# *Chapter VI*
# Finite Distributive Lattices

We continue our study of the finite ideals of $\mathscr{D}$ in this chapter by showing that every finite distributive lattice is isomorphic to an ideal of $\mathscr{D}$. This result is proved using techniques extending those introduced in Chap. V. Different trees are used, and we introduce *tables* which provide reduction procedures from the top degree of the ideal; these tables are obtained from representations of distributive lattices. As an application, we show that the set of minimal degrees forms an automorphism base for $\mathscr{D}$.

Many of the applications which we obtain in later chapters from the complete characterization of the countable ideals of $\mathscr{D}$ can be obtained from the fact that all countable distributive lattices are isomorphic to ideals of $\mathscr{D}$. We use Exercise 4.17 of this chapter to indicate how to obtain the characterization of distributive ideals of $\mathscr{D}$. This exercise allows the reader to proceed directly to Chap. VIII.2 from the end of this chapter.

The results of Appendix B.1 are needed for this chapter.

## *1. Usl Representations*

Tables built from lattice representations provide the starting point for defining the trees used in this chapter. We begin to motivate the use of such tables in this section. Recall that $\mathscr{S}_p$ is the set of all strings of integers $< p$.

The trees used in this chapter are $p$-branching trees, i.e. trees $T: \mathscr{S}_p \to \mathscr{S}_p$. We will refer to $p$-branching trees as *trees* during this section, dropping the words *p-branching*.

Let $\mathscr{L} = \langle L, \leqslant, \vee, \wedge \rangle$ be a finite lattice, with $L = \{u_0, \ldots, u_n\}$. We assume, without loss of generality, that for all $i, j \leqslant n$, if $i < j$ then $u_j \nleqslant u_i$ and $u_0$ and $u_n$ are, respectively, the least and greatest elements of $L$. We wish to construct a function $g_n: N \to N$ such that $\mathbf{D_L} = \langle \mathbf{D[0, g_n]}, \leqslant \rangle$ is a lattice which is isomorphic to $\mathscr{L}$ under the map $\psi: L \to \mathbf{D_L}$ given by $\psi(u_i) = \mathbf{g_i}$ for all $i \leqslant n$, where $\mathbf{D[0, g_n]} = \{\mathbf{g_0}, \ldots, \mathbf{g_n}\}$. As in the construction of a minimal degree, we will define a sequence of trees $\{T_i: i \in N\}$ and choose $g_n \in \bigcap\{T_i: i \in N\}$.

For all $i \leqslant n$ and $z \in N$, we can view $g_i \restriction z$ as a string. Thus, for example, if $\sigma = 2 * 0 * 3$, we write $\sigma \subset g_i$ if $g_i(0) = 2$, $g_i(1) = 0$ and $g_i(2) = 3$. We will concentrate on the construction of $g_n$, and will need a procedure by which to recover each $g_i$ from $g_n$. The procedure which we use is derived from *tables*. Figure 1.1 pictures a lattice together with a table for this lattice. We refer to this table in subsequent remarks.


| | $g_0$ | $g_1$ | $g_2$ |
|---|---|---|---|
| $\alpha_0$ | 0 | 0 | 0 |
| $\alpha_1$ | 0 | 0 | 1 |
| $\alpha_2$ | 0 | 1 | 2 |
| $\alpha_3$ | 0 | 1 | 3 |

Fig. 1.1

Suppose that we have specified that $\sigma = 2 * 0 * 3 \subset g_2$. We begin a partial diagram with columns for $g_0$, $g_1$ and $g_2$, list $\sigma$ in the $g_2$-*column*, and place dashes in all other columns for all rows which have an entry (see the left-hand side of Fig. 1.2). We now try to fill in the dashes so that each row of the diagram corresponds to a row of the table (see the right-hand side of Fig. 1.2). Hence for the first row, the entry in the $g_2$-*column* is 2, so we use $\alpha_2$ to fill in the row. The $g_0$-*column* now specifies that $0 * 0 * 0 \subset g_0$, and the $g_1$-*column* specifies that $1 * 0 * 1 \subset g_1$.

| $g_0$ | $g_1$ | $g_2$ |
|---|---|---|
| — | — | 2 |
| — | — | 0 |
| — | — | 3 |

→

| $g_0$ | $g_1$ | $g_2$ |
|---|---|---|
| 0 | 1 | 2 |
| 0 | 0 | 0 |
| 0 | 1 | 3 |

Fig. 1.2

Since $u_0 < u_1$, $g_1$ must uniquely determine $g_0$. The procedure used for this determination is similar to the determination of $g_0$ and $g_1$ from $g_2$. As an example, suppose that we have specified that $\sigma = 1 * 0 \subset g_1$. We begin with the left-hand

| $g_0$ | $g_1$ | $g_2$ |
|---|---|---|
| — | 1 | — |
| — | 0 | — |

→

| $g_0$ | $g_1$ | $g_2$ |
|---|---|---|
| 0 | 1 | 2 |
| 0 | 0 | 0 |

| $g_0$ | $g_1$ | $g_2$ |
|---|---|---|
| 0 | 1 | 2 |
| 0 | 0 | 1 |

| $g_0$ | $g_1$ | $g_2$ |
|---|---|---|
| 0 | 1 | 3 |
| 0 | 0 | 0 |

| $g_0$ | $g_1$ | $g_2$ |
|---|---|---|
| 0 | 1 | 3 |
| 0 | 0 | 1 |

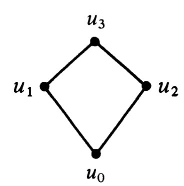
Fig. 1.3

diagram of Fig. 1.3 with dashes in the unknown places, but note that the procedure for filling in the diagram with tuples from the table is not unique; there are four possibilities, all listed in Fig. 1.3. However, all the possibilities have $0 * 0$ in the $g_0$-*column*. Thus if $1 * 0 \subset g_1$, then $0 * 0 \subset g_0$. This procedure can be used to have $g_1$ uniquely determine $g_0$. We note that the procedure can be reduced to a procedure which handles each row of the diagram separately.

Since $u_0$ is the smallest element of $L$, we will want $g_0$ to be recursive. We achieve this by requiring, for all tables, that *all entries in the* $g_0$*-column are the same.* Similarly, $g_n$ will be the greatest element of $L$, so should uniquely specify each $g_i$. We thus require, for all tables, that *all entries in the* $g_n$*-column are different.* And in order to guarantee that the recovery procedure described above will work, we require of all tables, that *if* $u_i \leqslant u_j$ *and coordinate j of a tuple in the table is specified as having the value m, then any two tuples in the table having the value m in coordinate j must have the same value in coordinate i.*

It is possible to satisfy the three conditions mentioned above with a table consisting of only one tuple. However, referring to the lattice of Fig. 1.1, this would allow us to recover $g_1$ and $g_2$ from $g_0$, contrary to our desire to have each $g_i$ occupy a different degree. We thus want our tables to have the flexibility to allow us to satisfy *diagonalization requirements* of the form $\Phi_e^{g_i} \neq g_j$ whenever $u_j \nleqslant u_i$. Thus if $u_j \nleqslant u_i$, we cannot allow the above procedure to uniquely determine $g_j$ from $g_i$. This will be the only constraint which we need to impose on tables to allow the satisfaction of diagonalization requirements. More precisely, we require, of all tables, that *if* $u_j \nleqslant u_i$, *then there are two tuples in the table which have the same value in the* $g_i$*-column but different values in the* $g_j$*-column.* As an example for the lattice in Fig. 1.1, suppose that we have specified that $\sigma = 1 * 0 \subset g_1$ and have computed $\Phi_e^{\sigma}(0)\downarrow = 2$. Then $g_2$ can be specified by any of the four possibilities in Fig. 1.3. The first two specifications of $g_2$ will not satisfy this requirement, but if we specify that $\tau = 3 * 0 \subset g_2$, then the recovery procedure from $\tau$ will specify that $\sigma \subset g_1$, and $\Phi_e^{\sigma}(0) = 2 \neq 3 = g_2(0)$.

We will be dealing with usls, so will also need a procedure for recovering suprema. This procedure differs from the recovery procedure for the ordering only when the lattice $\mathscr{L}$ has incomparable elements. We thus introduce, in Fig. 1.4, a new lattice together with a table for this lattice. We will use this lattice as an example, to describe the recovery procedure for suprema.




| | $g_0$ | $g_1$ | $g_2$ | $g_3$ |
|---|---|---|---|---|
| $\alpha_0$ | 0 | 0 | 0 | 0 |
| $\alpha_1$ | 0 | 0 | 1 | 1 |
| $\alpha_2$ | 0 | 1 | 0 | 2 |
| $\alpha_3$ | 0 | 1 | 1 | 3 |

Fig. 1.4

Since $u_1 \vee u_2 = u_3$, we must be able to recover $g_3$ from $g_1$ and $g_2$. Suppose that we have specified that $0 * 1 * 0 \subset g_1$ and $0 * 1 * 1 \subset g_2$. We proceed, in Fig. 1.5, as we did in Fig. 1.2, except that we begin by specifying both the $g_1$- and $g_2$-*columns*, and placing dashes in the other columns. For each row on the left-hand side of Fig. 1.5, we search through our table for a tuple which agrees with the given information, and fill in the rows as in the right-hand side of Fig. 1.5. For this example, there is only one way to complete the diagram, and the $g_3$-column of the result is $0 * 3 * 1$; hence $0 * 3 * 1 \subset g_3$. In order for this procedure to work in general, we require, of all tables, that *if* $u_i \vee u_j = u_k$ *and the* $g_i$ *and* $g_j$ *values of a tuple in the table are specified, then any other tuple of the table which has the same* $g_i$ *and* $g_j$ *values as the original tuple must also have the same* $g_k$ *value as the original tuple.*

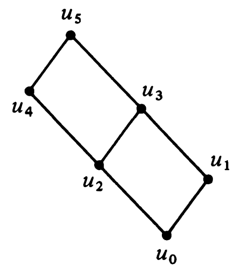
| $g_0$ | $g_1$ | $g_2$ | $g_3$ |
|---|---|---|---|
| – | 0 | 0 | – |
| – | 1 | 1 | – |
| – | 0 | 1 | – |

⟶

| $g_0$ | $g_1$ | $g_2$ | $g_3$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 3 |
| 0 | 0 | 1 | 1 |

Fig. 1.5

The supremum example given above may be misleading as, in general, there may be many possible ways to complete rows by following the procedure. We introduce another example in Fig. 1.6 in order to demonstrate what can happen in a more complicated setting. We will also use this example to give the reader practice with the recovery procedures.




| | $g_0$ | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ |
|---|---|---|---|---|---|---|
| $\alpha_0$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\alpha_1$ | 0 | 1 | 0 | 1 | 0 | 1 |
| $\alpha_2$ | 0 | 0 | 1 | 2 | 1 | 2 |
| $\alpha_3$ | 0 | 1 | 1 | 3 | 1 | 3 |
| $\alpha_4$ | 0 | 0 | 0 | 0 | 2 | 4 |
| $\alpha_5$ | 0 | 1 | 0 | 1 | 2 | 5 |
| $\alpha_6$ | 0 | 0 | 1 | 2 | 3 | 6 |
| $\alpha_7$ | 0 | 1 | 1 | 3 | 3 | 7 |

Fig. 1.6

We refer the reader to Fig. 1.7 for the following example. Suppose that we specify that $1 * 1 \subset g_1$ and $0 * 1 \subset g_2$. We fill in the dashes using rows of the table, and find that there are four possibilities. However $1 * 3$ appears in the $g_3$-*column* for every possibility, so we have specified that $1 * 3 \subset g_3$. Hence in this case, $g_1$ and $g_2$ have uniquely determined $g_3$.

The reader may find it helpful to practice with the following specifications. We refer to the table in Fig. 1.6.

| $g_0$ | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ |
|---|---|---|---|---|---|
| – | 1 | 0 | – | – | – |
| – | 1 | 1 | – | – | – |

→

| $g_0$ | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 3 | 1 | 3 |

| $g_0$ | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 2 | 5 |
| 0 | 1 | 1 | 3 | 1 | 3 |

| $g_0$ | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 3 | 3 | 7 |

| $g_0$ | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 2 | 5 |
| 0 | 1 | 1 | 3 | 3 | 7 |

Fig. 1.7

Suppose that we specify that $3 * 1 \subset g_3$. Find the string $\sigma$ of length 2 such that $\sigma \subset g_2$. How many possibilities are there for strings $\tau$ of length 2 such that $\tau \subset g_4$?

Suppose that we specify that $1 * 0 \subset g_1$ and $2 * 1 \subset g_4$. Find the string $\sigma$ of length 2 such that $\sigma \subset g_5$. How many possibilities are there for strings $\tau$ of length 2 such that $\tau \subset g_3$?

The tables we will use will also need to satisfy certain properties connected to the preservation of greatest lower bounds and extensions of maps. These properties will be motivated and discussed in Sect. 3 when the need for them arises. The existence of tables with such properties will be a corollary of certain representation theorems for lattices which are proved in Appendix B.

We treat the rows of a table as tuples of integers. In order to more easily talk about the agreement of rows on various columns, we introduce the following notation.

**1.1 Definition.** Let $\Theta \subseteq N^{n+1}$ be given. (Thus $\Theta$ is a set of $n + 1$-tuples of integers.) Let $\alpha, \beta \in \Theta$ be given such that $\alpha = \langle a_0, \ldots, a_n \rangle$ and $\beta = \langle b_0, \ldots, b_n \rangle$, and fix $j \leqslant n$. We say $\alpha \equiv_j \beta$ if $a_j = b_j$. We use $\alpha^{[j]}$ to denote $a_j$, *the jth coordinate of* $\alpha$.

We collect some of the properties which must be possessed by a table in the next definition.

**1.2 Definition.** Let $\mathscr{L} = \langle L, \leqslant, \vee, \wedge \rangle$ be a finite lattice, and let $\Theta \subseteq N^{n+1}$ be given. $\Theta$ is said to be a *usl table* for $\mathscr{L}$ if there is an enumeration $u_0, \ldots, u_n$ of the elements of $L$ such that:

(i) (Recursiveness property) $\forall \alpha, \beta \in \Theta(\alpha \equiv_0 \beta)$.

(ii) ($u_n$ computes everything) $\forall \alpha, \beta \in \Theta(\alpha \equiv_n \beta \rightarrow \alpha = \beta)$.

(iii) (Order preservation and one-oneness) $\forall i,j \leqslant n(u_i \leqslant u_j \leftrightarrow \forall \alpha, \beta \in \Theta(\alpha \equiv_j \beta \rightarrow \alpha \equiv_i \beta))$.

(iv) (Least upper bound property) $\forall i,j,k \leqslant n(i_i \vee u_j = u_k \leftrightarrow \forall \alpha, \beta \in \Theta(\alpha \equiv_i \beta \,\&\, \alpha \equiv_j \beta \leftrightarrow \alpha \equiv_k \beta))$.

The properties of usl tables allow for the construction of homomorphisms into $\mathscr{D}$.

**1.3 Definition.** Let $\mathscr{L} = \langle L, \leqslant, \vee \rangle$ and $\mathscr{L}^* = \langle L^*, \leqslant^*, \vee^* \rangle$ be usls. A *homomorphism* from $\mathscr{L}$ into $\mathscr{L}^*$ is a map $\psi: L \rightarrow L^*$ such that:

$$\forall a, b, c \in L(a \vee b = c \rightarrow \psi(a) \vee^* \psi(b) = \psi(c)).$$

Since, in any usl, $a \leqslant b \leftrightarrow a \vee b = b$, it follows from Definition 1.3 that for all $a, b \in L$, $a \leqslant b \rightarrow \psi(a) \leqslant^* \psi(b)$. Note that if $\psi$ is a usl homomorphism which is one-one and onto, then $\psi$ is an isomorphism.

Tables for lattices are used as follows. Let $\langle L, \leqslant, \vee, \wedge \rangle$ be a lattice and let $\Theta \subseteq N^{n+1}$ be a usl table for this lattice. Given a function $g = g_n$ as described earlier and $x \in N$, there will be a unique $\alpha_x \in \Theta$ such that $\alpha_x^{[n]} = g_n(x)$. We define $g_i(x) = \alpha_x^{[i]}$ for all $i \leqslant n$. Our next lemma shows that this definition of $\{g_i\}$ guarantees that the map $\psi: L \rightarrow \mathbf{D_L}$ is a usl homomorphism. Additional conditions which will be required of tables will enable us to show later that $\psi$ is an isomorphism onto an initial segment of $\mathscr{D}$ when $g_n$ is appropriately chosen.

**1.4 Homomorphism Lemma.** *Let $\Theta \subseteq N^{n+1}$ be a recursive usl table for the usl $\langle L, \leqslant, \vee \rangle$ where $L = \{u_0, \ldots, u_n\}$. Let $g: N \rightarrow N$ be given such that for all $x \in N$, there is exactly one $\alpha_x \in \Theta$ for which $g(x) = \alpha_x^{[n]}$. For all $i \leqslant n$, define $g_i(x) = \alpha_x^{[i]}$. Let $G = \{g_i : i \leqslant n\}$ and $\mathbf{G} = \{\mathbf{g_i} : i \leqslant n\}$. Then the map $\psi: \langle L, \leqslant, \vee \rangle \rightarrow \langle \mathbf{G}, \leqslant, \cup \rangle$ defined by $\psi(u_i) = \mathbf{g_i}$ for all $i \leqslant n$ is a usl homomorphism.*

*Proof.* Fix $i, j, k \leqslant n$ such that $u_i \vee u_j = u_k$. We first show that $g_k \leqslant_T g_i \oplus g_j$. Given $x \in N$, search for $\alpha \in \Theta$ such that $\alpha^{[i]} = g_i(x)$ and $\alpha^{[j]} = g_j(x)$. Such an $\alpha$ can be found recursively in $g_i \oplus g_j$ since $\Theta$ is recursive. By Definition 1.2(iv), for all $\beta \in \Theta$, if $\beta \equiv_i \alpha$ and $\beta \equiv_j \alpha$ then $\beta \equiv_k \alpha$, so $\alpha^{[k]} = \beta^{[k]}$. Hence $g_k(x) = \alpha^{[k]}$. We next show that $g_i \leqslant_T g_k$ and $g_j \leqslant_T g_k$ so that $g_i \oplus g_j \leqslant_T g_k$. Since $u_i \leqslant u_k$ and $u_j \leqslant u_k$, it follows from Definition 1.2(iii) that for all $\alpha, \beta \in \Theta$ if $\alpha \equiv_k \beta$ then $\alpha \equiv_i \beta$ and $\alpha \equiv_j \beta$. Hence for every $\alpha \in \Theta$ such that $\alpha^{[k]} = g_k(x)$, $\alpha^{[i]} = g_i(x)$ and $\alpha^{[j]} = g_j(x)$. Since $\Theta$ is recursive, we have given a procedure which computes $g_i$ and $g_j$ recursively from $g_k$. ▯

A preliminary strategy for embedding a finite lattice $\langle L, \leqslant, \vee, \wedge \rangle$ as an initial segment of $\mathscr{D}$ might proceed as follows:

*Step 1.* Find a recursive table for $\langle L, \leqslant, \vee, \wedge \rangle$, where $L = \{u_0, \ldots, u_n\}$.

*Step 2.* Construct a sequence of trees $\{T_i : i \in N\}$ and choose $g_n \in \bigcap\{T_i : i \in N\}$.

*Step 3.* Guarantee that for all $i, j \leqslant n$, if $u_i \neq u_j$ then $g_i \neq g_j$. The map $\psi$ of Lemma 1.4 will then be a usl isomorphism.

*Step 4.* Guarantee that for all $e \in N$, if $\Phi_e^{g_n}$ is total, then $\Phi_e^{g_n} \equiv_T g_i$ for some $i \leqslant n$. The set $\mathbf{G}$ of Lemma 1.4 will then be an initial segment, and hence an ideal, of $\mathscr{D}$.

This strategy almost works, but modifications are needed. In order to satisfy Step 2, we will require the table for the lattice to be finite. Problems arise in carrying out Step 3, which are circumvented by requiring that all trees be *uniform*. This restriction on the trees will force us to require additional properties of tables in order to be able to carry out Step 4. We will have to require that tables be *homogeneous* and *preserve greatest lower bounds*. These conditions are discussed in Sect. 3.

**1.5 Remarks.** Lattice representations have been studied by lattice theorists, but we know of no results concerning lattice representations which possess the homogeneity property which we require. Many theorems about ideals of $\mathscr{D}$ were proved implicitly using representations of particular lattices or classes of lattices, without specifying that this was the case. Thomason [1970] first noticed the importance of lattice representations in such proofs, paving the way for the complete characterization of the countable ideals of $\mathscr{D}$.

**1.6–1.7 Exercises**

**1.6** Let $\mathscr{L} = \langle L, \leqslant, \vee, \wedge \rangle$ be the lattice such that $L = \{u_0, \ldots, u_n\}$ with $u_0 < u_1 < \cdots < u_n$. Find a usl table for $\mathscr{L}$.

**1.7** Let $\mathscr{B}$ be the boolean algebra consisting of $2^m$ elements. Find a usl table for $\mathscr{B}$.

## 2. *Uniform Trees*

Many of the trees needed to construct countable ideals of $\mathscr{D}$ are introduced in this section.

For the next two sections, fix a finite lattice $\mathscr{L} = \langle L, \leqslant, \vee, \wedge \rangle$ which has a finite table $\Theta$. Without loss of generality, we may assume that $\{\alpha^{[n]} : \alpha \in \Theta\} = [0, p)$ where $L$ has $n + 1$ elements. Let $L = \{u_0, \ldots, u_n\}$ with $u_0$ being the least element of $L$ and $u_n$ being the greatest element of $L$. It follows from Definition 1.2(ii) that for all $q < p$, there is exactly one $\alpha \in \Theta$ such that $\alpha^{[n]} = q$. Thus we can identify each $\alpha \in \Theta$ with its $n$th coordinate, $\alpha^{[n]}$. The definition of $\equiv_i$ can thus be extended from $\Theta$ to the $n$th coordinates of $\Theta$. (The reader should note that this extension to integers $< p$ is dependent on $\Theta$.)

**2.1 Definition.** Let $q, r < p$ ($q$ and $r$ should be thought of as the $n$th coordinates of rows of $\Theta$) and $i \leqslant n$ be given. We say that $q \equiv_i r$ if there are $\alpha, \beta \in \Theta$ such that $\alpha^{[n]} = q$, $\beta^{[n]} = r$, and $\alpha \equiv_i \beta$.

The definition of $\equiv_i$ can be extended, argument by argument, to partial functions, and hence to strings and total functions.

**2.2 Definition.** Let $\theta, \theta^* : N \to [0, p)$ be partial functions, and let $i \leqslant n$ be given. We say that $\theta \equiv_i \theta^*$ if for all $x \in N$, if $\theta(x)\downarrow$ and $\theta^*(x)\downarrow$, then $\theta(x) \equiv_i \theta^*(x)$.

Trees are functions from strings into strings, so the notation of Definition 2.2 carries over to trees. Thus if $T: \mathscr{S}_p \to \mathscr{S}_p$ is a tree and $i \leqslant n$, then we write

$$T(\xi) \equiv_i T(\eta) \Leftrightarrow \forall x < \max(\{\text{lh}(T(\xi)), \text{lh}(T(\eta))\})(T(\xi)(x) \equiv_i T(\eta)(x)).$$

We thus compare $T(\xi)$ and $T(\eta)$, argument by argument, and require that for each argument $x$ such that $x < \mathrm{lh}(T(\xi))$ and $x < \mathrm{lh}(T(\eta))$, if $T(\xi)(x) = y$ and $T(\eta)(x) = z$, then there are $\alpha$ and $\beta$ in the table such that $\alpha^{[n]} = y$, $\beta^{[n]} = z$, and $\alpha \equiv_i \beta$.

Uniform trees, which are introduced in the next definition, are needed in order to make the homomorphism into the degrees which is given in Lemma 1.4 a one-one map. Thus given $i,j \leqslant n$ such that $u_i \nleqslant u_j$ and $e \in N$, we must make sure that $\Phi_e^{g_j} \neq g_i$. We use 1.2(iii) to produce an $x \in N$ on which, for some $q,r < p$, $T(q) \equiv_j T(r)$ but $T(q)(x) \not\equiv_i T(r)(x)$ (here, $T$ is the tree which we want to refine in order to force $\Phi_e^{g_j} \neq g_i$). We thus try to compute $\Phi_e^{g_j}(x)$ on some branch $g$ of $T$ which extends $T(q)$, and if we find, for this branch, that $\Phi_e^{g_j}(x) = T(q)(x)$ (and so that we have failed to satisfy the given diagonalization requirement), then we need to switch to a branch $h \subset T$ such that $h \supset T(r)$ and $h \equiv_j g$. It will then follow that $\Phi_e^{g_j}(x) = \Phi_e^{h_j}(x) = T(q)(x) \neq T(r)(x)$. Clause (ii) of the next definition will enable us to make this switch. The other clauses will facilitate later proofs.

**2.3 Definition.** A tree $T$ is *uniform* if it has the following properties:

(i) $\forall \sigma, \tau \in \mathscr{S}_p(\mathrm{lh}(\sigma) = \mathrm{lh}(\tau) \rightarrow \mathrm{lh}(T(\sigma)) = \mathrm{lh}(T(\tau)))$.

(ii) $\forall \sigma, \tau \in \mathscr{S}_p \forall i \leqslant n(\sigma \equiv_i \tau \leftrightarrow T(\sigma) \equiv_i T(\tau))$.

(iii) $\forall \sigma, \tau \in \mathscr{S}_p \forall q < p \forall x \in N(\mathrm{lh}(\sigma) = \mathrm{lh}(\tau) \,\&\, \mathrm{lh}(T(\sigma)) \leqslant x < \mathrm{lh}(T(\sigma * q))$
$\rightarrow T(\sigma * q)(x) = T(\tau * q)(x))$.

A better way of picturing uniform trees is in terms of admissible $p + 1$-tuples of functions. There is one function $f$ which specifies the height of the levels of the tree, i.e., $f(0) = 0$ and $f(m + 1) = \mathrm{lh}(T(\sigma))$ where $\mathrm{lh}(\sigma) = m$. The other functions, $\theta_j(m)$ specify the string placed as the $j$th branch of level $m$ of the tree, i.e., the $\tau$ such that $T(\sigma) * \tau = T(\sigma * j)$ where $\mathrm{lh}(\sigma) = m$. By Definition 2.3(iii), this $\tau$ is independent of $\sigma$ as long as $\mathrm{lh}(\sigma) = m$. Figure 2.1 pictures such a binary tree.

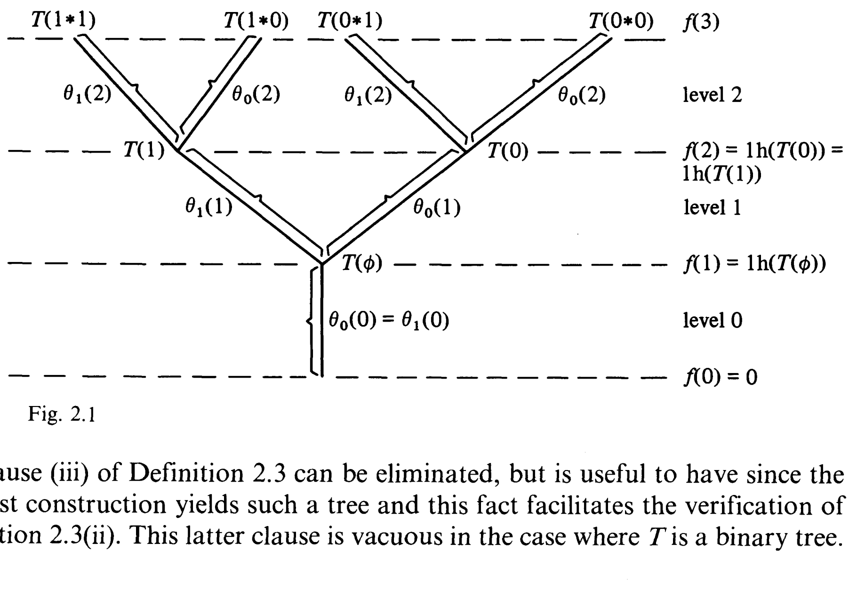



Fig. 2.1

Clause (iii) of Definition 2.3 can be eliminated, but is useful to have since the simplest construction yields such a tree and this fact facilitates the verification of Definition 2.3(ii). This latter clause is vacuous in the case where $T$ is a binary tree.

Definition 2.3(i), which requires that the levels of the tree be uniform, is used heavily in the proof of the Computation Lemma. The crucial point will be that if we want to prove that $\Phi_e^{g_n} \equiv_T g_i$, then the recovery of $g_i$ from $\Phi_e^{g_n}$ proceeds by induction on levels.

The first tree which we define is the identity $p$-branching tree.

**2.4 Definition.** The *identity p-branching tree* $\mathrm{Id}_p: \mathcal{S}_p \to \mathcal{S}_p$ is defined by $\mathrm{Id}_p(\sigma) = \sigma$ for all $\sigma \in \mathcal{S}_p$.

**2.5 Remark.** $\mathrm{Id}_p$ is a recursive uniform tree.

Uniform $p$-branching trees are the conditions of our notion of forcing. These are ordered, as before, by the subtree relation. However, when we refine a tree by taking one of its subtrees, we must make sure that the subtree is uniform, and so an allowable condition.

**2.6 Definition.** Let $T$ and $T^*$ be trees. We say that $T^*$ is a *uniform subtree* of $T$ (write $T^* \subseteq_u T$) if $T^*$ is a subtree of $T$ which is uniform.

Frequently, when we construct a subtree $T^*$ of a uniform tree $T$ in this chapter, the following property will be satisfied:

(1) $\quad \forall q < p \, \forall \sigma, \tau \in \mathcal{S}_p (T^*(\sigma) = T(\tau) \to T(\tau * q) \subseteq T^*(\sigma * q)).$

This property will facilitate showing that $T^*$ is a uniform subtree of $T$ because, given the uniformity of $T$, (1) together with 2.3(i) for $T^*$ implies the $\leftarrow$ direction of 2.3(ii). We thus have:

**2.7 Lemma.** *Let $T$ be a uniform tree, and let $T^*$ be a subtree of $T$ which satisfies* (1), 2.3(i) *and* (iii), *and*

(i) $\quad \forall \sigma, \tau \in \mathcal{S}_p \forall i \leqslant n (\sigma \equiv_i \tau \to T^*(\sigma) \equiv_i T^*(\tau)).$

(*Note that* (i) *differs from 2.3(ii) in that the equivalence is replaced by an implication.*) *Then $T^* \subseteq_u T$.*

*Proof.* Let $\sigma, \tau \in \mathcal{S}_p$ and $i \leqslant n$ be given such that $\sigma \not\equiv_i \tau$. We show that $T^*(\sigma) \not\equiv_i T^*(\tau)$. Since $\sigma \not\equiv_i \tau$, there is an $x < \min(\{\mathrm{lh}(\sigma), \mathrm{lh}(\tau)\})$ such that $\sigma(x) \not\equiv_i \tau(x)$. Let $\xi = \sigma \upharpoonright x$ and $\eta = \tau \upharpoonright x$. Then $\mathrm{lh}(\xi) = \mathrm{lh}(\eta)$, so by 2.3(i) for $T^*$, $\mathrm{lh}(T^*(\xi)) = \mathrm{lh}(T^*(\eta))$. Thus there are $\sigma_0, \tau_0 \in \mathcal{S}_p$ such that $\mathrm{lh}(\sigma_0) = \mathrm{lh}(\tau_0)$, $T(\sigma_0) = T^*(\xi)$, and $T(\tau_0) = T^*(\eta)$. By (1), $T^*(\sigma) \supseteq T(\sigma_0 * \sigma(x))$ and $T^*(\tau) \supseteq T(\tau_0 * \tau(x))$. But $\sigma_0 * \sigma(x) \not\equiv_i \tau_0 * \tau(x)$, and so by 2.3(ii) for $T$, $T(\sigma_0 * \sigma(x)) \not\equiv_i T(\tau_0 * \tau(x))$. Hence $T^*(\sigma) \not\equiv_i T^*(\tau)$. ∎

We will frequently use the following types of subtrees of a tree $T$ to force requirements.

**2.8 Definition.** Let a tree $T$ and $\sigma \in \mathcal{S}_p$ be given. Define $\mathrm{Ext}_p(T, \sigma)$, the subtree of $T$ *extending* $T(\sigma)$, by $\mathrm{Ext}_p(T, \sigma)(\tau) = T(\sigma * \tau)$ for all $\tau \in \mathcal{S}_p$. Thus the branches of $\mathrm{Ext}_p(T, \sigma)$ are those branches of $T$ which extend $T(\sigma)$.

**2.9 Remark.** $\mathrm{Ext}_p(T, \sigma) \subseteq T$ and if $T$ is uniform then $\mathrm{Ext}_p(T, \sigma) \subseteq_u T$. Furthermore, for all $h: N \to N$, if $T$ is recursive in $h$ then $\mathrm{Ext}_p(T, \sigma)$ is recursive in $h$.

We now construct subtrees which will force the map $\psi$ of Lemma 1.4 to be an isomorphism. These are the trees used for diagonalization, and were alluded to in motivating the definition of uniform tree. Their purpose is to insure that $g_i \neq \Phi_e^{g_j}$ whenever $u_i \nleq u_j$. The following useful definitions are given first.

Given a string $\sigma \in \mathscr{S}_p$, we treat $\sigma$ as an initial segment of the function $g$. Hence, using the table $\Theta$, we can compute the initial segments $g_i \upharpoonright x$ for each $i \leqslant n$ and $x < \mathrm{lh}(\sigma)$. We call these initial segments $\sigma^{\langle i \rangle}$. Formally, we introduce this notation in the next definition.

**2.10 Definition.** Given $\sigma \in \mathscr{S}_p$ and $i \leqslant n$, let $\sigma^{\langle i \rangle} \in \mathscr{S}_p$ be defined as follows: $\mathrm{lh}(\sigma^{\langle i \rangle}) = \mathrm{lh}(\sigma)$, and for each $x < \mathrm{lh}(\sigma)$, let $\alpha_x \in \Theta$ be such that $\sigma(x) = \alpha_x^{[n]}$; define $\sigma^{\langle i \rangle}(x) = \alpha_x^{[i]}$. The notation $f^{\langle i \rangle}$ for functions is defined similarly.

We will sometimes have to redefine a string by changing its beginning. Notation is now introduced for this operation.

**2.11 Definition.** Let $\sigma, \tau, \rho \in \mathscr{S}_p$ be given such that $\sigma \subset \rho$ and $\mathrm{lh}(\sigma) = \mathrm{lh}(\tau)$. We define the string $\mathrm{tr}(\sigma \to \tau; \rho)$, *the transfer of* $\sigma$ *into* $\tau$ *below* $\rho$, by

$$\mathrm{tr}(\sigma \to \tau; \rho)(x) = \begin{cases} \tau(x) & \text{if } \; x < \mathrm{lh}(\tau) \\ \rho(x) & \text{if } \; \mathrm{lh}(\tau) \leqslant x < \mathrm{lh}(\rho) \\ \uparrow & \text{otherwise.} \end{cases}$$

**2.12 Definition.** Let $T$ be a tree and let $e \in N$ and $i, j \leqslant n$ be given. $T$ is $\langle e, i, j \rangle$-*differentiating* if for all $g \subset T$, $\Phi_e^{g_j} \neq g_i$.

**2.13 Lemma.** *Let $T$ be a uniform tree and let $e \in N$ and $i, j \leqslant n$ be given. Assume that $u_i \nleq u_j$. Then there is a tree $T^* \subseteq_u T$ such that $T^*$ is $\langle e, i, j \rangle$-differentiating. Furthermore, for all $h: N \to N$, if $T$ is recursive in $h$ then $T^*$ is recursive in $h$.*

*Proof.* First assume that there are $\sigma \in \mathscr{S}_p$ and $x \in N$ such that for all $\tau \in \mathscr{S}_p$, if $\sigma \subseteq \tau$ then $\Phi_e^{T(\tau)^{\langle j \rangle}}(x)\uparrow$. In this case, fix such a $\sigma$ and let $T^* = \mathrm{Ext}_p(T, \sigma)$. The lemma now follows from Remark 2.9. Otherwise,

(2) $$\forall \sigma \in \mathscr{S}_p \forall x \in N \exists \tau \in \mathscr{S}_p (\sigma \subseteq \tau \;\&\; \Phi_e^{T(\tau)^{\langle j \rangle}}(x)\downarrow).$$

The table given by Definition 1.2(iii) allows us to fix $\alpha, \beta \in \Theta$ such that $\alpha \equiv_j \beta$ but $\alpha \not\equiv_i \beta$. Let $\alpha^{[n]} = r$ and $\beta^{[n]} = q$. Since $T$ is uniform, it follows from Definition 2.3(ii) that $T(r) \equiv_j T(q)$ but $T(r) \not\equiv_i T(q)$, so there is an $x < \mathrm{lh}(T(r))$ such that $T(r)^{\langle i \rangle}(x) \neq T(q)^{\langle i \rangle}(x)$. By (2), we can fix $\tau \in \mathscr{S}_p$ such that $r \subseteq \tau$ and $\Phi_e^{T(\tau)^{\langle j \rangle}}(x)\downarrow = z$. Let $\rho = \mathrm{tr}(r \to q; \tau)$. Then $\tau \equiv_j \rho$ so $\Phi_e^{T(\rho)^{\langle j \rangle}}(x)\downarrow = z$. Since $T(\tau)^{\langle i \rangle}(x) \neq T(\rho)^{\langle i \rangle}(x)$, we can let $\sigma = \tau$ if $T(\tau)^{\langle i \rangle}(x) \neq z$, and $\sigma = \rho$ otherwise. We now define $T^* = \mathrm{Ext}_p(T, \sigma)$ and note that $T^*$ is $\langle e, i, j \rangle$-differentiating. The lemma now follows from Remark 2.9. ∎

**2.14 Definition.** Let $\mathrm{Diff}_p(T, e, i, j)$ be the $\langle e, i, j \rangle$-differentiating subtree of $T$ defined in Lemma 2.13.

The initial segments results which are proved in this chapter can be relativized through the use of pointed trees. These trees are defined exactly as in Lemma V.4.2. Narrow subtrees, which are used to construct degrees which are not in $\mathbf{L}_1$ are also defined exactly as in Lemma V.3.9. It is, however, more difficult to construct uniform $e$-total trees. Such trees were used to control the double jumps of the degrees constructed. We now indicate how to construct these trees.

**2.15 Lemma.** *Let $e \in N$ be given, and fix a uniform tree $T$. Assume that*

(i) $\forall x \in N \forall \sigma \in \mathcal{S}_p \exists \tau \in \mathcal{S}_p(\sigma \subseteq \tau \,\&\, \Phi_e^{T(\tau)}(x)\downarrow)$.

*Then there is an e-total $T^* \subseteq_u T$. Furthermore, for all $h: N \to N$, if $T$ is recursive in $h$ then $T^*$ is recursive in $h$.*

*Proof.* We define $T^*$ by induction on $\mathrm{lh}(\sigma)$. Let $\{\sigma_j : j \in N\}$ be a one-one recursive correspondence of $N$ with $\mathcal{S}_p$.

*Stage* 0. Find the least $j \in N$ such that $\Phi_e^{T(\sigma_j)}(0)\downarrow$. Such a $j$ must exist by (i). Let $T^*(\emptyset) = T(\sigma_j)$.

*Stage* $m + 1$. The induction step is an iteration of the operation performed at Stage 0, dovetailed with the dummying in of strings in order to preserve uniformity. Let $\rho_0, \ldots, \rho_r$ be the set of strings in $\mathcal{S}_p$ such that $\mathrm{lh}(\rho_i) = m + 1$. We perform a subinduction on $\{s : s \leqslant r + 1\}$. At substage $s$, we will define $\{\tau_q^s : q \leqslant r\}$ such that for all $q \leqslant r$, $T^*(\rho_q^-) \subset \tau_q^0 \subseteq \cdots \subseteq \tau_q^s$ so as to preserve the uniformity of $T^*$ and to force the convergence of $\Phi_e^{\tau_{s-1}^{s-1}}(m + 1)$. We assume the following induction hypotheses:

(3) $\forall \xi, \eta \in \mathcal{S}_p(\mathrm{lh}(\xi) = \mathrm{lh}(\eta) = m \to \mathrm{lh}(T^*(\xi)) = \mathrm{lh}(T^*(\eta)))$.

(4) $\forall t, q \leqslant r \forall i < s(\mathrm{lh}(\tau_q^i) = \mathrm{lh}(\tau_t^i))$.

(5) $\forall \xi \in \mathcal{S}_p(\mathrm{lh}(\xi) = m \to T^*(\xi) \subset T)$.

(6) $\forall q \leqslant r \forall i < s(\tau_q^i \subset T)$.

(3)–(6) are easily verified at the end of Stage 0.

*Substage* 0. For all $q \leqslant r$, fix the unique $i_q \in N$ such that $\rho_q = \rho_q^- * i_q$. By (5), $T^*(\rho_q^-) = T(\eta_q)$ for some $\eta_q \in \mathcal{S}_p$. Set $\tau_q^0 = T(\eta_q * i_q)$ for all $q \leqslant r$. Note that (4) and (6) hold since $T$ is uniform.

*Substage* $s$, $1 \leqslant s \leqslant r + 1$. Find the least $i \in N$ such that $\Phi_e^{\sigma_i}(m + 1)\downarrow$ and $\tau_{s-1}^{s-1} \subseteq \sigma_i \subset T$. By (i) and (6), such an $i$ must exist. For each $q \leqslant r$, define

$$\tau_q^s = \mathrm{tr}(\tau_{s-1}^{s-1} \to \tau_q^{s-1}; \sigma_i).$$

Since $T$ is uniform, (4) and (6) hold with $s + 1$ in place of $s$.

If $s = r + 1$, set $T^*(\rho_q) = \tau_q^{s+1}$ for all $q \leqslant r$. By (4) and (6), we note that (3) and (5) follow for $s + 1$ in place of $s$.

This concludes the construction of $T^*$. (3)–(6) can be used to show that $T^* \subseteq_u T$; we leave the verification of this fact to the reader. It is easily shown that $T^*$ is $e$-total. Furthermore, for all $h: N \to N$, if $T$ is recursive in $h$, then $T^*$ is recursive in $h$. ▯

**2.16 Definition.** Let $\mathrm{Tot}_p(T, e)$ be the $e$-total uniform subtree of $T$ constructed in the proof of Lemma 2.15.

**2.17 Remarks.** Spector [1956] used admissible triples in order to construct a minimal degree. Shoenfield [1966] simplified the construction of a minimal degree by using trees instead of admissible triples. Subsequently, Hugill [1969] and Lachlan [1968] recast Spector's construction in terms of uniform trees. Admissible triples were also used by Miller and Martin [1968] to construct hyperimmune-free degrees.

**2.18 Exercise.** Verify the claim that $\mathrm{Tot}_p(T, e)$ is a uniform subtree of $T$ and is $e$-total.

## 3. Splitting Trees

The splitting trees which are used to construct distributive initial segments of $\mathscr{D}$ are introduced in this section. In order to prove that such trees exist, we require additional properties of tables. These properties are introduced as they are needed.

Given $e \in N$ and a uniform tree $T$, we want to find $i \leqslant n$ such that for all $g \subset T$, if $\Phi_e^g$ is total, then $\Phi_e^g \equiv_T g_i$. We will be able to do this if $T$ is *e-splitting for i*. This fact is the content of the Computation Lemma, which we prove after introducing some useful terminology.

**3.1 Definition.** Let $\sigma, \tau, \rho \in \mathscr{S}_p$ and $e, i \in N$ be given such that $\langle \tau, \rho \rangle$ is an $e$-splitting of $\sigma$. We say that $\langle \tau, \rho \rangle$ *e-splits $\sigma$ mod i* if $\tau \equiv_i \rho$.

**3.2 Definition.** Let $T$ be a tree, and let $e \in N$ and $i \leqslant n$ be given. $T$ is said to be an *e-splitting tree for i* if:

(i) $\forall \sigma \in \mathscr{S}_p \forall q, r < p(q \not\equiv_i r \rightarrow \langle T(\sigma * q), T(\sigma * r) \rangle$ $e$-splits $T(\sigma))$.

(ii) $\forall \sigma, \tau \in \mathscr{S}_p(\langle T(\sigma), T(\tau) \rangle$ $e$-splits $T(\emptyset) \rightarrow \sigma \not\equiv_i \tau)$.

**3.3 Computation Lemma.** *Let $T$ be a uniform tree and let $e \in N$ and $i \leqslant n$ be given such that $T$ is an e-splitting tree for i. Let $h: N \rightarrow N$ be given such that $T$ is recursive in h. Then for every branch g of T, if $\Phi_e^g$ is total, then $\Phi_e^g \leqslant_T g_i \oplus h$ and $g_i \leqslant_T \Phi_e^g \oplus h$, where $g_i$ is defined as in Lemma 1.4.*

*Proof.* Let $T, e, i$ and $h$ be given as in the hypothesis of the lemma. Fix $g \subset T$ such that $\Phi_e^g$ is total. We first show how to compute $\Phi_e^g$ using a $g_i \oplus h$ oracle. Given $x \in N$, search for $\sigma \in \mathscr{S}_p$ such that $T(\sigma) \equiv_i g$ and $\Phi_e^{T(\sigma)}(x)\downarrow$. Such a $\sigma$ must exist since $g \subset T$ and $\Phi_e^g$ is total, and can be found through the use of a $g_i \oplus h$ oracle. Now if $\tau \in \mathscr{S}_p$, $T(\tau) \subset g$, and $\Phi_e^{T(\tau)}(x)\downarrow$, then $T(\tau) \equiv_i g \equiv_i T(\sigma)$, and so by Definition 3.2(ii), $\Phi_e^{T(\tau)}(x) = \Phi_e^{T(\sigma)}(x) = \Phi_e^g(x)$.

We now show how to compute $g_i$ using a $\Phi_e^g \oplus h$ oracle. We proceed by induction on the levels of $T$. At stage $s$ in the induction, we define $\sigma_s \in \mathscr{S}_p$ such that $\mathrm{lh}(\sigma_s) = s$ and $T(\sigma_s) \equiv_i g$. Since $g_i(x) = T(\sigma_s)^{\langle i \rangle}(x)$ for all $x < \mathrm{lh}(T(\sigma_s))$, this allows us to compute $g_i$.

*Stage* 0. Since $T(\emptyset) \subset g$, we can set $\sigma_0 = \emptyset$.

*Stage* $s+1$. By induction, we are given $\sigma_s$ such that $\mathrm{lh}(\sigma_s) = s$ and $T(\sigma_s) \equiv_i g$. By Definition 3.2(i), it follows that for every $q < p$ such that $T(\sigma_s * q) \not\equiv_i g$, there is an $x \in N$ such that $\Phi_e^{T(\sigma_s * q)}(x)\downarrow \neq \Phi_e^g(x)$. Hence one by one, we can eliminate various $q < p$ as potential candidates for the satisfaction of $T(\sigma_s * q) \subset g$, until all remaining candidates are in the same $\equiv_i$ class. By Definition 3.2(ii), if $\mathrm{lh}(\sigma) = s$ and $T(\sigma * q) \subset g$, then $q$ will not be eliminated during this procedure. Hence if we fix any $r < p$ remaining after the elimination process has been completed, it will be the case that $T(\sigma_s * r) \equiv_i g$. Let $\sigma_{s+1} = \sigma_s * r$. Note that the elimination process can be carried out through the use of a $\Phi_e^g \oplus h$ oracle. ▯

Given a uniform tree $T$ and $e \in N$, we can apply the Computation Lemma if we can build an $e$-splitting tree $T^* \subseteq_u T$ for some $k \leqslant n$. $i$ will be chosen so that $u_k = \wedge \mathscr{U}_\sigma$ where $\mathscr{U}_\sigma = \{u_j \in L$: there are no $e$-splittings mod $j$ of $T(\sigma)\}$ for some $\sigma \in \mathscr{S}_p$. In order to apply the Computation Lemma, we need to know that $u_k \in \mathscr{U}_\sigma$ i.e., that $\mathscr{U}_\sigma$ is closed under $\wedge$. Thus if $u_k = u_i \wedge u_j$ and there are no $e$-splittings of $T(\sigma)$ mod $i$ and none mod $j$, then there are no $e$-splittings of $T(\sigma)$ mod $k$. This is proved by assuming that $\langle T(\tau), T(\rho)\rangle$ $e$-split $T(\sigma)$ mod $k$, and interpolating $\tau = \tau_0, \tau_1, \ldots, \tau_v = \rho$ such that $\tau_0 \equiv_i \tau_1 \equiv_j \tau_2 \equiv_i \cdots \equiv_j \tau_v$ and for $m = 1, \ldots, v$, $\Phi_e^{T(\tau_m)}(x)\downarrow$ where $x$ is chosen so that $T(\tau)$ and $T(\rho)$ $e$-split on $x$. We will then have a contradiction, having produced an $e$-splitting of $T(\sigma)$ mod $i$ or an $e$-splitting of $T(\sigma)$ mod $j$. We will prove a lemma which tells us that we can always find such interpolants. The proof of the lemma relies on the fact that we are using a lattice table rather than a usl table.

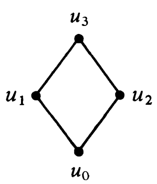
**3.4 Definition.** The usl table $\Theta$ for $\mathscr{L}$ is a *lattice table* if it satisfies the following additional *greatest lower bound preservation property*:

$$\forall i,j,k \leqslant n(u_i \wedge u_j = u_k \leftrightarrow \forall \alpha, \beta \in \Theta(\alpha \equiv_k \beta \leftrightarrow$$

$$\exists \gamma_0, \ldots, \gamma_m \in \Theta(\alpha = \gamma_0 \equiv_i \gamma_1 \equiv_j \gamma_2 \equiv_i \cdots \equiv_j \gamma_m = \beta))).$$

Recall the table for the diamond lattice which we reproduce in Fig. 3.1 below as an example. Note that $u_0 = u_1 \wedge u_2$ and $\alpha_0 \equiv_0 \alpha_3$ but $\alpha_0 \not\equiv_1 \alpha_3$ and $\alpha_0 \not\equiv_2 \alpha_3$. The single interpolant $\alpha_1$ can be chosen in this case. For $\alpha_0 \equiv_1 \alpha_1 \equiv_2 \alpha_3$.

Henceforth, unless otherwise indicated, we assume that $\Theta$ is a lattice table for $\mathscr{L}$.




| | $g_0$ | $g_1$ | $g_2$ | $g_3$ |
|---|---|---|---|---|
| $\alpha_0$ | 0 | 0 | 0 | 0 |
| $\alpha_1$ | 0 | 0 | 1 | 1 |
| $\alpha_2$ | 0 | 1 | 0 | 2 |
| $\alpha_3$ | 0 | 1 | 1 | 3 |

Fig. 3.1

**3.5 GLB Interpolation Lemma.** *Let $i, j, k \leqslant n$ and $\tau, \rho \in \mathscr{S}_p$ be given such that $u_i \wedge u_j = u_k$, $\mathrm{lh}(\tau) = \mathrm{lh}(\rho)$ and $\tau \equiv_k \rho$. Then there is a sequence $\tau = \tau_0, \tau_1, \ldots, \tau_m = \rho$*

*of elements of* $\mathcal{S}_p$ *such that*

$$\mathrm{lh}(\tau_0) = \mathrm{lh}(\tau_1) = \cdots = \mathrm{lh}(\tau_m) \qquad \textit{and} \qquad \tau_0 \equiv_i \tau_1 \equiv_j \tau_2 \equiv_i \cdots \equiv_j \tau_m.$$

*Proof.* We proceed by induction on $\mathrm{lh}(\tau)$. The lemma is trivial for $\tau = \emptyset$. Assume that the lemma holds whenever $\mathrm{lh}(\tau) = s$. Let $i, j, k \leqslant n$ and $\tau, \rho \in \mathcal{S}_p$ be given satisfying the hypothesis of the lemma with $\mathrm{lh}(\tau) = s + 1$. Fix $q, r < p$ such that $\tau = \tau^- * q$ and $\rho = \rho^- * r$. By induction, there are interpolants $\tau^- = \rho_0, \rho_1, \ldots, \rho_v = \rho^-$, all of length $s$, such that $\rho_0 \equiv_i \rho_1 \equiv_j \rho_2 \equiv_i \cdots \equiv_j \rho_v$. By Definition 3.4, there are $q = q_0, \ldots, q_w = r$ such that each $q_x$ is $< p$ and $q_0 \equiv_i q_1 \equiv_j q_2 \equiv_i \cdots \equiv_j q_w$. It is now easily checked that the sequence $\rho_0 * q_0$, $\rho_1 * q_0, \ldots, \rho_v * q_0$, $\rho_v * q_1, \ldots, \rho_v * q_w$ has the desired properties. ▯

The GLB Interpolation Lemma is used to prove the following important proposition.

**3.6 Proposition.** *Let $T$ be a uniform tree, and let $e \in N$ and $i, j, k \leqslant n$ be given such that $u_i \wedge u_j = u_k$. Assume that there is an e-splitting* mod $k$ *on $T$, and that*

$$\forall \sigma \in \mathcal{S}_p \forall x \in N \exists \tau \in \mathcal{S}_p(\sigma \subseteq \tau \,\&\, \Phi_e^{T(\tau)}(x)\downarrow).$$

*Then either $T$ has an e-splitting* mod $i$ *or $T$ has an e-splitting* mod $j$.

*Proof.* Let $\langle T(\tau), T(\rho)\rangle$ be an $e$-splitting mod $k$ on $x$. Without loss of generality, we may assume that $\mathrm{lh}(\tau) = \mathrm{lh}(\rho)$ since $T$ is uniform. By the GLB Interpolation Lemma, we can fix a sequence $\tau = \rho_0, \rho_1, \ldots, \rho_m = \rho$ of strings, all of the same length, such that for all $c < m$, there is a $d \in \{i, j\}$ for which $\rho_c \equiv_d \rho_{c+1}$. Define $\tau_0 = \rho_0$, and assuming that $c < m$ and $\tau_c$ has been defined, let $\tau_{c+1}$ be the least $\sigma$ (under some fixed recursive one-one correspondence of $\mathcal{S}_p$ with $N$) such that $\sigma \supseteq \mathrm{tr}(\rho_c \to \rho_{c+1}; \tau_c)$ and $\Phi_e^{T(\sigma)}(x)\downarrow$. Then there is a least $c < m$ such that $\langle T(\tau_c), T(\tau_{c+1})\rangle$ $e$-splits on $x$. For some $d \in \{i, j\}$, $\tau_c \equiv_d \tau_{c+1}$. Since $T$ is uniform, we have produced an $e$-splitting mod $i$ or an $e$-splitting mod $j$ on $T$. ▯

Let $T$ be a uniform tree, and let $e \in N$ and $i \leqslant n$ be given so that the following conditions hold:

(1) $\forall \sigma \in \mathcal{S}_p \forall j \leqslant n(u_j \ngeq u_i \to \exists \tau, \rho \in \mathcal{S}_p(\sigma \subseteq \tau \,\&\, \sigma \subseteq \rho \,\&$
$\langle T(\tau), T(\rho)\rangle$ $e$-split $T(\sigma)$ mod $j$)).

(2) $\forall \sigma, \tau \in \mathcal{S}_p(\sigma \equiv_i \tau \to \langle T(\sigma), T(\tau)\rangle$ is not an $e$-splitting).

(3) $\forall \sigma \in \mathcal{S}_p \forall x \in N \exists \tau \in \mathcal{S}_p(\tau \supseteq \sigma \,\&\, \Phi_e^{T(\tau)}(x)\downarrow)$.

Under these circumstances, we will want to build an $e$-splitting subtree $T^* \subseteq_u T$ for $i$. The construction of such a tree proceeds level by level. At each level, we iterate a certain basic procedure which, when completed, will guarantee that $\langle T^*(\sigma), T^*(\tau)\rangle$ is an $e$-splitting for the particular $\sigma$ and $\tau$ chosen with $\sigma \not\equiv_i \tau$. Conditions (1) and (3) allow us to carry out this procedure, as we show in the next lemma.

**3.7 Lemma.** *Let $T$ be a uniform tree, and let $e \in N$ and $i \leqslant n$ be given so that* (1) *and* (3) *hold. Let $\sigma_0, \sigma_1 \in \mathcal{S}_p$ and $j \leqslant n$ be given such that* $\mathrm{lh}(\sigma_0) = \mathrm{lh}(\sigma_1)$, $\sigma_0 \equiv_j \sigma_1$, *and*

$u_j \not\geq u_i$. *Then there are* $\tau_0, \tau_1 \in \mathcal{S}_p$ *such that* $\mathrm{lh}(\tau_0) = \mathrm{lh}(\tau_1)$, $\sigma_0 \subseteq \tau_0$, $\sigma_1 \subseteq \tau_1$, *and* $\langle T(\tau_0), T(\tau_1)\rangle$ *is an e-splitting* mod $j$.

*Proof.* Search for an $e$-splitting mod $j$, $\langle T(\xi_0), T(\xi_1)\rangle$ of $T(\sigma_0)$, which must exist by (1). (See Fig. 3.2; strings marked with the same symbol are equal.) Let $\langle T(\xi_0), T(\xi_1)\rangle$ $e$-split on $x$, and let $\eta = \mathrm{tr}(\sigma_0 \to \sigma_1; \xi_0)$. Search for $\rho \supseteq \eta$ such that $\Phi_e^{T(\rho)}(x)\downarrow$; such a $\rho$ must exist by (3). Let $\xi_k^* = \mathrm{tr}(\eta \to \xi_k; \rho)$ for $k = 0, 1$. Then $\langle T(\xi_k^*), T(\rho)\rangle$ is an $e$-splitting mod $j$ for some $k \in \{0, 1\}$. Fix this $k$ and let $\tau_0 = \xi_k^*$ and $\tau_1 = \rho$ to complete the proof of the lemma. ▯

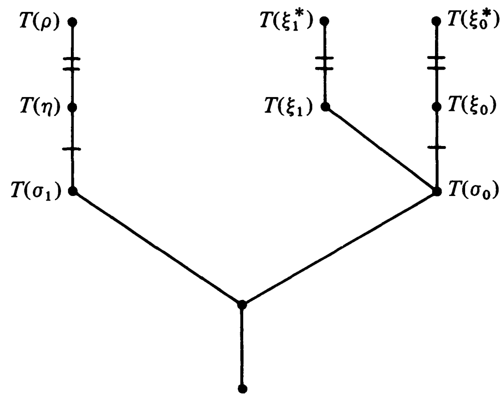



Fig. 3.2

We continue the discussion preceding Lemma 3.7. Suppose that $T$ satisfies (1)–(3), and that we want to build an $e$-splitting subtree $T^*$ of $T$ for $i$. Suppose that $T^*(\xi^*)$ has been defined for all $\xi^* \in \mathcal{S}_p$ such that $\mathrm{lh}(\xi^*) \leqslant m$, and we want to define $T^*(\sigma^*)$ where $\mathrm{lh}(\sigma^*) = m + 1$. Let $T^*(\sigma^{*-}) = T(\xi)$.

Given $\tau^* \in \mathcal{S}_p$ such that $\mathrm{lh}(\tau^*) = m + 1$, $\tau^* \not\equiv_i \sigma^*$, and $T^*(\tau^{*-}) = T(\eta)$, we will need to define $T^*$ so that $\langle T^*(\sigma^*), T^*(\tau^*)\rangle$ is an $e$-splitting, while preserving the uniformity of $T^*$. Thus we will have to find $\sigma \supseteq \xi$ and $\tau \supseteq \eta$ such that $\langle T(\sigma), T(\tau)\rangle$ is an $e$-splitting mod $j$ ($u_j$ is the greatest element of $L$ such that $\sigma^* \equiv_j \tau^*$), in order to erect this $e$-splitting on the uniform tree $T^*$ with $T(\sigma) \subseteq T^*(\sigma^*)$ and $T(\tau) \subseteq T^*(\tau^*)$. (The procedure which we follow will be iterated to take care of all $\sigma^*$ and $\tau^*$ satisfying the above conditions.)

Lemma 3.7 allows us to find $\sigma$ and $\tau$ as in the above paragraph. However, we will still have to define $T^*(\rho^*)$ for $\rho^* \in \mathcal{S}_p$ with $\mathrm{lh}(\rho^*) = m + 1$ and $\rho^* \notin \{\sigma^*, \tau^*\}$. In order to be able to define $T^*(\rho^*)$ while preserving the uniformity of $T^*$, we require that the table be *homogeneous*. The following example is used to demonstrate the need for the homogeneity property. The reader should refer to Fig. 3.3 while following the example.

Consider the diamond lattice of Fig. 3.1 together with its corresponding table. Let $T = \mathrm{Id}_p$, and suppose that we are trying to build an $e$-splitting subtree $T^*$ of $T$ for $i = 2$, after having specified that $T^*(\emptyset) = \emptyset$. Since $0 \not\equiv_2 1$, $\langle T^*(0), T^*(1)\rangle$ must be an $e$-splitting. Since $0 \equiv_1 1$, the uniformity of $T^*$ requires that $T^*(0) \equiv_1 T^*(1)$. We use Lemma 3.7 and find, for example, that $\langle 0 * 3, 1 * 2\rangle = \langle T(0 * 3), T(1 * 2)\rangle$ is an $e$-splitting. We then want to specify that $0 * 3 \subseteq T^*(0)$ and $1 * 2 \subseteq T^*(1)$. We will also have to specify the strings of length 2 which are to be initial segments of $T^*(2)$ and $T^*(3)$. We begin by forming a diagram (see the left-hand side of Fig. 3.3) whose columns correspond to the tuples of the table. We have already specified the initial segments of length 2 corresponding to the tuples $\alpha_0$ and $\alpha_1$ as $0 * 3$ and $1 * 2$ respectively ($\alpha_i$ and $i$ are identified), so we place these strings in the designated columns and fill in the remaining columns with dashes. We now need a way to fill in the dashes while preserving the uniformity of $T^*$; thus for all $k \leqslant n$, $i, j < p$ and $r \leqslant 1$, we must satisfy the condition

$$\alpha_i \equiv_k \alpha_j \rightarrow f_r(\alpha_i) \equiv_k f_r(\alpha_j)$$

where $f_r(\alpha_i)$ is the integer placed in row $r$, column $i$, of the diagram. The right-hand side of Fig. 3.3 indicates how to define $f_0$ and $f_1$ for the given example. We now have to check that each $f_i$ preserves congruences. Thus $0 \equiv_2 2$ and $0 * 3 \equiv_2 2 * 1$. We leave it to the reader to check the remaining cases.

| | $\alpha_0$ | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ |
|---|---|---|---|---|
| $f_0$ | 0 | 1 | – | – |
| $f_1$ | 3 | 2 | – | – |

$\longrightarrow$

| | $\alpha_0$ | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ |
|---|---|---|---|---|
| $f_0$ | 0 | 1 | 2 | 3 |
| $f_1$ | 3 | 2 | 1 | 0 |

Fig. 3.3

The homogeneity property which will allow us to fill in the dashes as in Fig. 3.3 in a congruence preserving way is now defined.

**3.8 Definition.** The lattice table $\Theta$ for $\mathscr{L}$ is *homogeneous* if for all $\alpha_0, \alpha_1, \beta_0, \beta_1 \in \Theta$, whenever

(i) $\qquad \forall k \leqslant n(\alpha_0 \equiv_k \alpha_1 \rightarrow \beta_0 \equiv_k \beta_1)$

then there is a function $f: \Theta \rightarrow \Theta$ such that for all $j \in \{0, 1\}$ and $\alpha, \beta \in \Theta$

(ii) $\qquad f(\alpha_j) = \beta_j$

and

(iii) $\qquad \forall k \leqslant n(\alpha \equiv_k \beta \rightarrow f(\alpha) \equiv_k f(\beta))$.

For the rest of this section, assume that $\Theta$ is a homogeneous table. This homogeneity property can be lifted to $\mathscr{S}_p$ as follows:

**3.9 Lemma.** *Let* $\sigma_0, \sigma_1, \ldots, \sigma_{p-1} \in \mathscr{S}_p$ *be given such that* $\mathrm{lh}(\sigma_0) = \mathrm{lh}(\sigma_1) = \cdots = \mathrm{lh}(\sigma_{p-1})$ *and for all* $q, r < p$ *and all* $k \leqslant n$, *if* $q \equiv_k r$ *then* $\sigma_q \equiv_k \sigma_r$. Let $u, v < p$ *and* $\tau_u, \tau_v \in \mathscr{S}_p$ *be given such that* $\mathrm{lh}(\tau_u) = \mathrm{lh}(\tau_v)$, $\sigma_u \subseteq \tau_u$, $\sigma_v \subseteq \tau_v$, *and for all* $k \leqslant n$, *if* $u \equiv_k v$ *then* $\tau_u \equiv_k \tau_v$. *Then there exist* $\rho_0, \rho_1, \ldots, \rho_{p-1} \in \mathscr{S}_p$ *such that* $\mathrm{lh}(\rho_0) = \mathrm{lh}(\rho_1) = \cdots = \mathrm{lh}(\rho_{p-1})$, $\tau_u = \rho_u$, $\tau_v = \rho_v$, $\sigma_j \subseteq \rho_j$ *for all* $j < p$, *and for all* $q, r < p$ *and* $k \leqslant n$, *if* $q \equiv_k r$ *then* $\rho_q \equiv_k \rho_r$.

*Proof*. By choice of $\tau_u$ and $\tau_v$, for each $k \leqslant n$ and $x \in N$ such that $\text{lh}(\sigma_u) \leqslant x < \text{lh}(\tau_u)$, if $u \equiv_k v$ then $\tau_u(x) \equiv_k \tau_v(x)$. Since $\Theta$ is homogeneous, for each such $x$ there is a map $f_x$: such that $f_x(u) = \tau_u(x), f_x(v) = \tau_v(x)$ and for all $q, r < p$ and $k \leqslant n$, if $q \equiv_k r$ then $f_x(q) \equiv_k f_x(r)$. For each $q < p$, define

$$\rho_q(x) = \begin{cases} \sigma_q(x) & \text{if } \ x < \text{lh}(\sigma_q) \\ f_x(q) & \text{if } \ \text{lh}(\sigma_q) \leqslant x < \text{lh}(\tau_u) \\ \uparrow & \text{otherwise.} \end{cases}$$

Then $\{\rho_q : q < p\}$ has the desired properties. ▯

We now have all the pieces needed to construct an $e$-splitting subtree for $i$. These trees are now constructed as subtrees of uniform trees.

**3.10 Lemma.** *Let $T$ be a uniform tree, and let $e \in N$ and $i \leqslant n$ be given so that* (1)–(3) *hold. Then there is a tree $T^* \subseteq_u T$ which is e-splitting for i. Furthermore, for all $h: N \to N$, if $T$ is recursive in $h$ then $T^*$ is recursive in $h$.*

*Proof*. We proceed by induction on the levels of $T^*$. At stage $s$ of the induction, we define $T^*(\sigma)$ for all $\sigma \in \mathcal{S}_p$ such that $\text{lh}(\sigma) = s$. We begin, at stage 0, by setting $T^*(\emptyset) = T(\emptyset)$.

*Stage* $s + 1$. Let $\{\langle \eta_m, q_m, r_m \rangle : m < m_0\}$ list all $\langle \eta, q, r \rangle \in \mathcal{S}_p \times [0, p)^2$ such that $\text{lh}(\eta) = s$ and $q \not\equiv_i r$. For $m < m_0$, let $\xi_m \in \mathcal{S}_p$ be the string such that $T(\xi_m) = T^*(\eta_m)$. We perform a subinduction on $\{m : m \leqslant m_0\}$, defining, at step $m$, a sequence $\{\rho_j^m \in \mathcal{S}_p : j < p\}$ which satisfies:

(4) $\quad \text{lh}(\rho_0^m) = \text{lh}(\rho_1^m) = \cdots = \text{lh}(\rho_{p-1}^m)$.

(5) $\quad \forall j < p(m > 0 \to \rho_j^m \supseteq \rho_j^{m-1})$.

(6) $\quad \forall q, r < p \, \forall k \leqslant n(q \equiv_k r \to \rho_q^m \equiv_k \rho_r^m)$.

(7) $\quad m > 0 \to \langle T(\xi_{m-1} * \rho_{q_{m-1}}^m), T(\xi_{m-1} * \rho_{r_{m-1}}^m) \rangle$ is an $e$-splitting.

We begin, for $m = 0$, setting $\rho_j^0 = j$ for all $j < p$.

*Substage* $m + 1$. Let $u_{k_m}$ be the greatest element of $L$ such that $q_m \equiv_{k_m} r_m$. Such a $k_m$ must exist, as is seen immediately from the Least Upper Bound Property for tables (Definition 1.2(iv)). By Lemma 3.7, there are $\tau_{q_m}, \tau_{r_m} \in \mathcal{S}_p$ such that $\tau_{q_m} \supseteq \rho_{q_m}^m$, $\tau_{r_m} \supseteq \rho_{r_m}^m$, and $\langle T(\xi_m * \tau_{q_m}), T(\xi_m * \tau_{r_m}) \rangle$ is an $e$-splitting mod $k_m$. By (4) and (6), the hypotheses for Lemma 3.9 are satisfied, so Lemma 3.9 produces $\{\rho_j^{m+1} : j < p\}$ satisfying (4)–(6) with $m + 1$ in place of $m$. Since $\rho_{q_m}^{m+1} = \tau_{q_m}$ and $\rho_{r_m}^{m+1} = \tau_{r_m}$, (7) is also satisfied.

Once the subinduction is completed, we define $T^*(\eta * q) = T(\xi * \rho_q^{m_0})$ for all $\eta \in \mathcal{S}_p$ such that $\text{lh}(\eta) = s$ and all $q < p$, where $\xi$ is defined by $T(\xi) = T^*(\eta)$.

Since the subinduction satisfies (4)–(6) and by the definition of $T^*$, we note that $T^*$ is a uniform tree. Since the subinduction satisfies (5) and (7), $T^*$ is an $e$-splitting tree for $i$. (Note that this is the only place where (2) is used.) Furthermore, if $T$ is recursive in $h$, then the construction of $T^*$ can be carried out recursively in $h$. ▯

**3.11 Definition.** Let $T$ be a uniform tree and let $e \in N$ and $i \leqslant n$ be given so that (1)–(3) are satisfied. Then Lemma 3.10 constructs $T^* \subseteq_u T$ such that $T^*$ is an $e$-splitting tree for $i$. We give that tree $T^*$ a name, $\mathrm{Sp}_p(T, e, i)$.

The final result of this section shows that if $T$ is a uniform tree and $e \in N$, then there is some $i \leqslant n$ such that $T$ has an $e$-splitting subtree for $i$, provided that $T$ satisfies (3).

**3.12 Lemma.** *Let $T$ be a uniform tree satisfying* (3) *and let $e \in N$ be given. Then there is an $i \leqslant n$ and a tree $T^* \subseteq_u T$ such that $T^*$ is e-splitting for i. Furthermore, for all $h: N \to N$, if $T$ is recursive in $h$ then $T^*$ is recursive in $h$.*

*Proof.* For each $\sigma \in \mathscr{S}_p$, let $\mathscr{U}_\sigma = \{u_k \in L$: there are no $e$-splittings mod $k$ on $\mathrm{Ext}_p(T, \sigma)\}$. Note that if $\sigma \subseteq \tau$ then $\mathscr{U}_\sigma \subseteq \mathscr{U}_\tau$. Since $L$ is finite, we can fix $\sigma \in \mathscr{S}_p$ such that $\mathscr{U}_\sigma = \mathscr{U}_\tau$ for all $\tau \supseteq \sigma$. Let $u_i = \wedge \mathscr{U}_\sigma$. By Proposition 3.6, $u_i \in \mathscr{U}_\sigma$, hence (1) and (2) hold for $\mathrm{Ext}_p(T, \sigma)$. Hence by Lemma 3.10 $\mathrm{Sp}_p(\mathrm{Ext}_p(T, \sigma), e, i)$ is the desired tree. ▯

**3.13 Remark.** The notation for marking trees as in Fig. 3.2 was introduced by Epstein [1975].

## 4. Initial Segments of $\mathscr{D}$

We will show, in this section, that if $\mathscr{L}$ is a lattice which has a homogeneous lattice table, then $\mathscr{L}$ is isomorphic to an ideal of $\mathscr{D}$. In particular, we will show that every finite distributive lattice is isomorphic to an ideal of $\mathscr{D}$. This fact is then used to show that $\mathrm{Th}(\mathscr{D})$ is undecidable.

**4.1 Notation.** Let $\mathscr{L}$ be a usl. We use $\mathscr{L} \hookrightarrow^* \mathscr{D}$ to denote the assertion that $\mathscr{L}$ is isomorphic to an initial segment of $\mathscr{D}$.

Let $\mathscr{L} = \langle L, \leqslant, \vee \rangle$ be a usl such that $\mathscr{L} \hookrightarrow^* \mathscr{D}$, and let $f$ be the embedding map. Then $L$ is a lattice if and only if $f(L)$ is a lattice.

We have been using the words *ideal* and *initial segment* almost interchangeably in this chapter. We note that an ideal of $\mathscr{D}$ is an initial segment of $\mathscr{D}$ which is closed under $\cup$.

The embedding results of this chapter are corollaries of the following theorem.

**4.2 Theorem.** *Let $\mathscr{L}$ be a lattice which has a finite homogeneous lattice table. Then $\mathscr{L} \hookrightarrow^* \mathscr{D}$.*

*Proof.* Let $\mathscr{L} = \langle L, \leqslant, \vee, \wedge \rangle$ be a lattice with finite homogeneous table $\Theta$. Let $L = \{u_0, \ldots, u_n\}$ where $u_0$ and $u_n$ are, respectively, the least and greatest elements of $L$. Without loss of generality, we can renumber $\Theta$ so that $\{\alpha^{[n]}: \alpha \in \Theta\} = [0, p)$. We construct a function $g: N \to N$ and embed $\mathscr{L}$ into $\mathscr{D}$ with the embedding map sending $u_i$ to $g^{\langle i \rangle}$ for all $i \leqslant n$. By Lemma 1.4, it suffices to construct $g$ so that the following requirements are satisfied for each $e \in N$:

(1) $P_{e,i,j}: g^{\langle i\rangle} \not\leq_T \Phi_e^{g^{\langle j\rangle}}$, where $i,j \leq n$ and $u_i \not\leq u_j$.

(2) $Q_e$: If $\Phi_e^g$ is total, then $\Phi_e^g \equiv_T g^{\langle i\rangle}$ for some $i \leq n$.

Let $\mathscr{R} = \{R_e : e \in N\}$ be the set of all requirements mentioned in (1) and (2).

Our notion of forcing is $\langle \mathscr{T}_p, \subseteq \rangle$ where $\mathscr{T}_p$ is the class of all recursive uniform $p$-branching trees and $\subseteq$ is the subtree relation. We say that $T$ forces $R_e$ (write $T \Vdash R_e$) if every branch $g$ of $T$ satisfies $R_e$. For each $R \in \mathscr{R}$, let $C_R = \{T \in \mathscr{T}_p : T \Vdash R\}$ and let $\mathscr{C} = \{C_R : R \in \mathscr{R}\}$. We show that each $C_R$ is a dense set.

Let $T \in \mathscr{T}_p$, $e \in N$ and $i,j \leq n$ be given such that $u_i \not\leq u_j$. By Lemma 2.13, there is an $\langle e,i,j\rangle$-differentiating tree $T^* = \mathrm{Diff}_p(T,e,i,j) \subseteq_u T$ such that $T^* \in \mathscr{T}_p$. Since $T^*$ is $\langle e,i,j\rangle$-differentiating, $T^* \Vdash P_{e,i,j}$.

Let $T \in \mathscr{T}_p$ and $e \in N$ be given. Assume first that

(3) $\forall \sigma \in \mathscr{S}_p \forall x \in N \exists \tau \in \mathscr{S}_p(\tau \supseteq \sigma \;\&\; \Phi_e^{T(\tau)}(x)\downarrow)$.

By Lemma 3.12, there is then an $i \leq n$ and a tree $T^* \subseteq_u T$ such that $T^*$ is $e$-splitting for $i$. It then follows from the Computation Lemma that $T^* \Vdash Q_e$. If (3) fails to hold, then there are $x \in N$ and $\sigma \in \mathscr{S}_p$ such that for all $\tau \in \mathscr{S}_p$, if $\tau \supseteq \sigma$ then $\Phi_e^{T(\tau)}(x)\uparrow$. Fix such a $\sigma$ and let $T^* = \mathrm{Ext}_p(T,\sigma)$. By Remark 2.9, $T^* \subseteq_u T$ so $T^* \in \mathscr{T}_p$. In this case, for all $g \subset T^*$, $\Phi_e^g$ is not total, so $T^* \Vdash Q_e$.

We have thus shown that $\mathscr{C}$ is a collection of dense sets. By Theorem II.2.8, there exists a $\mathscr{C}$-generic set $G$. By Lemma V.1.9, we can choose a function $g \in \bigcap G$. By the definition of $\Vdash$ on $\mathscr{R}$, all requirements are satisfied by this $g$. ▯

The proof of Theorem 4.2 can be modified, using techniques introduced in Chap. V, to yield more information about embeddings $\mathscr{L} \hookrightarrow^* \mathscr{D}$. Results of this form are left as exercises for the reader.

The diamond lattice is an example of a lattice which satisfies the hypothesis of Theorem 4.2. Additional examples are now discussed.

**4.3 Definition.** The lattice $\langle L, \leq, \vee, \wedge \rangle$ is *distributive* if for all $x,y,z \in L$:

(i) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

(ii) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

The following theorem is proved in Appendix B.1.11.

**4.4 Theorem.** *Let $\mathscr{L}$ be a finite distributive lattice. Then $\mathscr{L}$ has a finite homogeneous lattice table.*

Theorem 4.2 and Theorem 4.4 combine to yield the following corollary.

**4.5 Corollary.** *Let $\mathscr{L}$ be a finite distributive lattice. Then $\mathscr{L} \hookrightarrow^* \mathscr{D}$.*

Other lattices to which Theorem 4.2 applies can be found in the exercises and in Thomason [1970].

Corollary 4.5 yields important information about the decidability of $\mathrm{Th}(\mathscr{D})$.

**4.6 Theorem.** $\mathrm{Th}(\mathscr{D})$ *is undecidable.*

*Proof.* Let $\mathscr{L}^{\#}$ be the language for finite posets introduced in Chap. II.3.8. Then there is a formula $\theta(x)$ of $\mathscr{L}^{\#}$ which is satisfied in $\mathscr{D}$ by $\mathbf{d} \in \mathbf{D}$ if and only if $\mathscr{D}[\mathbf{0}, \mathbf{d}]$ is a distributive lattice. Let $S$ be the set of all sentences of $\mathscr{L}^{\#}$ which are true of all distributive lattices, and let $F$ be the set of all sentences of $\mathscr{L}^{\#}$ which are true of all finite distributive lattices. Then $S \subseteq F$. Ershov and Taitslin [1963] have shown that there is no recursive set $R$ such that $S \subseteq R \subseteq F$.

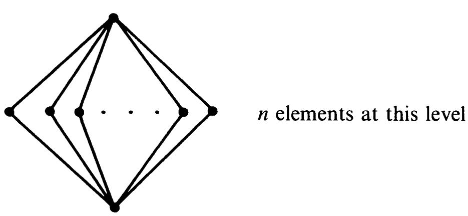
Let $\sigma$ be any sentence of $\mathscr{L}^{\#}$. Let $\sigma'(x)$ be the formula obtained from $\sigma$ by restricting all quantifiers of $\sigma$ to elements $\leqslant x$, and let $\sigma''$ be the sentence $\forall x(\theta(x) \to \sigma'(x))$. Let $H = \{\sigma : \mathscr{D} \models \sigma''\}$. By Corollary 4.5, $S \subseteq H \subseteq F$. Thus Th($\mathscr{D}$) must be undecidable, else $H$ would be recursive. ▯

It will follow from Exercise 4.17 that the sets $H$ and $S$ in the proof of Theorem 4.6 are identical.

**4.7 Remarks.** Theorem 4.2 is due to Thomason [1970]. Corollary 4.5 and Theorem 4.6 were proved by Lachlan [1968].

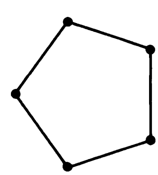
Thomason's proof of Theorem 4.2 was the first result about initial segments of $\mathscr{D}$ to explicitly focus on the use of representations of lattices, and so paved the way for further theorems of the form $\mathscr{L} \hookrightarrow^* \mathscr{D}$ which we prove in the next two chapters. Much work on theorems of the form $\mathscr{L} \hookrightarrow^* \mathscr{D}$ was done in the period between Spector [1956] and Thomason [1970], and most of these results were subsumed by Thomason's results. With the exception of Lerman [1969], these results used binary trees, so clause (ii) of Definition 2.3 (uniform trees) was vacuous, but clause (iii) was crucial in that setting. Large numbers of branches at a given node of a tree were thus not used; rather, many levels of a binary tree were used to recover functions having degree in the initial segment. Thus for the diamond lattice, if $g$ was the degree constructed, then $g_1$ was defined by $g_1(x) = g(2x)$ for all $x \in N$, and $g_2$ similarly was just the odd part of $g$.

We list the initial segments results which preceded Thomason [1970] with references. The three element chain; Titgemeyer [1962]. The diamond lattice; Sacks [1963]. Finite boolean algebras and some infinite ones; Rosenstein [1968] and Shoenfield (unpublished). Countable linearly ordered sets; Hugill [1969]. Countable distributive lattices; Lachlan [1968]. $1-3-1$ (see Fig. 4.1); Shoenfield (unpublished). $1-n-1$ where $n-1$ is a prime power; Lerman [1969]. The pentagon (see Fig. 4.2); Lerman [1969].




$1-n-1$ (or the $n$th chinese lantern)

Fig. 4.1



pentagon

Fig. 4.2

The embedding of the pentagon as an initial segment of $\mathscr{D}$ used a weakly homogeneous lattice table. Thomason notes that his proof can be modified to replace homogeneity with weak homogeneity, but this was still not sufficient to prove $\mathscr{L} \hookrightarrow^* \mathscr{D}$ for all finite lattices $\mathscr{L}$. We prove this theorem in the next section, passing to sequential tables for lattices.

**4.8–4.18 Exercises**

**4.8** Let $\mathscr{L} = \langle L, \leqslant, \vee, \wedge\rangle$ be a lattice which has a finite homogeneous lattice table. Show that there is an isomorphism $\theta$ taking $L$ onto an initial segment of $\mathbf{D}$ such that if $u_n$ is the greatest element of $L$ and $\mathbf{d} = \theta(u_n)$ then $\mathbf{d}^{(2)} = \mathbf{0}^{(2)}$ and $\mathbf{d} \notin \mathbf{GL}_1$.

**4.9** Let $\mathscr{L}$ be a lattice with a finite homogeneous lattice table. Show that there are $2^{\aleph_0}$ distinct embeddings $\mathscr{L} \hookrightarrow^* \mathscr{D}$.

Let $\mathscr{L}$ be a lattice and let $\mathbf{d} \in \mathbf{D}$ be given. We call $\mathbf{a} \in \mathbf{D}$ an *$\mathscr{L}$-cover* of $\mathbf{d}$ if $\mathscr{L} \simeq \mathscr{D}[\mathbf{d}, \mathbf{a}]$.

**4.10** Let $\mathscr{L}$ be a lattice which has a finite homogeneous lattice table and let $\mathbf{d} \in \mathbf{D}$ be given. Show that there is an $\mathscr{L}$-cover $\mathbf{a}$ of $\mathbf{0}$ such that $\mathbf{a}^{(2)} = \mathbf{a} \cup \mathbf{0}^{(2)} = \mathbf{d} \cup \mathbf{0}^{(2)}$.

**4.11** Let $\mathscr{L}$ be a lattice which has a finite homogeneous lattice table and let $\mathbf{c} \in \mathbf{D}$ be given. Show that there is an $\mathscr{L}$-cover $\mathbf{a}$ of $\mathbf{c}$ such that $\mathbf{a}^{(2)} = \mathbf{c}^{(2)}$ and $\mathbf{a} \notin \mathbf{GL}_1(\mathbf{c})$.

**4.12** Let $\mathscr{L}$ be a lattice which has a finite homogeneous lattice table and let $\mathbf{d} \in \mathbf{D}$ be given. Show that $\mathbf{d}$ has $2^{\aleph_0}$ distinct $\mathscr{L}$-covers.

**4.13** Let $\mathscr{L}$ be a lattice which has a finite homogeneous lattice table and let $\mathbf{c}, \mathbf{d} \in \mathbf{D}$ be given. Show that there is an $\mathscr{L}$-cover $\mathbf{a}$ of $\mathbf{c}$ such that $\mathbf{a}^{(2)} = \mathbf{a} \cup \mathbf{c}^{(2)} = \mathbf{d} \cup \mathbf{c}^{(2)}$.

**4.14** Let $\mathbf{d} \in \mathbf{D}$ be given. Show that the following theories are undecidable:

(i) $\mathrm{Th}(\mathscr{D}[\mathbf{d}, \infty))$.

(ii) $\mathrm{Th}(\mathscr{D}[\mathbf{d}, \mathbf{d}^{(2)}))$.

(iii) $\mathrm{Th}(\langle \mathbf{D}^{(2)}(\mathbf{d}), \leqslant\rangle)$ where $\mathbf{D}^{(2)}(\mathbf{d}) = \{\mathbf{c} \in \mathbf{D} : \mathbf{c} \geqslant \mathbf{d} \;\&\; \mathbf{c}^{(2)} = \mathbf{d}^{(2)}\}$.

(iv) $\mathrm{Th}(\langle \mathbf{D}^*, \leqslant\rangle)$ where $\mathbf{D}^*$ is any subset of $\mathbf{D}$ for which $\mathbf{D}^{(2)}(\mathbf{d}) \subseteq \mathbf{D}^*$ for some $\mathbf{d} \in \mathbf{D}$.

Let $\mathscr{L}$ be a lattice with least element 0 and let $\mathbf{I}$ be a countable ideal of $\mathscr{D}$. We call $\mathbf{a} \in \mathbf{D}$ an $\mathscr{L}$-cover of $\mathbf{I}$ if $\mathscr{L} - \{0\} \simeq \langle \mathbf{D}(\mathbf{I}, \mathbf{a}], \leqslant\rangle$ where

$$\mathbf{D}(\mathbf{I}, \mathbf{a}] = \{\mathbf{d} \in \mathbf{D} : \mathbf{d} \leqslant \mathbf{a} \;\&\; \forall \mathbf{c} \in \mathbf{I}(\mathbf{d} > \mathbf{c})\}.$$

**4.15** Let $\mathscr{L}$ be a lattice with a finite homogeneous lattice table and let $\mathbf{I}$ be a countable ideal of $\mathscr{D}$. Show that $\mathbf{I}$ has an $\mathscr{L}$-cover.

**4.16** Let $\mathscr{L}$ be a lattice with a finite homogeneous lattice table and let $\mathbf{I}$ be a countable ideal of $\mathscr{D}$. Show that $\mathbf{I}$ has $2^{\aleph_0}$ distinct $\mathscr{L}$-covers.

***4.17** Let $\mathscr{L} = \langle L, \leqslant, \vee, \wedge\rangle$ be a countable distributive lattice with least element $u_0$ and greatest element $u_1$. For each $i \in N$, let $\mathscr{L}_i = \langle L_i, \leqslant, \vee, \wedge\rangle$ be a finite sublattice of $\mathscr{L}$ extending $\mathscr{L}_{i-1}$ and containing $u_0$ and $u_1$ such that

$L = \bigcup\{L_i : i \in N\}$. Given a lattice table $\Theta$ for $\mathscr{L}_{i+1}$, we let

$$\Theta \upharpoonright i = \{\langle \alpha^{[0]}, \ldots, \alpha^{[n(i)]}\rangle : \alpha \in \Theta\},$$

where $L_i = \{u_0, \ldots, u_{n(i)}\}$. A *sequential table* for $\mathscr{L}$ has the form $\{\Theta_{i,j} : i, j \in N\}$ where

(i) $\forall i, j \in N(\Theta_{i,j}$ is a finite homogeneous lattice table for $\mathscr{L}_i)$.

(ii) $\forall i \in N \exists j_0 \in N \forall j \geqslant j_0(\Theta_{i+1,j} \upharpoonright i \subseteq \Theta_{i,j})$.

(iii) $\forall i \in N(\{\langle \alpha, j\rangle : \alpha \in \Theta_{i,j}\}$ is recursive).

In Exercise 1.12 of Appendix B, we show that every countable distributive lattice with least and greatest elements has a sequential table. Use this fact to show that $\mathscr{L} \hookrightarrow^* \mathscr{D}$. (*Hint*: For all $i, j \in N$, define $f_i : N \to N$ by $f_i(j) = |\Theta_{i,j}|$. List all requirements, and satisfy the $n$th requirement using an $f_n^*$-branching tree where $f_n^* = f_n$ for all but finitely many $n$.)

**4.18** Let $n \in N$ be given such that $n = p^k - 1$ for some prime number $p$ and some $k \in N$. Show that $1 - n - 1 \hookrightarrow^* \mathscr{D}$. (*Hint*: It suffices, by Theorem 4.2, to show that $1 - n - 1$ has a finite homogeneous lattice table. Such a table can be obtained from the projective plane of order $p^k$ by numbering the points of the projective plane and letting two points be in the same $\equiv_i$ class if they both lie on the same line going through the $i$th point on the line at $\infty$. The line at $\infty$ does not correspond to such an equivalence relation. Two additional equivalence relations are added, one in which no two points are equivalent, and another where all points are equivalent. Schmerl has noted that an alternate proof can be given utilizing the finite field of order $p^k$. Number all pairs of point in the field. The $\equiv_i$ classes for $i$ in the field are defined by $\langle a, b\rangle \equiv_i \langle c, d\rangle$ if $a - c = i(b - d)$, together with an equivalence relation under which all points are equivalent.)

## 5. *An Automorphism Base for* $\mathscr{D}$

The methods introduced in this chapter will now be extended to show that the minimal degrees form an automorphism base for $\mathscr{D}$. We will, in fact, show that the minimal degrees generate $\mathbf{D}$ by proving that every $\mathbf{d} \in \mathbf{D}$ can be expressed as $(\mathbf{m_0} \cup \mathbf{m_1}) \cap (\mathbf{m_2} \cup \mathbf{m_3})$ where $\{\mathbf{m_i} : i \leqslant 3\}$ is a set of minimal degrees.

The fact that the minimal degrees generate $\mathbf{D}$ is a corollary of the following result: Given $\mathbf{a}, \mathbf{d} \in \mathbf{D}$ such that $\mathbf{a} \leqslant \mathbf{d}$, there are minimal degrees $\mathbf{m_0}$ and $\mathbf{m_1}$ such that $\mathbf{a} = \mathbf{d} \cap (\mathbf{m_0} \cup \mathbf{m_1})$. Let $A$ and $D$ be sets of degree $\mathbf{a}$ and $\mathbf{d}$ respectively. The minimal degree construction is modified to simultaneously construct sets $M_0$ and $M_1$ of minimal degree such that $A \leqslant_T M_0 \oplus M_1$ which satisfy the following requirements for $e, n \in N$:

$$R_{e,n} : \Phi_e^D = \Phi_n^{M_0 \oplus M_1} \ \& \ \Phi_e^D \text{ total} \rightarrow \Phi_e^D \leqslant_T A.$$

The condition, $A \leqslant_T M_0 \oplus M_1$, will be satisfied as follows: Let $Z = \{z \in N: M_0(z) \neq M_1(z)\}$ and let $\{z_i: i \in N\}$ be an enumeration of $Z$ in order of magnitude. $M_0$ and $M_1$ are constructed to satisfy the following requirements for $e \in N$:

$$S_e: e \in A \leftrightarrow M_0(z_e) = 1.$$

The satisfaction of $\{S_e: e \in N\}$ clearly implies that $A \leqslant_T M_0 \oplus M_1$. Special kinds of trees are introduced in order to satisfy these requirements. All trees in this section are binary trees.

**5.1 Definition.** Let $\tau, \rho \in \mathscr{S}_2$ be given. We say that $\langle \tau, \rho \rangle$ is *strongly uniform* if:

(i) $\mathrm{lh}(\tau) = \mathrm{lh}(\rho)$.

(ii) $\{x \in N: \tau(x)\downarrow \,\dot{\neq}\, \rho(x)\downarrow\}$ has exactly one element.

**5.2 Definition.** The tree $T$ is *strongly uniform* if $T$ is uniform and for all $\langle \sigma, \xi, \eta \rangle \in \mathscr{S}_2$, if $T(\sigma * 0) = T(\sigma) * \xi$ and $T(\sigma * 1) = T(\sigma) * \eta$ then $\langle \xi, \eta \rangle$ is strongly uniform.

Strongly uniform trees are used to enable us to satisfy $\{S_e: e \in N\}$. For if $T$ were not strongly uniform, we would not have the flexibility to determine $M_0(z_e)$ for some $e \in N$ without simultaneously determining $M_0(z_{e+1})$, possibly incorrectly insofar as the recovery of $A$ from $M_0 \oplus M_1$ is concerned.

Some of the trees introduced in previous sections are strongly uniform, and will be used again in this section.

**5.3 Remark.** (i) $\mathrm{Id}_2$ is strongly uniform.

(ii) If $T$ is strongly uniform and $\sigma \in \mathscr{S}_2$ then $\mathrm{Ext}_2(T, \sigma)$ is strongly uniform.

(iii) If $T$ is strongly uniform and $\mathrm{Tot}_2(T, e)$ exists, then $\mathrm{Tot}_2(T, e)$ is strongly uniform.

The forcing conditions which will be used are pairs of strongly uniform trees which look alike except for their values on $\emptyset$. To pass from one half of the pair to the other, we use the following kind of tree, which just modifies the definition of the first tree on $\emptyset$.

**5.4 Definition.** Let $T$ be a tree and let $\sigma \in \mathscr{S}_2$ be given such that $\mathrm{lh}(\sigma) = \mathrm{lh}(T(\emptyset))$. Define the tree $\mathrm{Tr}(T, \sigma)$ by

$$\mathrm{Tr}(T, \sigma)(\xi) = \mathrm{tr}(T(\emptyset) \to \sigma; T(\xi)).$$

**5.5 Remark.** It is easily verified that $\mathrm{Tr}(T, \sigma)$ is a tree, and that if $T$ is strongly uniform then $\mathrm{Tr}(T, \sigma)$ is strongly uniform. Furthermore, for all $h: N \to N$, if $T$ is recursive in $h$ then $\mathrm{Tr}(T, \sigma)$ is recursive in $h$.

As we define a new pair of trees to force a new requirement, we must make sure that we have not lost the ability to satisfy $\{S_e: e \in N\}$. Thus we only allow the use of *A-acceptable* pairs of trees.

**5.6 Definition.** Let $\sigma, \tau \in \mathscr{S}_2$ be given such that $\mathrm{lh}(\sigma) = \mathrm{lh}(\tau)$. Let $Z(\sigma, \tau) = \{x \in N : \sigma(x)\downarrow \neq \tau(x)\downarrow\}$ and let $\{z_i : i \leqslant k\}$ be an enumeration of $Z(\sigma, \tau)$ in order of magnitude. We then say that $\langle \sigma, \tau \rangle$ is *A-consistent* if for all $i \leqslant k$, $i \in A \leftrightarrow \sigma(z_i) = 1$. Let $M_0, M_1 \subseteq N$. We say that $\langle M_0, M_1 \rangle$ is *A-consistent* if for all $\sigma_0, \sigma_1 \in \mathscr{S}_2$ such that $\mathrm{lh}(\sigma_0) = \mathrm{lh}(\sigma_1)$, $\sigma_0 \subseteq M_0$ and $\sigma_1 \subseteq M_1$, $\langle \sigma_0, \sigma_1 \rangle$ is $A$-consistent.

**5.7 Definition.** Let $T_0$ and $T_1$ be trees and let $A \subseteq N$ be given. We say that $\langle T_0, T_1 \rangle$ is *A-acceptable* if the following conditions hold:
- (i) $\langle T_0(\emptyset), T_1(\emptyset) \rangle$ is $A$-consistent.
- (ii) $T_0$ is strongly uniform.
- (iii) $T_1 = \mathrm{Tr}(T_0, T_1(\emptyset))$.

The following remark will be useful in obtaining $A$-acceptable pairs of trees.

**5.8 Remark.** Let $\langle T_0, T_1 \rangle$ be an $A$-acceptable pair of trees, and let $T_i^*$ be a strongly uniform subtree of $T_i$ for a fixed $i \in \{0, 1\}$. Define

$$T_{1-i}^* = \mathrm{Tr}(T_i^*, \mathrm{tr}(T_i(\emptyset) \to T_{1-i}(\emptyset); T_i^*(\emptyset))).$$

Then $T_{1-i}^*$ is a strongly uniform subtree of $T_{1-i}$ and $\langle T_0^*, T_1^* \rangle$ is an $A$-acceptable pair of trees. Furthermore, if $T_i^*$ is recursive, then $T_{1-i}^*$ is recursive.

The use of strongly uniform trees complicates the proof that the requirements $\{Q_{e,i} : e \in N \,\&\, i \leqslant 1\}$ are satisfied, where

$$Q_{e,i}\colon \text{If } \Phi_e^{M_i} \text{ is total, then either } \Phi_e^{M_i} \text{ is recursive or } M_i \leqslant_T \Phi_e^{M_i}.$$

The other requirements which must be satisfied are the following for $e \in N$ and $i \leqslant 1$:

$$P_{e,i}\colon M_i \neq \Phi_e.$$

The next lemma is used to show that all requirements in $\{S_e : e \in N\}$ can be satisfied. We first define the ordering which is placed on acceptable pairs of trees.

**5.9 Definition.** Let $\langle T_0, T_1 \rangle$ and $\langle T_0^*, T_1^* \rangle$ be $A$-acceptable pairs of trees. Then $\langle T_0^*, T_1^* \rangle \subseteq \langle T_0, T_1 \rangle$ if $T_0^* \subseteq T_0$ and $T_1^* \subseteq T_1$.

**5.10 Lemma.** *Let $e \in N$ be given, and let $\langle T_0, T_1 \rangle$ be an $A$-acceptable pair of recursive trees. Then there is an $A$-acceptable pair $\langle T_0^*, T_1^* \rangle \subseteq \langle T_0, T_1 \rangle$ of recursive trees such that $Z(T_0^*(\emptyset), T_1^*(\emptyset))$ has at least e elements.*

*Proof.* We proceed by induction on $\{e : e \in N\}$. The lemma holds trivially for $e = 0$. Let $\langle T_0, T_1 \rangle$ be an $A$-acceptable pair of recursive trees. By induction, we may assume that $Z = Z(T_0(\emptyset), T_1(\emptyset))$ has at least $e - 1$ elements. If $Z$ has at least $e$ elements, then we are done. Otherwise, fix the unique $x < \mathrm{lh}(T_0(0))$ such that $T_0(0)(x) \neq T_1(0)(x)$. For $i \leqslant 1$, define

$$\sigma_0 = \begin{cases} 0 & \text{if } (e - 1 \in A \,\&\, T_0(0)(x) = 1) \text{ or } (e - 1 \notin A \,\&\, T_0(0)(x) = 0) \\ 1 & \text{otherwise,} \end{cases}$$

and define $T_0^* = \mathrm{Ext}_2(T_0, \sigma_0)$ and $T_1^* = \mathrm{Ext}_2(T_0, 1 - \sigma_0)$. We leave it to the reader to verify that, for $i \leqslant 1$, $T_i^*$ has the desired properties. ▯

The strong uniformity of $\langle T_0, T_1 \rangle$ was crucial in the proof of Lemma 5.10. For otherwise, it would have been possible for $Z$ to have $e - 1$ elements while $Z^* = Z(T_0^*(\emptyset), T_1^*(\emptyset))$ could have more than $e$ elements; thus the coding of $A(e-1)$ into $Z^*$ might force us to code $e$ into $Z^*$ incorrectly.

The next lemma is useful for showing that all requirements in $\{R_{e,n} : e, n \in N\}$ can be satisfied.

**5.11 Lemma.** *Let $e, n \in N$ be given, and let $\langle T_0, T_1 \rangle$ be an A-acceptable pair of recursive trees. Then there is an A-acceptable pair $\langle T_0^*, T_1^* \rangle$ of recursive trees such that $\langle T_0^*, T_1^* \rangle \subseteq \langle T_0, T_1 \rangle$ and*

(i) $\forall M_0, M_1 \subseteq N(\langle M_0, M_1 \rangle$ *A-consistent* $\& M_0 \subseteq T_0 \& M_1 \subseteq T_1 \rightarrow$ $R_{e,n}$ *is satisfied by* $M_0$ *and* $M_1$.

*Proof.* We may assume that $\Phi_e^D$ is total, else the lemma follows once we set $T_i^* = T_i$ for $i \leqslant 1$. We proceed by cases.

*Case 1.* There are $x \in N$ and $\sigma_0, \sigma_1 \in \mathscr{S}_2$ such that $\mathrm{lh}(\sigma_0) = \mathrm{lh}(\sigma_1)$, $\langle T_0(\sigma_0), T_1(\sigma_1) \rangle$ is $A$-consistent, and $\Phi_n^{T_0(\sigma_0) \oplus T_1(\sigma_1)}(x)\downarrow \neq \Phi_e^D(x)$. Let $T_i^* = \mathrm{Ext}_2(T_i, \sigma_i)$ for $i \leqslant 1$. It is easily verified that $\langle T_0^*, T_1^* \rangle$ has the desired properties.

*Case 2.* Otherwise. Set $T_i^* = T_i$ for $i \leqslant 1$. Fix $M_0$ and $M_1$ as in the hypothesis of (i). If $\Phi_n^{M_0 \oplus M_1}$ is not total, then (i) holds. Suppose that $\Phi_n^{M_0 \oplus M_1}$ is total. Fix $x \in N$. To compute $\Phi_e^D(x)$, search for $\sigma_0, \sigma_1 \in \mathscr{S}_2$ such that $\mathrm{lh}(\sigma_0) = \mathrm{lh}(\sigma_1)$, $\langle T_0(\sigma_0), T_1(\sigma_1) \rangle$ is $A$-acceptable, and $\Phi_n^{T_0(\sigma_0) \oplus T_1(\sigma_1)}(x)\downarrow$. Such $\sigma_0$ and $\sigma_1$ must be found since $\Phi_n^{M_0 \oplus M_1}$ is total, and can be found recursively in $A$. Since Case 1 was not followed, $\Phi_e^D(x) = \Phi_n^{T_0(\sigma_0) \oplus T_1(\sigma_1)}(x)$, so $\Phi_e^D \leqslant_T A$. ▯

Proposition 5.15 which follows shortly is used to show that all requirements in $\{Q_{e,i} : e \in N \& i \leqslant 1\}$ can be satisfied. It will suffice to prove the following: Given $e, n \in N$ and a strongly uniform recursive $e$-total tree $T$, there is a strongly uniform recursive $T^* \subseteq T$ such that either $T^*$ is an $e$-splitting tree or there are no $e$-splittings of $T^*(\emptyset)$ on $T^*$. Similar theorems were proved earlier in the book without the requirement that $T^*$ be strongly uniform, a condition which complicates the proof. Instead of proceeding by cases, we try to construct a strongly uniform $e$-splitting $T^* \subseteq T$, and if we fail, we show that there is a strongly uniform recursive $T^* \subseteq T$ with no $e$-splittings of $T^*(\emptyset)$ on $T^*$. Before giving this construction, we present some useful definitions and a lemma.

**5.12 Definition.** Let $m, e \in N$, $\tau, \rho \in \mathscr{S}_2$, and $S \subseteq \mathscr{S}_2$ be given such that for all $\xi \in S$, $m = \mathrm{lh}(\xi) < \mathrm{lh}(\tau) = \mathrm{lh}(\rho)$. We say that $\langle \tau, \rho \rangle$ *induces a simultaneous e-splitting for S* if for all $\xi \in S$, $\langle \mathrm{tr}(\tau \upharpoonright m \rightarrow \xi; \tau), \mathrm{tr}(\rho \upharpoonright m \rightarrow \xi; \rho) \rangle$ $e$-splits $\xi$.

The construction of an $e$-splitting subtree $T^*$ of $T$ will proceed level by level. To construct level $n$ of $T^*$, we will need a simultaneous strongly uniform $e$-splitting on $T$ of $\{T^*(\sigma) : \mathrm{lh}(\sigma) = n\}$. We construct such a simultaneous $e$-splitting gradually, adding one more $\sigma$ such that $\mathrm{lh}(\sigma) = n$ to the set which is simultaneously $e$-split at each step. We construct a *right e-splitting tree* for the small set on which we simultaneously $e$-split, and show that if we cannot enlarge the simultaneously $e$-split set, then we can find a subtree of $T$ on which there are no $e$-splittings.

**5.13 Definition.** Let $T$ be a strongly uniform tree, and let $e \in N$ and $S \subseteq \mathscr{S}_2$ be given. Assume that for all $\xi, \eta \in S$, $\mathrm{lh}(\xi) = \mathrm{lh}(\eta)$. Let $T(S) = \{T(\xi): \xi \in S\}$, and for each $k \in N$, let $0_k$ be the string of length $k$ such that $0_k(x) = 0$ for all $x < k$. We say that $T$ is *right e-splitting for S* if for all $n \in N$ and $\xi \in S$, $\langle T(\xi * 0_n * 0), T(\xi * 0_n * 1)\rangle$ is an $e$-splitting.

The following lemma is the crucial lemma used to show that we can satisfy all requirements in $\{Q_{e,i}: e \in N \,\&\, i \leqslant 1\}$.

**5.14 Lemma.** *Let $T$ be a strongly uniform recursive e-total tree, and let $e, m \in N$, $\sigma \in \mathscr{S}_2$, and $S \subseteq \mathscr{S}_2$ be given. Assume that $T$ is right e-splitting for $S$ and that for all $\xi \in S$, $\mathrm{lh}(\xi) = \mathrm{lh}(\sigma) = m$. Assume also that there is no strongly uniform e-splitting of $T(\sigma)$ on $T$ which induces a simultaneous e-splitting for $T(S)$. Let $T^* = \mathrm{Ext}_2(T, \sigma)$. Then there are no e-splittings on $T^*$.*

*Proof.* Fix $e, m, S, T$ and $\sigma$ as in the hypothesis of the lemma. It suffices to show that there are no $e$-splittings of $T(\sigma)$ on $T$. We refer the reader to Fig. 5.1. Assume that such $e$-splittings exist in order to obtain a contradiction. Let $\langle T(\tau), T(\rho)\rangle$ be an $e$-splitting of $T(\sigma)$ on $x$. Since $T$ is $e$-total, there must be an $n \in N$ such that $n > m$ and for all $\lambda \in \mathscr{S}_2$ such that $\mathrm{lh}(\lambda) = n$, $\Phi_e^{T(\lambda)}(x)\downarrow$. Without loss of generality, we may assume that $\mathrm{lh}(\tau) = \mathrm{lh}(\rho) = n$. Fix $k \in N$ such that $m + k = n$. Fix the largest $j < k$ for which there are $\xi, \eta \in \mathscr{S}_2$ such that

$$(1) \qquad \xi \supseteq \sigma * 0_j * 0, \quad \eta \supseteq \sigma * 0_j * 1, \quad \mathrm{lh}(\xi) = \mathrm{lh}(\eta) = n \quad \text{and} \quad \langle T(\xi), T(\eta)\rangle$$
$$e\text{-splits } T(\sigma) \text{ on } x.$$

Note that such a $j$ must exist since one of $\langle T(\sigma * 0_k), T(\tau)\rangle$ or $\langle T(\sigma * 0_k), T(\rho)\rangle$ $e$-splits $T(\sigma)$ on $x$. Fix $\xi$, $\eta \in \mathscr{S}_2$ which satisfy (1). Let $\tau^* = \mathrm{tr}(\eta \restriction m + j + 1 \to \xi \restriction m + j + 1; \eta)$ and let $\rho^* = \eta$. By choice of $j, \langle T(\tau^*), T(\xi)\rangle$ cannot $e$-split $T(\sigma)$ on $x$, hence by choice of $n$, $\langle T(\tau^*), T(\rho^*)\rangle$ must $e$-split $T(\sigma)$ on $x$. Note that $\langle T(\tau^*), T(\rho^*)\rangle$ is a strongly uniform $e$-splitting of $T(\sigma)$. Furthermore, for all $\delta \in S$, $\delta * 0_j * 0 \subseteq \mathrm{tr}(\tau^* \restriction m \to \delta; \tau^*)$ and $\delta * 0_j * 1 \subseteq \mathrm{tr}(\rho^* \restriction m \to \delta; \rho^*)$, so as $T$ is

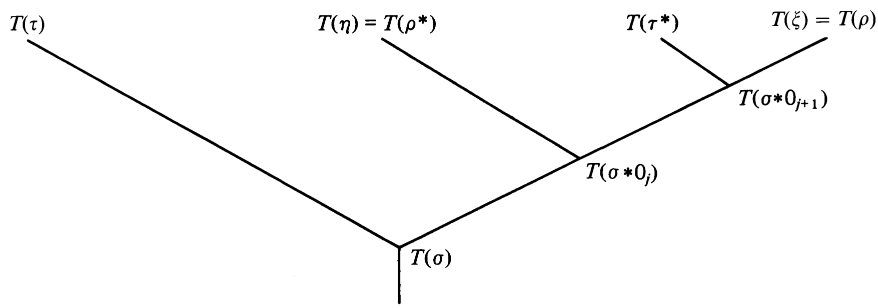



Fig. 5.1

right $e$-splitting for $S$, $\langle T(\tau^*), T(\rho^*)\rangle$ is a strongly uniform $e$-splitting of $T(\sigma)$ which induces a simultaneous $e$-splitting for $T(S)$. This contradicts the hypothesis of the lemma. ▯

We now show that all requirements in $\{Q_{e,i}: e \in N \,\&\, i \leqslant 1\}$ can be satisfied.

**5.15 Proposition.** *Let $T$ be a strongly uniform recursive tree and let $e \in N$ and $i \leqslant 1$ be given. Then there is a strongly uniform recursive subtree $T^*$ of $T$ such that one of the following conditions holds:*

(i) $\forall M \subseteq T^*(\Phi_e^M$ *is not total*$)$.

(ii) $\forall M \subseteq T^*(\Phi_e^M$ *is recursive*$)$.

(iii) $\forall M \subseteq T^*(M \leqslant_T \Phi_e^M)$.

*Proof.* Fix $e, i$ and $T$ as in the hypothesis of the proposition. We will construct $T^*$ level by level, and, at level $s$, will define a set $S_s$ of strings of length $s$. As long as we are able to make $T^*$ a strongly uniform $e$-splitting tree through level $s$, $S_s$ will be $\{\sigma \in \mathscr{S}_2: \mathrm{lh}(\sigma) = s\}$. If we first discover, at level $s$, that it is impossible to make $T^*$ a strongly uniform $e$-splitting tree, we specify a maximal subset $S_s$ of $\{\sigma \in \mathscr{S}_2: \mathrm{lh}(\sigma) = s\}$ for which we can find a strongly uniform $e$-splitting $\langle T^*(\sigma * 0), T^*(\sigma * 1)\rangle$ simultaneous for $T^*(S_s)$. We then try to construct $T^*$ so that for all $\sigma \in S_s$ and $k \geqslant 0$, $\mathrm{Ext}_2(T^*, \sigma * 0_k)$ is a right $e$-splitting tree. As long as this is possible at levels above $s$, we let $S_t = \{\sigma * 0_{t-s}: \sigma \in S_s\}$. If this becomes impossible, we find a maximal $S_t \subset \{\sigma * 0_{t-s}: \sigma \in S_s\}$ for which, for all $\sigma \in S_t$ and $k \geqslant 0$, $\mathrm{Ext}_2(T^*, \sigma * 0_k)$ is a right $e$-splitting tree through level $k$. We will then show that $\lim_s |S_s|$ exists and find a string $\sigma$ such that $\mathrm{Ext}_2(T, \sigma)$ has no $e$-splittings.

If there are $\sigma \in \mathscr{S}_2$ and $x \in N$ such that

(2) $\forall \tau \in \mathscr{S}_2(\tau \supseteq \sigma \to \Phi_e^{T(\tau)}(x)\uparrow)$

fix such a $\sigma$ and let $T^* = \mathrm{Ext}_2(T, \sigma)$. By Remark 5.3(ii) and Remark 2.9, $T^*$ is a strongly uniform recursive subtree of $T$. (i) follows immediately from (2).

If no $\sigma$ and $x$ satisfying (2) exist, then by Lemma 2.15 and Remark 5.3(iii), $T^+ = \mathrm{Tot}_2(T, e)$ is defined and is a strongly uniform recursive subtree of $T$. We proceed by induction on $N$, simultaneously constructing $T_s^*$, $S_s$ and $\sigma_s$ at stage $s \in N$ such that either $T^* = \bigcup\{T_s^*: s \in N\}$ satisfies (iii) or $\mathrm{Ext}_2(T^*, \sigma_s)$ satisfies (ii) in place of $T^*$ for some $s \in N$. Fix a recursive one-one correspondence $\{\langle \tau_i^0, \tau_i^1\rangle: i \in N\}$ of $N$ with $\mathscr{S}_2^2$. We will assume the following induction hypotheses at stage $s + 1$.

(3) $\forall \sigma \in \mathscr{S}_2(0 \leqslant \mathrm{lh}(\sigma) = s - 1 \to \langle T_s^*(\sigma * 0), T_s^*(\sigma * 1)\rangle$ is strongly uniform). (This condition and the next condition are required to witness the fact that $T_s^*$ can be extended to a strongly uniform tree.)

(4) $\forall \sigma, \tau \in \mathscr{S}_2 \forall i \leqslant 1 \forall x \in N(0 \leqslant \mathrm{lh}(\sigma) = \mathrm{lh}(\tau) = s - 1 \,\&\, \mathrm{lh}(T_s^*(\sigma)) \leqslant x < \mathrm{lh}(T_s^*(\sigma * i)) \to T_s^*(\sigma * i)(x) = T_s^*(\tau * i)(x) \,\&\, \mathrm{lh}(T_s^*(\sigma)) = \mathrm{lh}(T_s^*(\tau)))$.

(5) $\forall \sigma \in S_s(\mathrm{lh}(\sigma) = s)$.

(6) $\langle T_s^*(0_{s-1} * 0),\ T_s^*(0_{s-1} * 1)\rangle$ induces a simultaneous $e$-splitting for $T^*(S_s^-)$ where $S_s^- = \{\sigma^- : \sigma \in S_s\}$.

(7) If $\sigma_s$ is defined, then there is no strongly uniform $e$-splitting of $T_s^*(\sigma_s)$ on $T^+$ which induces a simultaneous $e$-splitting for $T_s^*(S_s^-)$.

*Stage* 0. Let $S_0 = \{\emptyset\}$ and let $\sigma_0$ be undefined. Define

$$T_0^*(\delta) = \begin{cases} T^+(\delta) & \text{if } \delta = \emptyset \\ \uparrow & \text{otherwise.} \end{cases}$$

(3)–(7) are easily verified for $s = 0$.

*Stage* $s + 1$. The path which the construction takes is determined by the truth or falsity of the following two conditions.

(8) $\exists \tau, \rho \in \mathscr{S}_2(\langle \tau, \rho\rangle$ is strongly uniform and $\langle T^+(\tau), T^+(\rho)\rangle$ induces a simultaneous $e$-splitting for $T^*(S_s))$.

(9) $S_s = \{\sigma : \text{lh}(\sigma) = s\}$.

*Case 1.* (8) is true. Fix the least $i \in N$ such that (8) holds for $\langle \tau_i^0, \tau_i^1\rangle$ in place of $\langle \tau, \rho\rangle$. Define

$$(10) \qquad T_{s+1}^*(\delta) = \begin{cases} T_s^*(\delta) & \text{if } \text{lh}(\delta) \leqslant s \\ T^+(\text{tr}(\tau_i^j \restriction s \to \delta^-; \tau_i^j)) & \text{if } \text{lh}(\delta) = s + 1 \,\&\, \delta = \delta^- * j \,\&\, j \leqslant 1 \\ \uparrow & \text{otherwise.} \end{cases}$$

If (9) is true, let $S_{s+1} = \{\sigma : \text{lh}(\sigma) = s + 1\}$ and if (9) is false, let $S_{s+1} = \{\sigma * 0 : \sigma \in S_s\}$. Let $\sigma_{s+1}$ be undefined. (3)–(7) are easily verified with $s + 1$ in place of $s$.

*Case 2.* (8) is false. Then there are $R \subset S_s$ and $\xi \in S_s - R$ such that (8) holds for $R$ in place of $S_s$ but (8) fails to hold for $R \cup \{\xi\}$ in place of $S_s$. Fix such $R$ and $\xi$. Proceed as in Case 1, stopping after (10) and replacing $S_s$ with $R$. Let $\sigma_{s+1} = \xi$ and $S_{s+1} = \{\sigma * 0 : \sigma \in R\}$. (3)–(7) are easily verified with $s + 1$ in place of $s$.

By (3) and (4), $T^* = \bigcup\{T_s^* : s \in N\}$ is strongly uniform. If (8) and (9) are true at all stages $s > 0$, then by (6), $T^*$ is a strongly uniform recursive $e$-splitting tree, and so (iii) follows from the Computation Lemma. Otherwise, let $s > 0$ be the least stage at which either (8) or (9) is false. (9) cannot be false at stage $s$, else (8) would have been false at some stage $t < s$. We now note that (9) is false at all stages $t > s$. Hence for all $t \geqslant r > s$, $|S_t| \leqslant |S_r|$, so $\lim_t |S_t|$ exists. Fix the least stage $r$ such that for all $t \geqslant r$, $|S_t| = |S_r|$. Note that $\sigma_r$ is defined. By (6) and (7), $T^*$ is right $e$-splitting for $T^*(S_r^-)$ but there are no strongly uniform $e$-splittings of $T^*(\sigma_r)$ on $T^+$ which induce simultaneous $e$-splittings for $T^*(S_r^-)$. By Lemma 5.14, there are no $e$-splittings on $\text{Ext}_2(T^*, \sigma_r)$. Hence by the Computation Lemma, (ii) holds for $\text{Ext}_2(T^*, \sigma_r)$ in place of $T^*$. ▯

We use the next theorem to show that the set of minimal degrees generates **D**.

**5.16 Theorem.** *Let* $\mathbf{a}, \mathbf{d} \in \mathbf{D}$ *be given such that* $\mathbf{a} \leqslant \mathbf{d}$. *Then there are minimal degrees* $\mathbf{m}_0$ *and* $\mathbf{m}_1$ *such that* $\mathbf{a} = \mathbf{d} \cap (\mathbf{m}_0 \cup \mathbf{m}_1)$.

*Proof.* Fix sets $A \in \mathbf{a}$ and $D \in \mathbf{d}$. Let $\mathscr{R} = \{P_{e,i}: e \in N \,\&\, i \leqslant 1\} \cup \{Q_{e,i}: e \in N \,\&\, i \leqslant 1\} \cup \{R_{e,n}: e, n \in N\} \cup \{S_e: e \in N\}$. Our notion of forcing is $\mathscr{F} = \langle F, \subseteq \rangle$ where $F = \{\langle T_0, T_1 \rangle : \langle T_0, T_1 \rangle$ is an $A$-acceptable pair of recursive trees$\}$ and $\subseteq$ is defined as in Definition 5.9. Note that $\mathscr{F}$ is a poset with greatest element $\langle \mathrm{Id}_2, \mathrm{Id}_2 \rangle$.

For each $R \in \mathscr{R}$, define

$$\langle T_0, T_1 \rangle \Vdash R \leftrightarrow \forall M_0 \subseteq T_0 \, \forall M_1 \subseteq T_1 (\langle M_0, M_1 \rangle \ A\text{-consistent} \rightarrow M_0 \text{ and } M_1 \text{ satisfy } R).$$

For each $R \in \mathscr{R}$, let $C_R = \{\langle T_0, T_1 \rangle \in F : \langle T_0, T_1 \rangle \Vdash R\}$ and let $\mathscr{C} = \{C_R : R \in \mathscr{R}\}$. Assume, for the moment, that for each $R \in \mathscr{R}$, $C_R$ is a dense set. By the Existence Theorem for $\mathscr{C}$-generic Sets (Theorem II.2.8), we may choose a $\mathscr{C}$-generic set $G$. By Lemma V.1.9, there is an $A$-consistent pair $\langle M_0, M_1 \rangle$ such that for all $i \leqslant 1$, $M_i \subseteq T_i$ for all $\langle T_0, T_1 \rangle \in G$. Since $\langle M_0, M_1 \rangle$ is $A$-consistent and $S_e$ is satisfied for all $e \in N$, $A \leqslant_T M_0 \oplus M_1$. Since $R_{e,n}$ is satisfied for all $e, n \in N$, $\mathbf{d} \cap (\mathbf{m_0} \cup \mathbf{m_1}) \leqslant \mathbf{a}$ where $\mathbf{m_0}$ and $\mathbf{m_1}$ are the degrees of $M_0$ and $M_1$ respectively. Finally, since $P_{e,i}$ and $Q_{e,i}$ are satisfied for all $e \in N$ and $i \leqslant 1$, $M_0$ and $M_1$ are sets of minimal degree.

Finally, we verify that for all $R \in \mathscr{R}$, $C_R$ is dense. Fix $R \in \mathscr{R}$. If $R = R_{e,n}$, then $C_R$ is dense by Lemma 5.11. If $R = S_e$, then $C_R$ is dense by Lemma 5.10. Assume that $R = Q_{e,i}$ and let $\langle T_0, T_1 \rangle \in F$ be given. Let $T_i^*$ be a strongly uniform subtree of $T_i$ as in Proposition 5.15, and let

$$(11) \qquad T_{1-i}^* = \mathrm{Tr}(T_i^*, \mathrm{tr}(T_i(\emptyset) \rightarrow T_{1-i}(\emptyset); T_i^*(\emptyset))).$$

By Remark 5.8, $\langle T_0^*, T_1^* \rangle \subseteq \langle T_0, T_1 \rangle$ and $\langle T_0^*, T_1^* \rangle \in F$. By Proposition 5.15, $\langle T_0^*, T_1^* \rangle \Vdash Q_{e,i}$. Hence $C_R$ is dense in this case. Assume that $R = P_{e,i}$ and let $\langle T_0, T_1 \rangle \in F$ be given. Let $L$ be the two element lattice with $u_0 < u_1$. Let $T_i^* = \mathrm{Diff}_2(T_i, e, 1, 0)$ and define $T_{1-i}^*$ by (11). By Remark 5.8, $\langle T_0^*, T_1^* \rangle \subseteq \langle T_0, T_1 \rangle$ and $\langle T_0^*, T_1^* \rangle \in F$. (Note that since $\mathrm{Diff}_2(T, e, 1, 0) = \mathrm{Ext}_2(T, \sigma)$ for some $\sigma \in \mathscr{S}_2$, by Remark 5.3(ii), $T_i^*$ is strongly uniform.) By Lemma 2.13, $\langle T_0^*, T_1^* \rangle \Vdash P_{e,i}$. Hence $C_R$ is dense in this last case. ▯

**5.17 Corollary.** *Let* $\mathbf{a} \in \mathbf{D}$ *be given. Then there are minimal degrees* $\{\mathbf{m_i} : i \leqslant 3\}$ *such that* $\mathbf{a} = (\mathbf{m_0} \cup \mathbf{m_1}) \cap (\mathbf{m_2} \cup \mathbf{m_3})$.

*Proof.* Apply Theorem 5.16 with $\mathbf{d} = \mathbf{a}$ to obtain minimal degrees $\mathbf{m_0}$ and $\mathbf{m_1}$ such that $\mathbf{m_0} \cup \mathbf{m_1} \geqslant \mathbf{a}$. Apply Theorem 5.16 again with $\mathbf{d} = \mathbf{m_0} \cup \mathbf{m_1}$ to obtain minimal degrees $\mathbf{m_2}$ and $\mathbf{m_3}$ such that $\mathbf{a} = (\mathbf{m_0} \cup \mathbf{m_1}) \cap (\mathbf{m_2} \cup \mathbf{m_3})$. ▯

**5.18 Corollary.** *The set of minimal degrees forms an automorphism base for* $\mathscr{D}$.

*Proof.* Immediate from Corollary 5.17. ▯

We note that the construction of the sets $M_0$ and $M_1$ in Theorem 5.16 can be carried out by an oracle of degree $\mathbf{0}^{(2)} \cup \mathbf{a}$. Hence every degree $\mathbf{b} \geqslant \mathbf{0}^{(2)}$ is the least upper bound of a pair of minimal degrees.

**5.19 Remarks.** Strongly uniform trees were introduced by Lachlan [1971]. Theorem 5.16 and its corollaries are due to Jockusch and Posner [1981].

**5.20–5.23 Exercises**

**5.20** Show that the set of minimal degrees in $\overline{\mathbf{GL}_1}$ forms an automorphism base for $\mathscr{D}$.

**5.21** Let $\mathbf{a} \in \mathbf{D}$ be given. Show that the set of minimal covers of $\mathbf{a}$ forms an automorphism base for $\mathscr{D}[\mathbf{a}, \infty)$.

**5.22** Let $\mathbf{I}$ be a countable ideal of $\mathscr{D}$. Show that the set of minimal upper bounds for $\mathbf{I}$ forms an automorphism base for $\mathscr{D}[\mathbf{I}, \infty)$.

The following exercise is due to Jockusch. The proof we sketch was found by Shore. A weaker result was proved by Manaster [1971] with $\Sigma_n^0$ replaced with $\Delta_{n+1}^0$. It is not known whether the result holds for $n = 2$.

**5.23** Let $n \geqslant 3$ be given. Show that there is a minimal degree which is the degree of a set in $\Sigma_n^0 - \Delta_n^0$. (*Hint*: Construct a tree $\mathscr{T}$ of strongly uniform recursive trees (letting $T_\sigma = \mathscr{T}(\sigma)$) of degree $\leqslant \mathbf{0}^{(2)}$ such that each path through $\mathscr{T}$ forces all minimal degree requirements. $\mathscr{T}$ must also have the property that for all $\sigma, \tau \in \mathscr{S}_2$, there is exactly one $x \in N$ such that $T_{\sigma * 0}(\tau)(x) \neq T_{\sigma * 1}(\tau)(x)$, and for this $x$, $T_{\sigma * i}(\tau)(x) = i$ for $i \leqslant 1$. Choose a path through $\mathscr{T}$ corresponding to the set $\emptyset^{(n)}$. Show that the minimal degree constructed along this path has the desired properties.)