Part B

Countable Ideals of Degrees

Chapter V
Minimal Degrees

We now turn from the study of embeddings and extensions of embeddings into $\mathscr{D}$ to the study of countable ideals of $\mathscr{D}$. We will eventually characterize the countable ideals of $\mathscr{D}$ as those countable usls which have a least element. We begin, in this chapter, by showing that the simplest non-trivial usl, the chain consisting of two elements, is realized as an ideal of $\mathscr{D}$.

The methods of proof used in this part are the methods of forcing, but the notions of forcing used differ from those which we have previously encountered. The conditions of forcing here are trees, or equivalently, perfect closed sets in a certain topological space.

1. Binary Trees

We introduce some of the trees which will be used as forcing conditions in this section.

We recall some of the definitions dealing with strings for the reader who may be beginning the book at this point. A *string* is a finite sequence of integers, and is frequently treated as a function from a finite initial portion of N into N. $\mathscr{S}$ is the set of all strings. Given strings σ and τ, the string $\sigma * \tau$ is the finite sequence σ followed by the finite sequence τ; $\mathrm{lh}(\sigma)$, the *length* of σ, is the number of elements in the sequence σ; and we use the notation $\sigma \mid \tau$ if for some $x < \min(\{\mathrm{lh}(\sigma), \mathrm{lh}(\tau)\})$, the xth element of σ and the xth element of τ are different.

For the most part, we will be dealing with specific sets of strings. Thus given a function $f\colon N \to N$, we let $\mathscr{S}_f$ be the set of all strings σ such that for all $x < \mathrm{lh}(\sigma)$, $\sigma(x) < f(x)$. If f is a constant function, then we replace f by the unique number in its range. Thus $\mathscr{S}_2$ is the set of all *binary* strings, i.e., finite sequences of 0s and 1s.

1.1 Definition. Let $f\colon N \to N$ be given. An *f-tree* is a map $T\colon \mathscr{S}_f \to \mathscr{S}_f$ such that:

(i) $\forall \sigma, \tau \in \mathscr{S}_f (\sigma \subset \tau \to T(\sigma) \subset T(\tau))$.

(ii) $\forall \sigma, \tau \in \mathscr{S}_f (\sigma \mid \tau \to T(\sigma) \mid T(\tau))$.

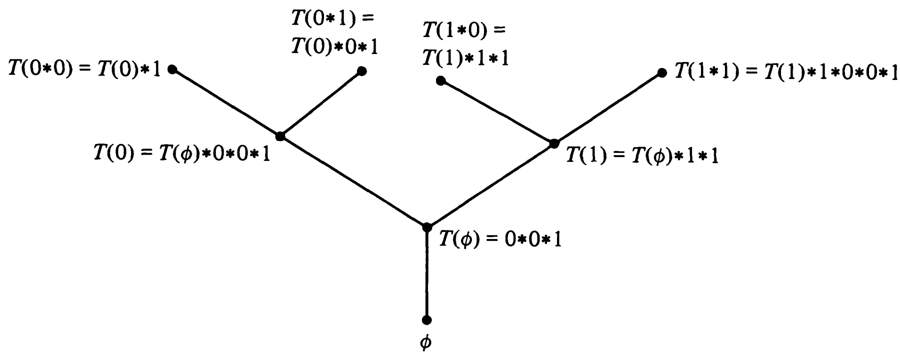

Fig. 1.1

In this chapter, we will restrict our attention to 2-trees or *binary trees*, i.e., the set of trees mapping $\mathscr{S}_2$ into $\mathscr{S}_2$. We will drop the word binary in this chapter, using the word *tree* with the understanding that we mean binary tree. Figure 1.1 pictures the bottom levels of a typical tree.

Without loss of generality, it can also be assumed that trees preserve lexicographical ordering; thus if σ lexicographically precedes τ, then $T(\sigma)$ must lexicographically precede $T(\tau)$. Although all trees which we use have this property, we do not make this assumption. It is, however, convenient to keep this assumption in mind when drawing pictures of trees.

1.2 Definition. Let T be a tree, and let $\sigma \in \mathscr{S}$ and $f: N \to N$ be given. We say that σ *is on* T (write $\sigma \subset T$) if $\sigma = T(\tau)$ for some $\tau \in \text{dom}(T)$. We say that σ is *compatible* with T if there is a $\tau \supseteq \sigma$ such that $\tau \subset T$. We say that f *is on* T (write $f \subset T$) if for all $\sigma \subset f$, σ is compatible with T. If $f \subset T$, we refer to f as a *branch* of T.

1.3 Remark. A tree is a function from a space into itself. Hence it makes sense to call a tree T *recursive* (the function T is recursive) or *recursive in* the function g (the function T is recursive in g).

We now define a simple tree which appears as a condition of our notion of forcing.

1.4 Definition. The *identity binary tree* $\text{Id}_2: \mathscr{S}_2 \to \mathscr{S}_2$ is defined by $\text{Id}_2(\sigma) = \sigma$ for all $\sigma \in \mathscr{S}_2$.

1.5 Remark. Id_2 is a recursive tree.

Trees are conditions of our notions of forcing. We now define the ordering of these conditions.

1.6 Definition. Let T and T^* be trees. We call T^* a *subtree* of T (write $T^* \subseteq T$) if $\text{rng}(T^*) \subseteq \text{rng}(T)$.

We will frequently use the following types of subtrees of a tree T to force requirements.

1.7 Definition. Let T be a tree and let $\sigma \in \mathscr{S}_2$ be given. Define $\text{Ext}_2(T, \sigma)$, the subtree of T *extending* $T(\sigma)$, by $\text{Ext}_2(T, \sigma)(\tau) = T(\sigma * \tau)$ for all $\tau \in \mathscr{S}_2$. Thus the branches of $\text{Ext}_2(T, \sigma)$ are those branches of T which extend $T(\sigma)$.

1.8 Remark. $\mathrm{Ext}_2(T, \sigma)$ is a subtree of T. Furthermore, for all $g: N \to N$, if T is recursive in g then $\mathrm{Ext}_2(T, \sigma)$ is recursive in g.

Our notions of forcing will be of the form $\langle \mathscr{T}, \subseteq \rangle$ where $\mathscr{T}$ is a collection of trees. We will prove theorems by specifying a set of requirements to be satisfied ($\models$) by a function f. Given a generic set G for a dense set of conditions, we will define the set A_G by

(1) $\qquad f \in A_G \Leftrightarrow \forall T \in G(f \subset T).$

The definition of forcing ($\Vdash$) for a requirement R will always have the property that for any tree T,

(2) $\qquad$ if $T \Vdash R$ then $\forall f \subset T(f \models R)$,

so the Satisfaction Lemma will hold. We will then have to verify the Density Lemma.

(1) and (2) combine to show that every $f \in A_G$ satisfies all requirements which give rise to the generic set G. A major difference between this type of forcing and forcing with perfect closed sets in Set Theory (see Sacks [1971]) is that we try to force only certain rather simple requirements, while the use of forcing in Set Theory is to force all sentences in an appropriate language.

If our notion of forcing is of the form $\langle \mathscr{T}, \subseteq \rangle$ where $\mathscr{T}$ is a set of f-trees and f never takes the value 0, then it will always be the case that $A_G \neq \emptyset$, so we will not have to worry about forcing this to be the case.

1.9 Lemma. *Let $\langle \mathscr{T}, \subseteq \rangle$ be a notion of forcing where $\mathscr{T}$ is a set of f-trees ordered by the subtree operation. Assume that for all $x \in N$, $f(x) \neq 0$. Let $\mathscr{C}$ be a collection of dense sets, let G be a $\mathscr{C}$-generic set, and define A_G as in* (1). *Then $A_G \neq \emptyset$.*

Proof. We proceed by induction on n, defining a sequence $\{\sigma_i \in \mathscr{S}_f : i \in N\}$ such that $\sigma_0 \subseteq \sigma_1 \subseteq \cdots$, $\mathrm{lh}(\sigma_n) = n$, and for all $i \in N$ and $T \in G$, σ_i is compatible with T. We begin by defining $\sigma_0 = \emptyset$. Suppose that σ_n has been defined. Define $S \subseteq N$ by

$$i \in S \Leftrightarrow i < f(n) \,\&\, \exists T \in G \ (\sigma_n * i \text{ is not compatible with } T).$$

If there is some $j \in \{i: i < f(n)\} - S$, then $\sigma_{n+1} = \sigma_n * j$ is compatible with every $T \in G$. We thus assume that $S = \{i: i < f(n)\}$ and obtain a contradiction. Since $f(n) \geqslant 1$, $S \neq \emptyset$. For each $i \in S$, choose $T_i \in G$ such that $\sigma_n * i$ is not compatible with T_i. Since G is generic, there is a common refinement (subtree) T of the $\{T_i: i < f(n)\}$. But then for all $i < f(n)$, $\sigma_n * i$ is not compatible with T. Since every string compatible with a tree has an extension on that tree, this is impossible. ▯

In the next section, we will use trees to construct minimal degrees.

1.10 Remark. Shoenfield [1966] was the first to cast constructions of ideals of $\mathscr{D}$ in terms of trees.

1.11–1.12 Exercises

1.11 Let T be a tree and let $f: N \to \{0, 1\}$ be given. Show that $f \subset T$ if and only if there is a function $g: N \to \{0, 1\}$ such that, setting $\sigma_n = g \upharpoonright n$, $f = \bigcup\{T(\sigma_n): n \in N\}$.

1.12 (Lachlan) Give a topological proof of Lemma 1.9. (*Hint*: For each $n \in N$, place the discrete topology on $[0, f(n))$ and note that, under this topology, $[0, f(n))$ is a compact space. By Tychonoff's Theorem, the product of compact spaces is compact, so $P = \times\{[0, f(n)): n \in N\}$ is compact under the product topology. Identify every tree with its set of branches. Show that every tree is a closed subset of P, and every generic set has the finite intersection property. Use the compactness of P to obtain $f \in A_G$.)

2. Minimal Degrees

We begin our characterization of the countable ideals of $\mathscr{D}$ by constructing a minimal degree.

2.1 Definition. A degree **a** is *minimal* if $\mathbf{a} > \mathbf{0}$ and $\mathbf{D}(\mathbf{0}, \mathbf{a}) = \emptyset$.

By Corollary IV.3.6, all minimal degrees are in $\mathbf{GL_2}$. We now complement this result by showing that minimal degrees exist.

We will use forcing to construct a minimal degree. The next lemma gives a necessary and sufficient condition for a set $A \subseteq N$ to have minimal degree, in terms of the requirements P_e and Q_e for $e \in N$, where

(1) $\quad P_e$: $A \neq \Phi_e$

and

(2) $\quad Q_e$: If Φ_e^A is total then either Φ_e^A is recursive or $A \leqslant_T \Phi_e^A$.

2.2 Lemma. *Let $A \subseteq N$ be given, and let $\mathscr{R} = \{P_e: e \in N\} \cup \{Q_e: e \in N\}$. Then A is a set of minimal degree if and only if A satisfies all requirements in $\mathscr{R}$.*

Proof. Immediate from Definition 2.1 and the Enumeration Theorem. ▯

The notion of forcing which we use to construct a set A of minimal degree is $\langle \mathscr{T}, \subseteq \rangle$ where $\mathscr{T}$ is the set of all recursive (binary) trees and $\subseteq$ is the subtree relation. Note that $\langle \mathscr{T}, \subseteq \rangle$ has a greatest element, Id_2. Our aim will be to have A satisfy every requirement $R \in \mathscr{R}$. We will define forcing ($\Vdash$) in such a way so that for all $R \in \mathscr{R}$

(3) $\quad$ If $T \Vdash R$ then $\forall A \subset T(A \models R)$.

It will then follow from Lemma 1.9 and Lemma 2.2 that there is a set $A \in A_G$ and that every set in A_G has minimal degree.

We now begin the verification of (3) and the Density Lemma for every requirement $R \in \mathscr{R}$. We note that by the definition of A_G, the verification of (3) will prove the Satisfaction Lemma for any $\mathscr{C}$-generic set, where $\mathscr{C} = \{C_R: R \in \mathscr{R}\}$ and $C_R = \{T \in \mathscr{T}: T \Vdash R\}$. We first consider P_e for $e \in N$.

2.3 Lemma. *Let $T \in \mathscr{T}$ and $e \in N$ be given. Define*

$$T \Vdash P_e \Leftrightarrow \textit{either } \Phi_e \textit{ is not total or } \exists x \in N(\Phi_e(x)\downarrow \neq T(\emptyset)(x)\downarrow).$$

Then (3) *holds for $R = P_e$. Furthermore, there is a tree $T^* \subseteq T$ such that $T^* \Vdash P_e$.*

Proof. (3) is immediate since if $f \subset T$ then $T(\emptyset) \subset f$. Fix $T \in \mathscr{T}$ and $e \in N$. If Φ_e is not total, then $T \Vdash P_e$. Assume that Φ_e is total. By Definition 1.1(ii), there is an $x \in N$ such that $T(0)(x)\downarrow \neq T(1)(x)\downarrow$. Fix such an x and the least $\sigma \in \{0, 1\}$ such that $T(\sigma)(x) \neq \Phi_e(x)$. Let $T^* = \mathrm{Ext}_2(T, \sigma)$. By Remark 1.8, $T^* \in \mathscr{T}$ and $T^* \subseteq T$. Since $T^*(\emptyset) = T(\sigma)$, $T^* \Vdash P_e$. ▯

Before proving (3) and the Density Lemma for $\{Q_e : e \in N\}$, we introduce a new class of trees and prove a lemma about these trees. We recall the following definition.

2.4 Definition. Let $\sigma, \tau, \rho \in \mathscr{S}_2$ be given. We call $\langle \sigma, \tau \rangle$ an *e-splitting of ρ* if $\rho \subset \sigma$, $\rho \subset \tau$, and for some $x \in N$, $\Phi_e^\sigma(x)\downarrow \neq \Phi_e^\tau(x)\downarrow$. For such an x, we call $\langle \sigma, \tau \rangle$ an *e-splitting of ρ for x*. If T is a tree and $\sigma, \tau \subset T$, then we call $\langle \sigma, \tau \rangle$ an *e-splitting* of *ρ on T*.

2.5 Definition. Let $e \in N$ be given. A tree T is an *e-splitting tree* if for all $\sigma \in \mathscr{S}_2$, $\langle T(\sigma * 0), T(\sigma * 1) \rangle$ is an e-splitting of $T(\sigma)$.

The crucial fact about e-splitting trees is captured by the next lemma.

2.6 Computation Lemma. *Fix $e \in N$, $h: N \to N$ and a tree T such that T is recursive in h. Let $A \subset T$ be given such that Φ_e^A is total. Then*:

(i) *If there are no e-splittings on T, then $\Phi_e^A \leqslant_T h$.*

(ii) *If T is an e-splitting tree, then $A \leqslant_T \Phi_e^A \oplus h$.*

Proof. (i) Fix e, h, T and A as in the hypothesis of the lemma. Given $x \in N$, we compute $\Phi_e^A(x)$ by finding $\sigma \in \mathscr{S}_2$ such that $\sigma \subset T$ and $\Phi_e^\sigma(x)\downarrow$. Such σ must exist since Φ_e^A is total, and can be found recursively in h. Furthermore, there is a $\tau \in \mathscr{S}_2$ such that $\tau \subset T, \tau \subset A$ and $\Phi_e^A(x) = \Phi_e^\tau(x)\downarrow$. Since $\langle \sigma, \tau \rangle$ cannot be an e-splitting of $T(\emptyset)$ on x, $\Phi_e^\sigma(x) = \Phi_e^\tau(x) = \Phi_e^A(x)$.

(ii) Fix e, h, T and A as in the hypothesis of the lemma. Given $y \in N$, we compute $A(y)$ using the Φ_e^A and h oracles. If $y < \mathrm{lh}(T(\emptyset))$, then since $T(\emptyset) \subset A$, $A(y) = T(\emptyset)(y)$ is computed using the h oracle. Suppose, by induction, that we have found $\sigma \in \mathscr{S}_2$ such that $\mathrm{lh}(\sigma) = n$ and $T(\sigma) \subset A$, and so have computed $A(y)$ for all $y < \mathrm{lh}(T(\sigma))$. Since $\langle T(\sigma * 0), T(\sigma * 1) \rangle$ is an e-splitting of $T(\sigma)$ on some x, we can use the h oracle to find $x \in N$ such that $\Phi_e^{T(\sigma * 0)}(x)\downarrow \neq \Phi_e^{T(\sigma * 1)}(x)\downarrow$. Then $\Phi_e^A(x) = \Phi_e^{T(\sigma * j)}(x)$ for exactly one $j \in \{0, 1\}$. For this j, $T(\sigma * j) \subset A$. ▯

In order to apply the Computation Lemma, we must be able to build e-splitting subtrees whenever we cannot build subtrees without e-splittings. This is accomplished by the following theorem.

2.7 Existence Theorem for *e*-splitting Trees. *Let $e \in N$ and $T \in \mathscr{T}$ be given. Assume that*

(i) $\forall \rho \in \mathscr{S}_2 \exists \sigma, \tau \in \mathscr{S}_2(\langle T(\sigma), T(\tau) \rangle$ *is an e-splitting of* $T(\rho))$.

Then there is an e-splitting tree $T^* \subseteq T$. Furthermore, for any $h\colon N \to N$, if T is recursive in h then T^* is recursive in h.

Proof. We proceed by induction on $\mathrm{lh}(\sigma)$. Fix a recursive one-one correspondence of $\mathscr{S}_2^2 \times N$ with N, given by $\{\langle \tau_j^0, \tau_j^1, x_j \rangle\colon j \in N\}$. We begin by defining $T^*(\emptyset) = T(\emptyset)$. Fix $\sigma \in \mathscr{S}_2$ such that $\mathrm{lh}(\sigma) = n$. We may assume by induction that $T^*(\sigma)$ has already been defined. By (i), we can find the least $j \in N$ such that $\langle T(\tau_j^0), T(\tau_j^1) \rangle$ is an e-splitting of $T^*(\sigma)$ on x_j. Let $T^*(\sigma * k) = T(\tau_j^k)$ for $k \in \{0, 1\}$.
It is easily verified that T^* has the desired properties. ▯

2.8 Definition. Given $e \in N$ and $T \in \mathscr{T}$ such that 2.7(i) holds, we let $\mathrm{Sp}_2(T, e)$ be the e-splitting subtree of T constructed in the proof of Theorem 2.7.

Let $e \in N$ and $T \in \mathscr{T}$ be given. We say that $T \Vdash Q_e$ if either T is an e-splitting tree or there are no e-splittings on T. We can now verify (3) and prove the density lemma for $\{Q_e\colon e \in N\}$.

2.9 Lemma. *Let $T \in \mathscr{T}$ and $e \in N$ be given. Then there is a tree $T^* \in \mathscr{T}$ such that $T^* \subseteq T$, $T^* \Vdash Q_e$, and for all $A \subset T^*$, $A \models Q_e$.*

Proof. Fix $e \in N$ and $T \in \mathscr{T}$. If there is a $\sigma \in \mathscr{S}_2$ such that there are no e-splittings on $\mathrm{Ext}_2(T, \sigma)$, let $T^* = \mathrm{Ext}_2(T, \sigma)$. By Remark 1.8, $T^* \in \mathscr{T}$ and $T^* \subseteq T$. Note that T^* was chosen so that $T^* \Vdash Q_e$. It now follows from the Computation Lemma that for all $A \subset T^*$, if Φ_e^A is total then Φ_e^A is recursive, so $A \models Q_e$.

If no σ as above exists, then 2.7(i) holds, so by the Existence Theorem for e-splitting Trees, $\mathrm{Sp}_2(T, e)$ is defined: Let $T^* = \mathrm{Sp}_2(T, e)$ and note that $T^* \in \mathscr{T}$ and $T^* \subseteq T$. Again, T^* is chosen so that $T^* \Vdash Q_e$. It now follows from the Computation Lemma that for all $A \subset T^*$, if Φ_e^A is total then $A \leqslant_T \Phi_e^A$, so $A \models Q_e$. ▯

A set of minimal degree can now easily be constructed.

2.10 Theorem. *There is a minimal degree.*

Proof. By Lemma 2.2, it suffices to construct a set $A \subset N$ such that A satisfies all requirements in $\mathscr{R}$. For each $R \in \mathscr{R}$, let $C_R = \{T \in \mathscr{T}\colon T \Vdash R\}$, and let $\mathscr{C} = \{C_R\colon R \in \mathscr{R}\}$. By Lemma 2.3 and Lemma 2.9, (3) holds and C_R is a dense set for all $R \in \mathscr{R}$. Hence by the Existence Theorem for $\mathscr{C}$-generic Sets, we can fix a $\mathscr{C}$-generic set G, and by Lemma 1.9, we can fix $A \in A_G$. It now follows from (3) that A satisfies all requirements in $\mathscr{R}$. ▯

The requirements chosen for the proof of Theorem 2.10 were the obvious ones. However, it would have been sufficient to let $\mathscr{R} = \{Q_e\colon e \in N\}$. We chose the above presentation because it is more direct. We indicate how to show that each P_e is satisfied if we only force $\{Q_e\colon e \in N\}$ in Exercise 2.15.

Having constructed a minimal degree, we try to determine its location within **D**.

2.11 Theorem. *There is a minimal degree* $\mathbf{a} < \mathbf{0}^{(2)}$.

Proof. Construct a set of minimal degree directly by starting with Id_2 and constructing subtrees as before to force requirements in the order $P_0, Q_0, P_1, Q_1, \ldots$. Since $\{e\colon \Phi_e \text{ is total}\}$ is infinite, there are infinitely many subtrees in this sequence of trees $\mathrm{Id}_2 = T_0 \supseteq T_1 \supseteq \cdots$ such that $T_{i+1}(\emptyset) \supset T_i(\emptyset)$ (this will happen when forcing P_e for infinitely many e) so a unique set $A = \bigcup\{T_i(\emptyset): i \in N\}$ is constructed. We now note that the particular choice of $T_{i+1} \subseteq T_i$ can be determined by a $\emptyset^{(2)}$ oracle, and so complete the proof. In the case of P_e, T_{i+1} is defined in Lemma 2.3, and the choice of T_{i+1} depends only on knowing whether Φ_e is total. By Lemma IV.3.2, a $\emptyset^{(2)}$ oracle can make this determination. In the case of Q_e, T_{i+1} is defined in Lemma 2.9, and the choice of T_{i+1} depends on whether or not 2.7(i) is true. But 2.7(i) is a Π_2^0 sentence, uniformly in an index for the recursive tree T_i, so a $\emptyset^{(2)}$ oracle can again determine an index for T_{i+1}. ▯

Minimal degrees also exist below $\mathbf{0}'$, but are more difficult to construct. Such a construction is carried out in Chap. IX, and minimal degrees below recursively enumerable degrees and degrees in $\mathbf{GH_1}$ are constructed in Chaps. XI and IX respectively.

The above construction of a $\mathscr{C}$-generic set can be modified to obtain a large number of minimal degrees.

2.12 Theorem. *There are* $2^{\aleph_0}$ *minimal degrees.*

Proof. We construct a *tree of trees*, i.e. a function $\mathscr{F}: \mathscr{S}_2 \to \mathscr{T}$ such that for all $\sigma, \tau \in \mathscr{S}_2$:

$$\sigma \subset \tau \to \mathscr{F}(\sigma) \supset \mathscr{F}(\tau).$$

$$\sigma \mid \tau \to \mathscr{F}(\sigma)(\emptyset) \neq \mathscr{F}(\tau)(\emptyset).$$

We will write T_σ for $\mathscr{F}(\sigma)$. Let $\{R_i: i \in N\}$ be a recursive ordering of $\mathscr{R}$.

We begin by defining $T_\emptyset = \mathrm{Id}_2$. Assume, by induction on $\mathrm{lh}(\sigma)$, that T_σ has been defined, and fix $i \in \{0, 1\}$ and $n = \mathrm{lh}(\sigma)$. Let $T_{\sigma * i}$ be a subtree of $\mathrm{Ext}_2(T_\sigma, i)$ which forces R_n. It is easily verified that we have defined a tree of trees.

For all $B \subset N$, let G_B be the $\mathscr{C}$-generic set $\{T \in \mathscr{T}: \exists \sigma \subset B(T_\sigma \subseteq T)\}$, and let A_B be a set obtained from G_B. Then for all $B \subseteq N$, A_B has minimal degree. Furthermore, for all $B, C \subseteq N$, if $B \neq C$ then $A_B \neq A_C$. Since each degree contains only countably many sets, $\{A_B: B \subseteq N\}$ is a collection of minimal degrees of cardinality $2^{\aleph_0}$. ▯

The following corollary of Theorem 2.12 was used in Corollary II.4.6.

2.13. Corollary. $\mathscr{D}$ *has a maximal antichain of cardinality* $2^{\aleph_0}$.

The techniques of this section can be extended to control the double jumps of minimal degrees. We carry this out in the next section.

2.14 Remarks. Theorem 2.10 and Theorem 2.11 were proved by Spector [1956]. Theorem 2.12 and Corollary 2.13 were noted by Lacombe [1954]. Sacks [1971] was the first to notice the connection between Spector's proof and forcing. Exercise 2.15 first appeared in Epstein and Posner [1978].

2.15–2.16 Exercises

2.15 (Posner's Lemma) Show that it is sufficient to force the requirements $\{Q_e: e \in N\}$; $\{P_e: e \in N\}$ will then automatically be satisfied. (*Hint*: We suppose that $A \in A_G$ is such that $A = \Phi_e$, and obtain a contradiction. Define the partial recursive functional

$$\Psi(B)(x) = \begin{cases} x & \text{if } \exists\sigma \subset B(\sigma \not\subset A) \\ \uparrow & \text{otherwise.} \end{cases}$$

Then $\Psi = \Phi_i$ for some $i \in N$. Show that $A \in A_G$ implies that we failed to force Q_i.)

2.16 Construct a tree of trees such that, in the notation of Theorem 2.12, for all $B, C \subseteq N$, A_B is a set of minimal degree and if $B \neq C$ then A_B and A_C have different degrees.

3. Double Jumps of Minimal Degrees

The methods introduced in the last section rely on a $\emptyset^{(2)}$ oracle to determine how constructions are to be carried out. Thus the local version of the theorem asserting the existence of a minimal degree produces such a degree below $\mathbf{0}^{(2)}$, and more powerful techniques are needed to produce a minimal degree below $\mathbf{0}'$. For similar reasons, the methods introduced in the last section are not sufficiently powerful to characterize jumps of minimal degrees, but can be used to characterize double jumps of minimal degrees. A characterization of the jumps of minimal degrees is deferred until Chap. X.

We begin this section by constructing a minimal degree $\mathbf{a}$ such that $\mathbf{a}^{(2)} = \mathbf{0}^{(2)}$. The proof of this result can be modified to obtain a complete characterization of the double jumps of minimal degrees as $\mathbf{D}[\mathbf{0}^{(2)}, \infty)$. We leave this characterization as an exercise for the reader. We also begin to obtain a finer version of Corollary IV.3.6 which implies that all minimal degrees lie in $\mathbf{GL}_2$. Thus we construct a minimal degree in $\mathbf{GL}_2 - \mathbf{GL}_1$. There are also minimal degrees in $\mathbf{GL}_1$, but a proof of this fact requires methods which enable us to exercise some control over the jump of the minimal degree constructed, so is deferred until Chap. IX.

We now indicate how to construct a set A of minimal degree $\mathbf{a}$ for which $\mathbf{a}^{(2)} = \mathbf{0}^{(2)}$. By Lemma IV.3.2, it will suffice to insure that $\mathrm{Tot}(A) \leqslant_T \mathrm{Tot}(\emptyset)$. For each $e \in N$, we require that A lie on a recursive tree T such that either

(1) $\quad \forall B \subset T(\Phi_e^B \text{ is total})$

or

(2) $\quad \forall B \subset T(\Phi_e^B \text{ is not total}).$

The determination of whether (1) or (2) holds for a given $e \in N$ will be made, uniformly in e, by an oracle of degree $\mathbf{0}^{(2)}$. Hence by Lemma IV.3.2, it will be the case that $\mathbf{a}^{(2)} \leqslant \mathbf{0}^{(2)}$.

3.1 Definition. Let T be a tree and let $e \in N$ be given. T is said to be *e-total* if (1) holds and *completely e-partial* if (2) holds.

Note that there are trees which are neither e-total nor completely e-partial; for it is possible for Φ_e^B to be total for some, but not all branches of T.

For each $e \in N$, we establish a requirement U_e which asserts that either (1) or (2) holds. Thus a tree T will force U_e ($T \Vdash U_e$) if either T is e-total or T is completely e-partial. The following lemma allows us to build e-total trees whenever we need them to force the requirement U_e.

3.2 Lemma. *Let $e \in N$ be given, and fix a tree T. Assume that*

$$\text{(i)} \qquad \forall x \in N \forall \sigma \in \mathcal{S}_2 \exists \tau \in \mathcal{S}_2 (\tau \supseteq \sigma \,\&\, \Phi_e^{T(\tau)}(x)\downarrow).$$

Then there is an e-total $T^* \subseteq T$. Furthermore, for all $h: N \to N$, if T is recursive in h then T^* is recursive in h.

Proof. We proceed by induction on $\mathrm{lh}(\sigma)$ for $\sigma \in \mathcal{S}_2$. Fix a one-one recursive correspondence $\{\sigma_i : i \in N\}$ of N with $\mathcal{S}_2$. We begin by finding the least $i \in N$ such that $\Phi_e^{T(\sigma_i)}(0)\downarrow$ and setting $T^*(\emptyset) = T(\sigma_i)$.

Fix $\sigma \in \mathcal{S}_2$ such that $\mathrm{lh}(\sigma) = n$. We may assume by induction that $T^*(\sigma)$ has already been defined and that $T^*(\sigma) = T(\eta)$. Given $i \in \{0, 1\}$, find the least $j \in N$ such that $\Phi_e^{T(\sigma_j)}(n+1)\downarrow$ and $\eta * i \subseteq \sigma_j$, and set $T^*(\sigma * i) = T(\sigma_j)$.

This completes the construction of T^*. It is easily verified that T^* has the desired properties. ▯

3.3 Definition. Let $e \in N$ be given. Fix a tree T for which 3.2(i) holds. Then we let $\mathrm{Tot}_2(T, e)$ be the tree constructed in Lemma 3.2.

We can now prove the Density Lemma for $\{U_e : e \in N\}$.

3.4 Lemma. *Let $T \in \mathcal{T}$ and $e \in N$ be given. Then there is a tree $T^* \in \mathcal{T}$ such that $T^* \subseteq T$ and $T^* \Vdash U_e$.*

Proof. If 3.2(i) holds, then we let $T^* = \mathrm{Tot}_2(T, e)$ and note that by Lemma 3.2, T^* has the desired properties. If 3.2(i) fails to hold, then for some $\sigma \in \mathcal{S}_2$, $\mathrm{Ext}_2(T, \sigma)$ is completely e-partial. Fix such a σ, and let $T^* = \mathrm{Ext}_2(T, \sigma)$. By Remark 1.8, $T^* \subseteq T$ and $T^* \in \mathcal{T}$. Hence T^* has the desired properties. ▯

If we include the requirements $\{U_e : e \in N\}$ in the list of requirements for the construction of a minimal degree, then by Lemma 3.4, all these requirements will be forced. Suppose that we have a recursive list of all requirements and are given T and asked to force U_e. Then if A is the final set of minimal degree and $T = T_n$, then

$$\text{(3)} \qquad e \in \mathrm{Tot}(A) \Leftrightarrow T_{n+1} = \mathrm{Tot}_2(T_n, e) \Leftrightarrow 3.2\text{(i) holds.}$$

Since an oracle of degree $\mathbf{0}^{(2)}$ can decide whether 3.2(i) holds for T and e uniformly in T and e, it follows from (3) that the decision as to whether or not $e \in \mathrm{Tot}(A)$ can be made by an oracle of degree $\mathbf{0}^{(2)}$. Since $\mathbf{0}^{(2)} \leqslant \mathbf{a}^{(2)}$ for all $\mathbf{a} \in \mathbf{D}$, we have just proved:

3.5 Theorem. *There is a minimal degree* $\mathbf{a}$ *such that* $\mathbf{a}^{(2)} = \mathbf{0}^{(2)}$.

A proof combining the methods of Theorem 2.12 with those of Theorem 3.5 can be used to show that for any set $B \subseteq N$, the set A_B defined as in Theorem 2.12

satisfies $A_B^{(2)} \equiv_T A_B \oplus \emptyset^{(2)} \equiv_T B \oplus \emptyset^{(2)}$. We leave the details to the reader (Exercise 3.14), but state this result as our next corollary.

3.6 Corollary. *Let* $\mathbf{d} \in \mathbf{D}$ *be given. Then there is a minimal degree* $\mathbf{a}$ *such that* $\mathbf{a}^{(2)} = \mathbf{a} \cup \mathbf{0}^{(2)} = \mathbf{d} \cup \mathbf{0}^{(2)}$.

Corollary 3.6 allows us to characterize the double jumps of minimal degrees.

3.7 Corollary. *Let* $\mathbf{d} \in \mathbf{D}$ *be given. Then the following are equivalent*:

(i) $\mathbf{d} \geqslant \mathbf{0}^{(2)}$.

(ii) $\mathbf{d} = \mathbf{a}^{(2)}$ *for some minimal degree* $\mathbf{a}$.

We next prove that there is a minimal degree in $\mathbf{GL}_2 - \mathbf{GL}_1$. In fact, there are such degrees in $\mathbf{L}_2 - \mathbf{L}_1$, but a proof of this fact requires that we construct a minimal degree below $\mathbf{0}'$, so is not carried out until Chap. IX. By Corollary IV.3.6, it suffices to construct a minimal degree which is not in $\mathbf{GL}_1$, so it suffices to satisfy the following requirements for all $e \in N$:

$$V_e\colon A' \neq \Phi_e^{A \oplus \emptyset'}.$$

We show that for any $B \subseteq N$, we can force the requirements V_e^B in which B replaces $\emptyset'$ by using *narrow* subtrees.

Fix $e \in N$. The idea behind the satisfaction of V_e is as follows. Given any recursive tree T, there is a partial recursive functional Ψ such that

$$\Psi(A;x) = \begin{cases} 0 & \text{if } A \not\subset T \\ \uparrow & \text{otherwise.} \end{cases}$$

An algorithm for computing $\Psi(\sigma;x)$ for $\sigma \in \mathscr{S}_2$ is to check whether or not $\sigma \subset T$. If $\sigma \not\subset T$ then $\Psi(\sigma;x) = 0$ and if $\sigma \subset T$ then $\Psi(\sigma;x)\uparrow$. For a given tree T, $\Psi = \Phi_{n(T)}^A$ for some $n(T) \in N$. Furthermore, $n(T)$ can recursively be computed once we have an index for T as a partial recursive function.

If $\Phi_e^{A \oplus \emptyset'}(n(T))\uparrow$, then V_e is satisfied. The only problem occurs when $\Phi_e^{A \oplus \emptyset'}(n(T))\downarrow = 0$, thus predicting that $n(T) \notin A'$ for $A \subset T$. The convergence of $\Phi_e^{A \oplus \emptyset'}(n(T))$ requires only a commitment to a finite string $\sigma \subset A$. Thus if we have the ability to let A extend σ and then leave T, we could then change the value of $A'(n(T))$ to 1. In order to achieve the flexibility to leave T, we carry out this procedure on a *narrow subtree* of T instead of on T.

3.8 Definition. Let T and T^* be trees such that $T^* \subseteq T$. We say that T^* is a *narrow subtree* of T if for all $\sigma \subset T^*$ there is a string $\tau \subset T - T^*$ such that $\tau \supset \sigma$.

Narrow subtrees are easy to construct, and such a construction is carried out in the next lemma.

3.9 Lemma. *Fix a tree* T. *Then* T *has a narrow subtree* T^*. *Furthermore*, T^* *can be defined so that for all* $h\colon N \to N$, *if* T *is recursive in* h *then* T^* *is recursive in* h.

Proof. For each $n \in N$, let 0_n be the string of length n such that $0_n(x) = 0$ for all $x < n$. For each $\sigma \in \mathscr{S}_2$, define $T^*(\sigma) = T(\sigma \oplus 0_n)$ where $n = \mathrm{lh}(\sigma)$. It is easily verified that T^* has the desired properties. ▯

3.10 Definition. For any tree T, let Nar(T) be the narrow subtree of T defined in Lemma 3.9.

Given $T \in \mathcal{T}$ and $e \in N$, we say that $T \Vdash V_e$ exactly if for all $A \subset T$, $A \models V_e$. The Satisfaction Lemma for V_e is then trivial. We now prove the Density Lemma for V_e.

3.11 Lemma. *Let $T \in \mathcal{T}$ and $e \in N$ be given. Then there is a tree $T^* \in \mathcal{T}$ such that $T^* \subseteq T$ and $T^* \Vdash V_e$.*

Proof. Let $n = n(\text{Nar}(T))$ be the index for the functional Ψ defined before Definition 3.8 for the tree Nar(T). We ask if there are strings $\sigma \subset \text{Nar}(T)$ and $\tau \subset \emptyset'$ such that $\Phi_e^{\sigma \oplus \tau}(n)\downarrow = 0$. If no such strings exist, we let $T^* = \text{Nar}(T)$. Then for all $A \subset T^*$, $\Psi(A;n)\uparrow$ so $A'(n) = 0 \neq \Phi_e^{A \oplus \emptyset'}(n)$. If such strings exist, fix such a string σ. Fix $\xi, \eta \in \mathcal{S}_2$ such that $T(\xi) = \sigma$ and $\eta = \xi * 1 * 1$, and let $T^* = \text{Ext}_2(T, \eta)$. Then $T(\eta) \not\subset \text{Nar}(T)$, so for all $A \subset T^*$, $\psi(A;n)\downarrow$. Thus $A'(n) = 1 \neq 0 = \Phi_e^{A \oplus \emptyset'}(n)$. In either case, we see that T^* has the desired properties. ▯

Note that we could have replaced $\emptyset'$ with any set $B \subseteq N$ and the above proof would yield the Density Lemma for V_e^B. Furthermore, an index for T^* in Lemma 3.11 can be obtained from an index for T recursively in an oracle of degree $\mathbf{0}^{(2)}$. Hence we can add the set $\{V_e \colon e \in N\}$ to the set of requirements used in the construction of a minimal degree to obtain the following theorem.

3.12 Theorem. *There is a minimal degree* $\mathbf{a}$ *such that* $\mathbf{a}^{(2)} = \mathbf{0}^{(2)}$ *and* $\mathbf{a} \in \mathbf{GL}_2 - \mathbf{GL}_1$.

To this point, we have used only recursive trees as conditions of our notion of forcing. In the next section, we pass to non-recursive trees in order to obtain relativizations of results already proved, and to work above ideals instead of single degrees.

3.13 Remarks. The use of e-total trees and the proof of Theorem 3.5 are due to Miller and Martin [1968]. Theorem 3.12 was proved by Sasso [1974].

3.14–3.15 Exercises

***3.14** Let $\mathbf{d} \in \mathbf{D}$ be given. Construct a minimal degree $\mathbf{a}$ such that $\mathbf{a}^{(2)} = \mathbf{a} \cup \mathbf{0}^{(2)} = \mathbf{d} \cup \mathbf{0}^{(2)}$.

3.15 For all $n \geqslant 2$, show that there is a set A of minimal degree $\mathbf{a}$ such that $\mathbf{a} \leqslant \mathbf{0}^{(n-1)}$ but $\mathbf{a}' \not\leqslant \mathbf{0}^{(n-1)}$.

4. Minimal Covers and Minimal Upper Bounds

In this section, we introduce trees which code specified subsets of N into each of their branches. These trees are used to relativize results previously obtained about minimal degrees, and to obtain minimal upper bounds for countable subsets of $\mathbf{D}$.

4.1 Definition. Let T be a tree and let $B \subseteq N$ be given. T is said to be *B-pointed* if $T \leqslant_T B$ and for every branch A of T, $B \leqslant_T A$. T is said to be *pointed* if T is C-pointed for some $C \subseteq N$.

B-pointed trees are useful for obtaining results about $\mathscr{D}[\mathbf{b}, \infty)$ where B has degree $\mathbf{b}$. The following lemma will allow us to obtain B-pointed subtrees of given trees under suitable hypotheses.

4.2 Lemma. *Let $B, C \subseteq N$ be given and let T be a B-pointed tree. Then there is a $B \oplus C$-pointed subtree T^* of T.*

Proof. For each $n \in N$, let $\tau_n = C \restriction n$. Define $T^*(\sigma) = T(\sigma \oplus \tau_n)$ where $\mathrm{lh}(\sigma) = n$. Clearly $T^* \leqslant_T B \oplus C$. If $A \subset T^*$, then since $T^* \subset T$, $A \subset T$. Since T is B-pointed, $B \leqslant_T A$, so $T \leqslant_T A$. To recover C from A, we note that $n \in C$ if and only if there is a string $\xi \in \mathscr{S}_2$ of length $2n + 2$ such that $T(\xi) \subset A$ and $\xi(2n + 1) = 1$. Since $T \leqslant_T A$, $C \leqslant_T A$. ▯

4.3 Definition. Let $B, C \subseteq N$ be given and let T be a B-pointed tree. Let $\mathrm{Pt}(C, T)$ denote the $B \oplus C$-pointed subtree T^* of T defined in Lemma 4.2.

Note that $\mathrm{Pt}(\emptyset, T) = \mathrm{Nar}(T)$ for every tree T. We now relativize the definition of minimal degree.

4.4 Definition. Let $\mathbf{a}, \mathbf{d} \in \mathbf{D}$ be given. Then $\mathbf{a}$ is a *minimal cover* for $\mathbf{d}$ if $\mathbf{a} > \mathbf{d}$ and $\mathbf{D}(\mathbf{d}, \mathbf{a}) = \emptyset$.

Results about minimal degrees relativize to arbitrary degrees $\mathbf{d} \in \mathbf{D}$, becoming results about minimal covers of $\mathbf{d}$. Only minor changes in proofs need to be made. The set $\mathscr{T}$ of forcing conditions becomes the set of D-pointed subtrees of $\mathrm{Pt}(D, \mathrm{Id}_2)$ where D is a set of degree $\mathbf{d}$. The requirement P_e becomes $\Phi_e^D \neq A$, and the requirement Q_e becomes: If Φ_e^A is total then either Φ_e^A is recursive in D or A is recursive in $\Phi_e^A \oplus D$. And instead of forcing $V_e^{\emptyset'}$, we force $V_e^{D'}$. All the lemmas and theorems proved earlier in this chapter now go through with these changes. We list some of the relativized results, leaving the verification of the results to the reader.

4.5 Theorem. *Let $\mathbf{d} \in \mathbf{D}$ be given. Then there is a minimal cover $\mathbf{a}$ of $\mathbf{d}$ such that $\mathbf{a}^{(2)} = \mathbf{d}^{(2)}$ and $\mathbf{a} \in \mathbf{GL}_2(\mathbf{d}) - \mathbf{GL}_1(\mathbf{d})$.*

4.6 Theorem. *Let $\mathbf{c}, \mathbf{d} \in \mathbf{D}$ be given. Then there is a minimal cover $\mathbf{a}$ of $\mathbf{d}$ such that $\mathbf{a}^{(2)} = \mathbf{a} \cup \mathbf{d}^{(2)} = \mathbf{c} \cup \mathbf{d}^{(2)}$.*

4.7 Corollary. *Let $\mathbf{c}, \mathbf{d} \in \mathbf{D}$ be given. Then the following are equivalent:*
(i) $\mathbf{c} \geqslant \mathbf{d}^{(2)}$.
(ii) $\mathbf{c} = \mathbf{a}^{(2)}$ *for some minimal cover* $\mathbf{a}$ *of* $\mathbf{d}$.

4.8 Theorem. *For all $\mathbf{d} \in \mathbf{D}$, $\mathbf{d}$ has $2^{\aleph_0}$ minimal covers.*

4.9 Corollary. *For all $\mathbf{d} \in \mathbf{D}$, $\mathbf{D}[\mathbf{d}, \infty)$ has an antichain of cardinality $2^{\aleph_0}$.*

If we iterate the pointed subtree operation, then we can get minimal upper bounds for countable sets of degrees.

4.10 Definition. Let $\mathbf{S}$ be a countable set of degrees. Then $\mathbf{D}[\mathbf{S}, \infty) = \{\mathbf{d} \in \mathbf{D} : \mathbf{d} \geqslant \mathbf{c}$ for all $\mathbf{c} \in \mathbf{S}\}$. A *minimal upper bound* for $\mathbf{S}$ is a degree $\mathbf{c} \in \mathbf{D}[\mathbf{S}, \infty)$ such that there is no $\mathbf{d} \in \mathbf{D}[\mathbf{S}, \infty)$ for which $\mathbf{d} < \mathbf{c}$.

Since a minimal upper bound for $\mathbf{S}$ is the same as a minimal upper bound for the ideal generated by $\mathbf{S}$, we restrict ourselves to the case where $\mathbf{S}$ is an ideal. Most of the results which have just been mentioned for $\mathbf{D}[\mathbf{d}, \infty)$ for all $\mathbf{d} \in \mathbf{D}$ have counterparts for $\mathbf{D}[\mathbf{I}, \infty)$ for all countable ideals $\mathbf{I}$ of $\mathscr{D}$, if we replace minimal covers with minimal upper bounds. We take as the set of forcing conditions, the set of all trees which are B-pointed for some $B \subseteq N$ whose degree $\mathbf{b}$ lies in $\mathbf{I}$. We use appropriate modifications of previous requirements, except that the requirements $\{P_e: e \in N\}$ are replaced with

$$P_e^B: A \nleqslant_T \Phi_e^B$$

for all $e \in N$ and $B \subseteq N$ such that the degree of B lies in $\mathbf{I}$. By Lemma 4.2, the set of trees forcing each such requirement is dense, so we can construct a suitable generic set G and take a set $A \in \mathbf{A}_G$. The degree $\mathbf{a}$ of A will be the desired minimal upper bound. Hence:

4.11 Theorem. *Let* $\mathbf{I}$ *be a countable ideal of* $\mathscr{D}$. *Then* $\mathbf{I}$ *has a minimal upper bound.*

We now mention some additional results pertaining to minimal upper bounds whose proofs are left to the reader.

4.12 Theorem. *Let* $\mathbf{I}$ *be a countable ideal of* $\mathscr{D}$ *with no greatest element. Then* $\mathbf{I}$ *has* $2^{\aleph_0}$ *minimal upper bounds.*

Results about double jumps of minimal upper bounds can also be obtained, but depend on how effectively we can list a set $S \subseteq N^2$ such that $\{S^{[i]}: i \in N\}$ generates the ideal $\mathbf{I}$.

4.13 Theorem. *Let* $\mathbf{I}$ *be a countable ideal of* $\mathscr{D}$ *and let* $\mathbf{S} \subseteq \mathbf{D}$ *and* $\mathbf{d} \in \mathbf{D}$ *be given such that* $\mathbf{S}$ *generates* $\mathbf{I}$ *and* $\mathbf{S}$ *is uniformly of degree* $\leqslant \mathbf{d}$. *Then there is a minimal upper bound* $\mathbf{a}$ *for* $\mathbf{I}$ *such that* $\mathbf{a}^{(2)} \leqslant \mathbf{d}^{(2)}$ *and* $\mathbf{a} \notin \mathbf{GL}_1(\mathbf{d})$.

4.14 Remarks. The results on minimal covers were obtained as immediate corollaries of the results which they relativize. Theorem 4.11 was proved by Sacks [1963].

4.15–4.16 Exercises

***4.15** Write out a detailed proof of the theorem that every degree has a minimal cover.

***4.16** Write out a detailed proof that every countable ideal has a minimal upper bound.

5. Cones of Minimal Covers

5.1 Definition. The set of *arithmetical degrees*, $\mathbf{D}_{\mathbf{arith}}$, is $\{\mathbf{d} \in \mathbf{D}: \exists n \in N(\mathbf{d} \leqslant \mathbf{0}^{(\mathbf{n})})\}$. (Note that $\mathbf{D}_{\mathbf{arith}}$ is the set of all degrees of sets which lie in the arithmetical hierarchy.) A degree $\mathbf{d}$ is *arithmetical* if $\mathbf{d} \in \mathbf{D}_{\mathbf{arith}}$. We let $\mathscr{D}_{\mathbf{arith}}$ denote the poset $\langle \mathbf{D}_{\mathbf{arith}}, \leqslant \rangle$.

In this section, we use cones of minimal covers to show that $\mathscr{D}$ and $\mathscr{D}_{\mathbf{arith}}$ have different elementary theories.

5.2 Definition. Let $\mathbf{d} \in \mathbf{D}$ be given. Then $\mathbf{d}$ is a *minimal cover* if $\mathbf{d}$ is a minimal cover for some $\mathbf{b} \in \mathbf{D}$.

In Section 4, we showed that every $\mathbf{b} \in \mathbf{D}$ has a minimal cover. We now show that not every $\mathbf{a} \in \mathbf{D}$ is a minimal cover. The proof relies on two facts:

(1) Given $\mathbf{a}, \mathbf{b}, \mathbf{d} \in \mathbf{D}$, if $\mathbf{a}$ and $\mathbf{b}$ are both recursively enumerable in $\mathbf{d}$, then $\mathbf{a} \cup \mathbf{b}$ is recursively enumerable in $\mathbf{d}$.

(2) Given $\mathbf{a}, \mathbf{d} \in \mathbf{D}$, if $\mathbf{a}$ is recursively enumerable in $\mathbf{d}$, then $\mathbf{a}$ is not a minimal cover for $\mathbf{d}$.

(1) follows easily from Definition III.1.6 and (2) follows easily from Corollary III.6.3.

5.3 Theorem. *For all* $n \in N$, $\mathbf{0}^{(\mathbf{n})}$ *is not a minimal cover.*

Proof. Let $\mathbf{0}^{(\mathbf{0})}$ denote $\mathbf{0}$. Fix $n > 0$. We suppose that $\mathbf{0}^{(\mathbf{n})}$ is a minimal cover for $\mathbf{a}$ and obtain a contradiction. Fix the greatest $k < n$ such that $\mathbf{0}^{(\mathbf{k})} \leqslant \mathbf{a}$. (Note that such a k exists since $\mathbf{0}^{(\mathbf{0})} \leqslant \mathbf{a}$.) Then $\mathbf{0}^{(\mathbf{k+1})} \leqslant \mathbf{0}^{(\mathbf{n})}$ and $\mathbf{a} \cup \mathbf{0}^{(\mathbf{k+1})} > \mathbf{a}$, so since $\mathbf{0}^{(\mathbf{n})}$ is a minimal cover for $\mathbf{a}$, $\mathbf{a} \cup \mathbf{0}^{(\mathbf{k+1})} = \mathbf{0}^{(\mathbf{n})}$. By Theorem III.2.3(i), $\mathbf{0}^{(\mathbf{k+1})}$ is recursively enumerable in $\mathbf{0}^{(\mathbf{k})}$, so since $\mathbf{0}^{(\mathbf{k})} \leqslant \mathbf{a}$, $\mathbf{0}^{(\mathbf{k+1})}$ is recursively enumerable in $\mathbf{a}$. Also, $\mathbf{a}$ is recursively enumerable in $\mathbf{a}$. Hence by (1), $\mathbf{0}^{(\mathbf{n})} = \mathbf{a} \cup \mathbf{0}^{(\mathbf{k+1})}$ is recursively enumerable in $\mathbf{a}$. But then by (2), $\mathbf{0}^{(\mathbf{n})}$ is not a minimal cover for $\mathbf{a}$. ▯

Many other degrees are not minimal covers. Other examples will be constructed in Chap. VIII. Our next main result will show that there is a degree $\mathbf{d}$ such that every degree $\mathbf{a} \geqslant \mathbf{d}$ is a minimal cover. Some preliminary results must first be stated.

5.4 Definition. A set $\mathbf{S} \subseteq \mathbf{D}$ is a *cone* if there is a degree $\mathbf{d}$ such that $\mathbf{S} = \{\mathbf{a} \in \mathbf{D}: \mathbf{a} \geqslant \mathbf{d}\}$.

5.5 Definition. Expand the language for Recursion Theory introduced in Sect. III.1 by adding new variables $\{X_i: i \in N\}$, each to be interpreted as a function from N into N, or equivalently, as an element of N^N. The arithmetical hierarchy can now be defined as before, where no quantification is allowed over these new variables. We then call a set $\mathscr{Y} \subseteq N^N$ *arithmetical* if there is an arithmetical formula $S(X_i)$ of this expanded language such that for all $f \in N^N$, $S(f) \Leftrightarrow f \in \mathscr{Y}$.

The construction of a cone of minimal covers depends on the Axiom of Determinateness for arithmetical subsets of N^N. This axiom is now described.

5.6 Definition. Let $\mathscr{Y} \subseteq N^N$ be given. The *Gale-Stewart game* $G_{\mathscr{Y}}$ is played by two players, I and II, as follows: The players alternate moves, each picking an integer $n \in N$ when his turn comes, with player I beginning. The game ends with a sequence $\{n_i: i \in N\}$ which can be viewed as an element of N^N. Player I wins the game if this sequence is in $\mathscr{Y}$; player II wins otherwise. (Note that different players may win different plays of the game $G_{\mathscr{Y}}$.)

5.7 Definition. Let $\mathscr{Y} \subseteq N^N$ be given. A *strategy* is a function $g: \mathscr{S} \to N$. A player *follows the strategy* g if whenever it is that player's turn to specify the next integer in the sequence, and the string σ has been played up to that point of the game, then the player plays $g(\sigma)$ at that turn. The strategy g is a *winning strategy* for a given player if, under the assumption that the player follows g, the sequence constructed during the game is an element of $\mathscr{Y}$. (Hence if the player follows the strategy, then he will win no matter what the other player does.)

5.8 Definition. Let $\mathscr{Y} \subseteq N^N$ be given. The game $G_{\mathscr{Y}}$ is said to be *determinate* if one of the players has a winning strategy.

The Axiom of Determinateness states that for every $\mathscr{Y} \subseteq N^N$, $G_{\mathscr{Y}}$ is determinate. This axiom is known to contradict the Axiom of Choice. However, the Axiom of Determinateness is known to hold for a certain class of subsets of N^N, the Borel sets, a class which contains the arithmetical sets. We state this fact without proof.

5.9 Theorem (Martin [1975]). If $\mathscr{Y} \subseteq N^N$ is Borel then $G_{\mathscr{Y}}$ is determinate. Hence if $\mathscr{Y}$ is arithmetical, then $G_{\mathscr{Y}}$ is determinate.

5.10 Definition. A subset $\mathscr{Y} \subseteq N^N$ is *degree invariant* if whenever $A, B \subseteq N$ are given such that $A \in \mathscr{Y}$ and $B \equiv_T A$, then $B \in \mathscr{Y}$. A game $G_{\mathscr{Y}}$ is a *degree game* if $\mathscr{Y}$ is degree invariant.

The Axiom of Determinateness is used as follows to show that certain sets of degrees contain cones.

5.11 Lemma. *Let $G_{\mathscr{Y}}$ be a determinate degree game. Then if player I has a winning strategy for $G_{\mathscr{Y}}$ then $\mathscr{Y}$ contains a cone; and if player II has a winning strategy for $G_{\mathscr{Y}}$, then $N^N - \mathscr{Y}$ contains a cone. (A subset of N^N contains a cone if there is a cone of degrees such that all elements of N^N of degree in the cone are elements of the original subset of N^N.)*

Proof. Without loss of generality, we may assume that player I has a winning strategy g for $G_{\mathscr{Y}}$. Let $\mathbf{d}$ be the degree of g. We show that $\{f \in N^N : \mathbf{d} \leqslant \mathbf{f}\}$ is a subset of $\mathscr{Y}$. Fix $h \in N^N$ such that $\mathbf{d} \leqslant \mathbf{h}$. Consider the play of $G_{\mathscr{Y}}$ with player I playing the strategy g and player II playing $h(n)$ at his nth move. Let $f \in N^N$ be constructed by this play of the game. Then $h \leqslant_T f \leqslant_T h \oplus g$ so since $\mathbf{d} \leqslant \mathbf{h}$, $f \equiv_T h$. Since g is a winning strategy for player I, $f \in \mathscr{Y}$; and since $G_{\mathscr{Y}}$ is a degree game, $h \in \mathscr{Y}$. ▯

Theorem 5.9 and Lemma 5.11 combine to give the following result.

5.12 Theorem. **D** *contains a cone of minimal covers.*

Proof. Let $\mathscr{Y} = \{f \in N^N : \mathbf{f}$ is a minimal cover$\}$. We leave it to the reader to verify that $\mathscr{Y}$ is an arithmetical set. By Theorem 5.9, $G_{\mathscr{Y}}$ is determinate, so one of the players has a winning strategy. By Lemma 5.11, it suffices to show that player II cannot have a winning strategy for $G_{\mathscr{Y}}$.

We assume that player II has a winning strategy for $G_{\mathscr{Y}}$ and obtain a contradiction. Under this assumption, Lemma 5.11 implies that $N^N - \mathscr{Y}$ contains a set $\mathscr{C}$ such that $\mathscr{C} = \{\mathbf{C} : C \in \mathscr{C}\}$ is a cone. Fix $\mathbf{d} \in \mathscr{C}$. By Theorem 4.5, $\mathbf{d}$ has a minimal cover $\mathbf{b}$. Since $\mathbf{b} > \mathbf{d}$, $\mathbf{b} \in \mathscr{C}$ so if $f \in N^N$ has degree $\mathbf{b}$, $f \in \mathscr{Y}$. Since $G_{\mathscr{Y}}$ is a degree game, $\mathbf{b} \notin \mathscr{C}$, yielding the desired contradiction. ▯

Theorem 5.12 and Theorem 5.3 combine to specify a sentence in the language of posets which differentiates between $\mathscr{D}$ and $\mathscr{D}_{\mathbf{arith}}$.

5.13 Definition. Let $\mathscr{A}$ and $\mathscr{B}$ be structures which interpret the same language. Then we say that $\mathscr{A}$ and $\mathscr{B}$ are *elementarily equivalent* (write $\mathscr{A} \equiv \mathscr{B}$) if $\mathscr{A}$ and $\mathscr{B}$ satisfy exactly the same sentences in the language.

5.14 Corollary. *$\mathscr{D}$ and $\mathscr{D}_{\mathbf{arith}}$ are not elementarily equivalent.*

Proof. Let S be the sentence in the language of posets which asserts the existence of a cone of minimal covers, i.e., S is

$$\exists d \forall a \exists b (a \geqslant d \to b < a \,\&\, \forall c (b < c \leqslant a \to c = a)).$$

By Theorem 5.12, $\mathscr{D}$ satisfies S. By Theorem 5.3, $\mathscr{D}_{\mathbf{arith}}$ does not satisfy S. ∎

The following definitions are needed for the subsequent discussion.

5.15 Definition. Let $\mathbf{a}, \mathbf{d} \in \mathbf{D}$ be given. We say that $\mathbf{a}$ is a *strong minimal cover for* $\mathbf{d}$ if $\mathbf{a}$ is a minimal cover for $\mathbf{d}$ and $\mathbf{D[0, a)} = \mathbf{D[0, d]}$. We say that $\mathbf{a}$ is a *strong minimal cover* if $\mathbf{a}$ is a strong minimal cover for some $\mathbf{b} \in \mathbf{D}$.

5.16 Definition. $\mathbf{S} \subseteq \mathbf{D}$ is an *initial segment* of $\mathbf{D}$ if for all $\mathbf{d} \in \mathbf{S}$, $\mathbf{D[0, d]} \subseteq \mathbf{S}$.

Much of the remainder of Part B of this book is devoted to classifying the countable initial segments of $\mathscr{D}$. (This problem will be seen to be equivalent to classifying the countable ideals of $\mathscr{D}$.) Theorem 2.10 was the first non-trivial initial segment result to be proved. One might attempt to construct longer initial segments of $\mathscr{D}$ by using strong minimal covers. Unfortunately, it is not known whether every minimal degree has a strong minimal cover, although Simpson [1977] has shown that many minimal degrees do have strong minimal covers. The following theorem shows that a certain property of a degree guarantees that the degree will not have a strong minimal cover. By the Friedberg Jump Theorem, $\mathbf{0}'$ is a degree having this property.

5.17 Theorem. *Let $\mathbf{d} \in \mathbf{D}$ be given such that for all $\mathbf{c} > \mathbf{d}$ there is a $\mathbf{b} < \mathbf{c}$ such that $\mathbf{b} \cup \mathbf{d} = \mathbf{c}$. Then $\mathbf{d}$ does not have a strong minimal cover.*

Proof. Given $\mathbf{c} > \mathbf{d}$, find $\mathbf{b} < \mathbf{c}$ such that $\mathbf{b} \cup \mathbf{d} = \mathbf{c}$. Then $\mathbf{b} \nleqslant \mathbf{d}$ so $\mathbf{c}$ is not a strong minimal cover for $\mathbf{d}$. ∎

By the Friedberg Jump Theorem (relativized) the property given in the hypothesis of Theorem 5.17 is possessed by all degrees in $\mathbf{GH_0}$, so no $\mathbf{d} \in \mathbf{GH_0}$ has a strong minimal cover. Jockusch [1980] shows that 2-generic degrees have this property, and by Exercise IV.3.17, all degrees in $\overline{\mathbf{GL_2}}$ have this property. Shore has independently constructed degrees with this property which are not in $\mathbf{GH_0}$. Hence there are many degrees without strong minimal covers. Cooper has shown that there is a non-zero recursively enumerable degree with a strong minimal cover.

The proof of Theorem 5.17 easily implies the following result.

5.18 Corollary. *Let $\mathbf{d} \in \mathbf{D}$ be given such that for all $\mathbf{c} \geqslant \mathbf{d}$ there is a degree $\mathbf{b} < \mathbf{c}$ such that $\mathbf{b} \cup \mathbf{d} = \mathbf{c}$. Then no degree in $\mathbf{D(d, \infty)}$ is a strong minimal cover.*

5.19 Remarks. Theorem 5.3 was proved by Jockusch and Soare [1970]. In the same paper it was shown, under the assumption that all Σ^0_5 degree games are determinate, that Theorem 5.12 holds. This proof preceded Martin's [1975] proof of Borel Determinateness, so was not an outright theorem of Set Theory. After Paris [1972] proved that all Σ_4^0 games are determinate, Jockusch [1973] found a different game which was not a degree game but nevertheless yielded a cone of minimal covers. Subsequently, Harrington and Kechris [1975] found a Σ_1^0 game which implied the existence of a cone of minimal covers, and also computed a vertex for this cone, the degree of Kleene's $\mathcal{O}$ (the degree of a complete Π_1^1 set – the superscript 1 allows the use of set quantifiers). Jockusch and Shore [1983a] then found a smaller vertex, $\mathbf{0}^{(\omega)}$, for such a cone ($\mathbf{0}^{(\omega)}$ is the degree of $\emptyset^{(\omega)}$, where $\emptyset^{(\omega)}$ is defined by $(\emptyset^{(\omega)})^{[n]} = \emptyset^{(n)}$ for all $n \in N$). Lemma 5.11 was proved by Martin [1968].

5.20–5.22 Exercises

5.20 Show that $\{f \in N^N\colon \mathbf{f}$ is a minimal cover$\}$ is a Σ_5^0 set.

5.21 Show that the arithmetical degrees form a usl.

5.22 Let $\mathbf{b} \in \mathbf{D}$ be given. Show that $\mathbf{D}[\mathbf{b}, \infty)$ contains a cone of degrees which are minimal covers of degrees $\geqslant \mathbf{b}$.