of the following result which is proved in Appendix A.2.9.

4.5 Theorem. *The $\exists_2$-theory of finite lattices in the language of the predicate calculus with just one binary relation symbol is strongly undecidable. (A set Σ of sentences is strongly undecidable if there is no recursive set R of sentences such that $V \cap \Sigma \subseteq R \subseteq \Sigma$, where V is the set of all logically valid sentences, a non-recursive set.)*

4.6 Corollary. *The $\forall_3$-theory of $\mathscr{D}$ is undecidable.*

Proof. Let σ be an $\exists_2$ sentence in the language of the predicate calculus with just one binary relation symbol, $\leqslant$, adjoined. Let $\psi(x)$ be the formula which asserts that the elements $\leqslant x$ form a lattice, and let σ_1 be the formula which restricts all variables appearing in σ to elements $\leqslant x$. Let θ_σ be the sentence

$$\forall x(\psi(x) \rightarrow \sigma_1(x)).$$

Let Σ be the set of all $\exists_2$ sentences of our language which are true for all finite lattices, and let $R = \{\sigma : \theta_\sigma \text{ is true in } \mathscr{D}\}$. By Theorem 4.1, $V \cap \Sigma \subseteq R \subseteq \Sigma$, so R is not recursive. Hence the set of sentences $\{\theta_\sigma : \sigma \text{ is an } \exists_2 \text{ sentence of our language}\}$ is a recursive class of $\forall_3$ sentences of our language whose truth in $\mathscr{D}$ is not uniformly decidable. ▯

4.7 Remarks. Theorem 4.1, Corollary 4.2 and Corollary 4.3 were proved by Lerman [1971]. Theorem 4.4 is due independently to Shore [1978] and Lerman. Corollary 4.6 was proved by Schmerl.

4.8–4.17 Exercises

4.8 Let $\mathbf{a} \in \mathbf{D}$ be given, and let $\mathscr{L}$ be a finite lattice. Show that $\mathscr{L} \hookrightarrow^* \mathscr{D}[\mathbf{a}, \infty)$.

4.9 Let $\mathscr{L}$ be a finite lattice. Show that there is an isomorphism $\psi : \mathscr{L} \hookrightarrow^* \mathscr{D}$ such that for all $a \in L$, $\psi(a)^{(2)} = \mathbf{0}^{(2)}$ and $\psi(a) \notin \mathbf{GL}_1$.

4.10 Let $\mathscr{L}$ be a finite lattice. Show that there are $2^{\aleph_0}$ distinct embeddings $\mathscr{L} \hookrightarrow^* \mathscr{D}$.

4.11 Let $\mathscr{L}$ be a finite lattice, and let $\mathbf{d} \in \mathbf{D}$ be given. Show that there is an $\mathscr{L}$-cover $\mathbf{a}$ of $\mathbf{0}$ such that $\mathbf{a}^{(2)} = \mathbf{a} \cup \mathbf{0}^{(2)} = \mathbf{d} \cup \mathbf{0}^{(2)}$.

4.12 Let $\mathscr{L}$ be a finite lattice and let $\mathbf{c} \in \mathbf{D}$ be given. Show that there is an $\mathscr{L}$-cover $\mathbf{a}$ of $\mathbf{c}$ such that $\mathbf{a}^{(2)} = \mathbf{c}^{(2)}$ and $\mathbf{a} \notin \mathbf{GL}_1(\mathbf{c})$.

4.13 Let $\mathscr{L}$ be a finite lattice and let $\mathbf{d} \in \mathbf{D}$ be given. Show that $\mathbf{d}$ has $2^{\aleph_0}$ distinct $\mathscr{L}$-covers.

4.14 Let $\mathscr{L}$ be a finite lattice, and let $\mathbf{c}, \mathbf{d} \in \mathbf{D}$ be given. Show that there is an $\mathscr{L}$-cover $\mathbf{a}$ of $\mathbf{c}$ such that $\mathbf{a}^{(2)} = \mathbf{a} \cup \mathbf{c}^{(2)} = \mathbf{d} \cup \mathbf{c}^{(2)}$.

4.15 Let $\mathbf{d} \in \mathbf{D}$ be given. Show that $\forall_2 \cap \mathrm{Th}(\mathscr{D}[\mathbf{d}, \infty))$ is decidable but that $\forall_3 \cap \mathrm{Th}(\mathscr{D}[\mathbf{d}, \infty))$ is undecidable.

4.16 Let $\mathscr{L}$ be a finite lattice, and let **I** be a countable ideal of $\mathscr{D}$. Show that **I** has $2^{\aleph_0}$ distinct $\mathscr{L}$-covers.

4.17 Let **I** be a countable ideal of $\mathscr{D}$. Show that $\forall_2 \cap \mathrm{Th}(\mathscr{D}[\mathbf{I}, \infty))$ is decidable, but that $\forall_3 \cap \mathrm{Th}(\mathscr{D}[\mathbf{I}, \infty))$ is undecidable.