Part D

The Number of Models

In the remainder of this book, we calculate the possible spectra for two classes. If T is a countable superstable theory we will give the possible functions $I(\aleph_\alpha, \mathbf{S})$. If T is a countable ω-stable theory we will give the possible functions $I(\aleph_\alpha, \mathbf{AT})$. A number of the theorems extend to more general classes of theories or to different classes of models. We have tried to build a framework which handles these more general cases. Thus, there are theorems and exercises referring to them. Some of the extensions we touch on are $I(T, \mathbf{AT}_\kappa)$ for any uncountable κ, uncountable T, and small countable superstable T.

The calculation proceeds by first classifying the theories and then computing the spectra in each class. We begin with the fact, proved in Section IX.6, that if T is not superstable then $I(\kappa, AT) = 2^{|\kappa|}$. (We gave the proof only for regular κ). Although we did not prove it here, it is shown in Chapter VII of [Shelah 1978], that $I(\kappa, \mathbf{S}) = 2^\kappa$. This justifies our assumption in the remainder of this book that T is superstable. Chapter XIV collects some of the main tools used in the computation. In Chapter XV we distinguish the bounded from the unbounded, or multidimensional, theories. We classify the spectra of bounded theories and compute a lower bound for the spectrum of an unbounded theory. Thereafter, we need only analyze unbounded theories. We also introduce in Chapter XV the notion of an eventually nonisolated type which is crucial for the study of countable models.

In Chapter XVI we introduce a major dividing line, the dimensional order property (DOP). We prove that if T has the dimensional order property then T has 2^κ $\mathbf{S}$-models in every power $\kappa \geq 2^{|T|}$. We also find a flaw in our classification of the classes of models to study. That is, we will prove that a theory T has the DOP for all of the classes K introduced earlier or for none of them. But there is another variant, the ENI-DOP which must be investigated to deal with countable models.

In Chapter XVII, we see that if T does not have the dimensional order property then every model can be decomposed into a tree of small models. If for each model the tree is well-founded, T is called 'shallow'; then we are able to assign invariants to the models in the manner suggested in Section

I.1. If not the theory is called 'deep' and again the theory has the maximal number of models.

Thus, we establish the 'main gap'. Either for all uncountable cardinals κ, $I(\kappa, K) = 2^\kappa$ or for all $\alpha > 0$, $I(\aleph_\alpha, K) \leq \beth_{\aleph_1}(\alpha)$. In Chapter XVIII we undertake a more detailed analysis of ω-stable theories. We establish both the Vaught and Morley conjectures for ω-stable T. That is, we show that if T is ω-stable then T has either countably many or $2^{\aleph_0}$ countable models and the spectrum function is increasing on uncountable cardinals. Shelah extends the second result to all theories in [Shelah 198?].

These computations of the spectrum functions have an interesting sidelight. It is by no means evident that the spectrum function of a theory T does not depend on the axioms for set theory. Indeed, for uncountable theories there are examples where it does so depend (cf. [Shelah 1978] Chapter IX). However, we will establish in this book equivalences (for countable theories) between various spectrum functions and certain 'syntactic' conditions which are clearly absolute.

We restrict our attention to countable first order theories. For the case of **S**-models this restriction is totally unnecessary. In order to extend the results to arbitrary models of an uncountable theory, one must generalize the notion of ω-stability. Shelah has done this (by generalizing Morley's definition via rank rather than the spectrum of stability) but we do not deal with this generalization here.

Throughout Part D, K denotes an acceptable class. Unless expressly asserted otherwise, we assume it admits stationary strongly regular types. This assumption provides the obvious obstruction to extending the results in Part D on countable ω-stable theories to countable superstable theories. Shelah surmounts this obstacle in [Shelah 198?].

Chapter XIV

The Construction of Many Nonisomorphic Models

We describe in this chapter the methods by which we later construct many nonisomorphic models of theories with certain specified properties. The basic technique is to construct for each graph G from a large family of graphs a model M_G in such a way that if M_G is isomorphic to M_H then G is isomorphic to H. The tool for this will be to encode the points of the graph and the edges between them by the sizes of certain indiscernible sets. In the first section of this chapter we review the construction of maximal sets of pairwise nonisomorphic graphs. In the second section we discuss how to identify the elements or, often, equivalence classes of elements that we want to use as the points of a graph. That is, we show how to construct models with the dimensions of types over certain base sets prescribed in advance. The methods of Section 2 are, in general, unable to prescribe dimensions below $\lambda(\mathbf{I})$. In Section 3, we make this limitation more precise and single out the exceptions to it.

1. Many Nonisomorphic Graphs

Many theorems showing that certain theories have many non-isomorphic models will be established by coding certain well known complicated classes, like the class of all graphs, into each theory satisfying a certain condition. In this section we justify this procedure by showing that the classes of graphs which we will interpret later do have many nonisomorphic models. We begin by establishing some nomenclature.

1.1 Definition. A *graph* is a set with a symmetric binary relation. A *directed graph* is a set with an asymmetric binary relation.

It is easy to construct 2^λ non-isomorphic directed graphs of power λ. For any $X \subseteq \lambda$, let G_X be the disjoint union of copies of the $\alpha \in X$ and use $<$ as the binary relation. It is somewhat more difficult to construct 2^λ

symmetric graphs of power λ, although the basic strategy is the same. To make it easier to identify the graph we impose one further condition.

1.2 Definition. A *triangle* in a graph G is a triple of points such that each pair is connected by an edge.

We will define another notion of triangle in Chapter XV. It will be clear from context which notion is meant.

1.3 Proposition. For every cardinal λ there are 2^λ non-isomorphic symmetric graphs which contain no triangles.

Proof. (Fig. 1). The key to the proof is to construct for each ordinal α a connected symmetric graph G_α which contains no triangles such that if $\alpha \neq \beta$ then $G_\alpha \not\approx G_\beta$. The universe of G_α will contain

$$\alpha \cup \{a_{\beta,\gamma} : \beta < \gamma < \alpha\} \cup \{b^0_{\beta,\gamma} : \beta < \gamma < \alpha\} \cup \{b^1_{\beta,\gamma} : \beta < \gamma < \alpha\}.$$

We will encode the usual ordering on α by means of the auxiliary elements.

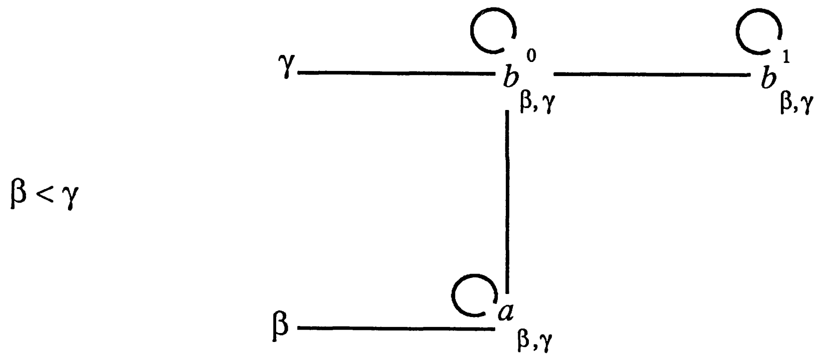

Fig. 1. Encoding orders in graphs

Let R be the symmetric closure of the following relation:

$$\begin{aligned}&\{\langle \beta, a_{\beta,\gamma}\rangle : \beta < \gamma < \alpha\} \cup \{\langle \gamma, b^0_{\beta,\gamma}\rangle : \beta < \gamma < \alpha\}\\ &\cup \{\langle a_{\beta,\gamma}, b^0_{\beta,\gamma}\rangle : \beta < \gamma < \alpha\} \cup \{\langle a_{\beta,\gamma}, a_{\beta,\gamma}\rangle : \beta < \gamma < \alpha\}\\ &\cup \{\langle b^1_{\beta,\gamma}, b^0_{\beta,\gamma}\rangle : \beta < \gamma < \alpha\} \cup \{\langle b^1_{\beta,\gamma}, b^1_{\beta,\gamma}\rangle : \beta < \gamma < \alpha\}.\end{aligned}$$

It isn't hard to verify that G_α is as required.

1.4 Exercise. Show that there are 2^λ nonisomorphic symmetric graphs of cardinality λ which contain no triangles but each point is connected to at least two others. (Hint: Put tails on the elements of the graph constructed for Proposition 1.3.)

1.5 Historical Notes. These results are all well-known. There is an elementary exposition in [Manaster 1972].

2. *Models with Prescribed Dimensions*

This section contains some of the most useful constructions in the entire book. We show how to construct models with specified dimensions for certain families of regular types. These constructions will be applied repeatedly to construct non-isomorphic models in the remainder of Part D.

The relation of orthogonality establishes a dichotomy on pairs of types. Roughly, if two types are not orthogonal, they have the same dimension; if they are orthogonal, their dimensions can vary arbitrarily. We devote this section to making this intuition precise.

We begin by showing that if the K-strongly regular type p is orthogonal to the K-strongly regular type q we can increase the dimension of p arbitrarily without increasing the dimension of q. Then we show that we can increase the dimension of a fixed q without increasing the dimension of any of a family of p's each orthogonal to q. In Theorem 2.4 we turn this argument on its head and show that we can fix the dimension of each type in a family X while increasing the dimensions of all types orthogonal to each $p \in X$. After some exercises illustrating these techniques, the remainder of the section considers sufficient conditions for the dimension of two types to be equal. Throughout this section we assume T is superstable.

2.1 Theorem. *Let T be superstable.*

i) *Suppose $p \perp q$ are K-strongly regular types which are each strongly based on subsets of $M \in K$. Then for any $\kappa \geq \max(|M|, \lambda_0(K))$ there is a model $N \in K$ with $\dim(p, N) = \dim(p, M)$ and $\dim(q, N) \geq \kappa$.*
ii) *Further, let X be a collection of K-strongly regular types each strongly based on a subset of M. Suppose $q \perp p$ for every $p \in X$. For each $\kappa \geq \max(|M|, \lambda_0(K))$ there is a model N with $|N| = \kappa$ such that $\dim(q, N) = \kappa$ and for each $p \in X$, $\dim(p, N) = \dim(p, M)$.*

Proof. (Fig. 2). i) Let E be an independent set of κ realizations of the nonforking extension of q to $S(M)$ and let N be K-prime over $M \cup E$. By XII.4.4, $\dim(p, N) = \dim(p, M) + \dim(p^M, N)$. But $\dim(p^M, N) = 0$ by Theorem X.4.6. Assertion ii) is easily shown by an increasing chain argument, repeatedly applying i).

2.2 Exercise. Prove Theorem 2.1 ii).

We now show how to combine these results to construct models with certain prescribed dimensions. The following definition codifies the situation of Theorem 2.1 ii).

2.3 Definition. i) If S is a family of stationary K-strongly regular types, the type q is *irrelevant* to S if $p \perp q$ for each $p \in S$.
Recall the following notation which is extremely useful to describe families of types.
ii) Suppose $p \in S(\overline{a} \cup A)$ and $t(\overline{b}; A) = t(\overline{a}; A)$. Then $p_{\overline{b}}$ denotes the image of p under an automorphism which maps $\overline{a}$ to $\overline{b}$ and fixes A.

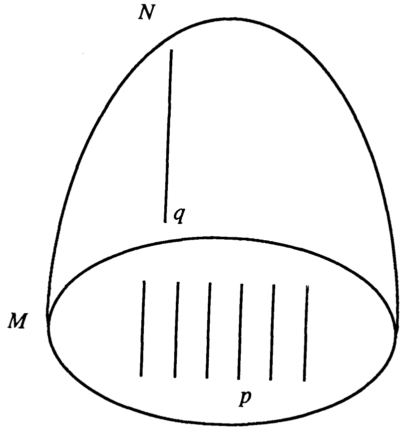

Fig. 2. Theorem XIV.2.1

It would be reasonable in i) to say q is orthogonal to S (and indeed Makkai in [Makkai 1984] does) but this seems to make the context bear too much burden as to which of three meanings of orthogonal is meant.

2.4 Theorem. *Let T be superstable. Let S be a family of pairwise orthogonal K-strongly regular stationary types over $A \subseteq M$. For any cardinal $\kappa \geq \max(\lambda_0(K), \kappa(T), |M|)$ there is a model $M_\kappa \supseteq M$ of T satisfying the following conditions.*

i) $|M_\kappa| = \kappa$.
ii) *If* $p \in S$, $\dim(p, M_\kappa) = \dim(p, M)$.
iii) *Suppose q is based on a subset of M_κ and q is irrelevant to S. Then* $\dim(q, M_\kappa)$ *is* κ.

Proof. Let $\langle q_i : i < \mu \rangle$ enumerate the K-strongly regular types which are strongly based on a subset of M and are irrelevant to S. Since T is superstable, $\mu \leq \kappa$. Define $\langle M_i : i < \mu \rangle$ by induction with $M_0 = M$. Take unions at limit ordinals and apply Theorem 2.1 ii) to choose M_{i+1} with $|M_{i+1}| = \kappa$, $\dim(q_i, M_{i+1}) = \kappa$, and for each $p \in S$, $\dim(p, M_{i+1}) = \dim(p, M_i)$. By induction, $\dim(p, M_i) = \dim(p, M)$ and by Theorem X.4.6 $\dim(p^{M_i}, M_{i+1}) = 0$. Let $N_0 = \bigcup_{i<\mu} M_i$. Construct N_i for $i < \omega$ by iterating this procedure. Then $M_\kappa = \bigcup_{i<\omega} N_i$ is the required model.

There are two uses of superstability in the preceding proof. The less important arises from the need to have $\mu \leq \kappa$. For this, one can assume that $\kappa \geq |M|^{<\kappa(T)}$ and $\kappa = \kappa^{<\kappa(T)}$. (The second of these requirements is needed for the iteration.) More essential for the application of the theorem is restriction to K-strongly regular types. The crucial use of regularity is to show (in the proof of Theorem 2.1) that

$$\dim(p, N) = \dim(p, M) + \dim(p^M, N).$$

2.5 Exercise. Let S be a family of stationary K-strongly regular types with each $p_{\overline{a}} \in S$ strongly based on $\overline{a} \in A$ and with $p_{\overline{a}} \perp t(A - \overline{a}; \overline{a})$.

Choose for each $p \in S$, a cardinal λ_p with $\lambda(\mathbf{I}) \leq \lambda_p \leq \kappa$. There is a model M_κ such that i) $\dim(p, M_\kappa) = \lambda_p$ if $p \in S$ and ii) $\dim(q, M_\kappa) = \kappa$ if q is irrelevant to S. (Hint: Let E_p be a set of λ_p independent realizations of p for each $p \in S$ and let $B = A \cup \bigcup_{p \in S} E_p$. Now if $p'_{\overline{a}}$ denotes the nonforking extension of $p_{\overline{a}}$ to $\overline{a} \cup E_{p_{\overline{a}}}$, $p'_{\overline{a}} \perp t(B - \overline{a}; \overline{a})$.)

The following exercise is easily proved by combining the techniques of Theorems 2.1 and 2.4.

2.6 Exercise. Let S be a family of stationary K-strongly regular types with each $p_{\overline{a}} \in S$ strongly based on $\overline{a} \in A$ and with $p_{\overline{a}} \perp t(A - \overline{a}; \overline{a})$. If λ_p, for each $p \in S$, denotes $\dim(p, M_p)$ where M_p is K-prime over the finite set on which p is strongly based and $\lambda_p \leq \kappa$, then there is a model M_κ such that i) $\dim(p, M_\kappa) = \lambda_p$ if $p \in S$ and ii) $\dim(q, M_\kappa) = \kappa$ if q is irrelevant to S.

Now we determine when two strongly regular types have the same dimension in a model N. There are several steps to this procedure. First we deal with an arbitrary pair of nonorthogonal K-strongly regular types over $M \in K$ with $M \subseteq N$. Then we restrict the types to be two copies of a type over an element $\overline{b}$. The relation between the dimensions of $p_{\overline{b}}$ and $p_{\overline{b}'}$ are seen to depend on properties of $\overline{b}$ and $\overline{b}'$. For this situation we deal first with the case that $\overline{b}$ and $\overline{b}'$ have the same strong type. Then we make the further assumption that the type of $\overline{b}$ is $\mathbf{I}$-isolated. With this assumption we no longer have to require the type to be stationary.

2.7 Theorem. *Suppose $p, q \in S(M)$, $M, N \in K$, $M \subseteq N$ and the pairs $(p, p_0), (q, q_0)$ are K-strongly regular. If $p \not\perp q$ then* $\dim(p, N) = \dim(q, N)$.

Proof. Let $E = \langle \overline{e}_\alpha : \alpha < \kappa \rangle$ be a basis for $p(N)$. Define by induction models N_α for $\alpha < \kappa$ with $N_0 = M$ and $N_{\alpha+1}$ a submodel of N which is K-prime over $N_\alpha \cup \overline{e}_\alpha$. By Exercise X.1.21, for each α, $N_\alpha \cap E = E_\alpha$. By Theorem XII.4.5 the nonforking extension of q to N_α is realized in $N_{\alpha+1}$. So $\dim(p, N) \leq \dim(q, N)$. Reversing the roles of p and q we finish.

This previous proof used in an essential way the hypothesis that p and q were types over K-models. If we relax this hypothesis we must weaken the conclusion somewhat. For cardinals κ μ, and λ, we write $\kappa = \mu \operatorname{mod}(\lambda)$ if $\kappa + \lambda = \mu + \lambda$.

2.8 Theorem. *If p, $q \in S(A)$ are K-strongly regular and $p \not\perp q$ then for any $N \in K$,* $\dim(p, N) = \dim(q, N) \operatorname{mod}(\lambda(\mathbf{I}))$.

Proof. Let $M \prec N$ be K-prime over A. By Theorem X.4.5 both $\dim(p, M)$ and $\dim(q, M)$ are less than or equal $\lambda(\mathbf{I})$. By Theorem XII.4.1 and Theorem XII.4.5 $\dim(p, N) = \dim(p, M) + \dim(p^M, N)$ and similarly for q. But $\dim(p^M, N) = \dim(q^M, N)$ by Theorem 2.7 so we finish.

We can remove the 'mod $\lambda(I)$' if we tighten our control on the types p and q.

2.9 Lemma. *Let $p \in S(A \cup \bar{b})$ be K-strongly regular and suppose $p \not\perp A$. Then for all $\bar{b}'$ realizing $stp(\bar{b}; A)$ and all $M \in K$ containing $A \cup \bar{b} \cup \bar{b}'$, $\dim(p, M) = \dim(p_{\bar{b}'}, M)$.*

Proof. (Fig. 3). Let p' denote $p_{\bar{b}'}$ and choose $N \prec M$ to be a K-prime model over $A \cup \bar{b} \cup \bar{b}'$. If $\bar{b} \downarrow_A \bar{b}'$ then $\{\bar{b}, \bar{b}'\}$ is a set of indiscernibles over A. Thus, N is K-prime over $A \cup \bar{b}' \cup \bar{b}$ (Note the change in order of $\bar{b}, \bar{b}'$.) and so $\dim(p, N) = \dim(p', N)$. By Theorem 2.7 and Theorem XII.4.4, we conclude $\dim(p, M) = \dim(p', M)$.

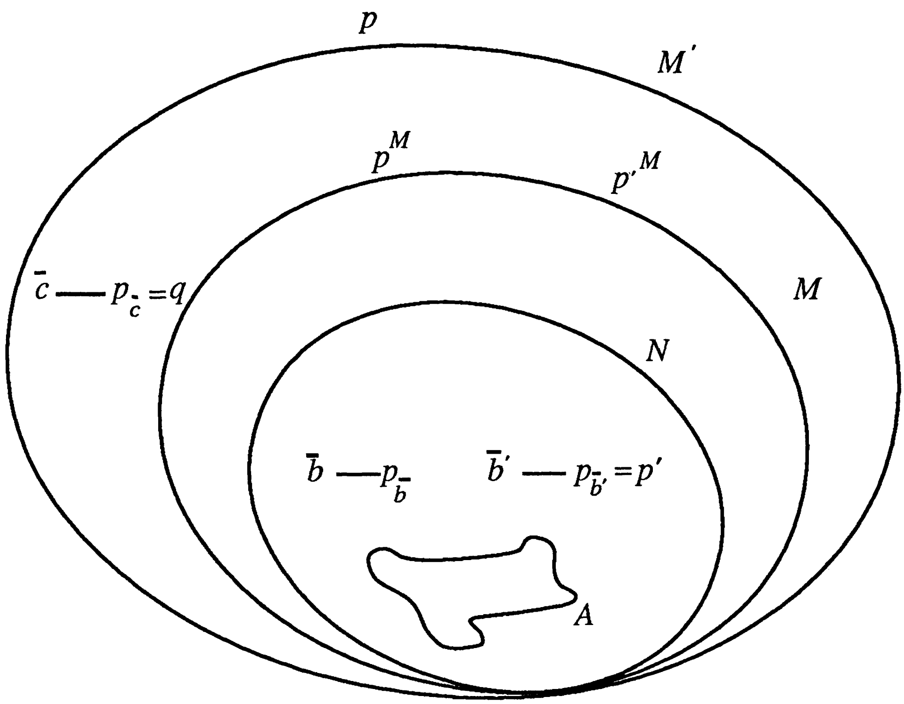

Fig. 3. Theorem XIV.2.9

Even if $(\bar{b} \not\downarrow \bar{b}'; A)$, we can reduce to the previous case as follows. Choose $\bar{c}$ realizing $stp(\bar{b}; A)$ with $\bar{c} \downarrow_A M$. Let $q = p_{\bar{c}}$ and choose M' which is K-prime over $M \cup \bar{c}$. Now let p^M and p'^M be the nonforking extensions to $S(M)$ of p and p' respectively. Since $p \not\perp A$, Theorem VI.2.22 implies $p \not\perp q$ and $p' \not\perp q$. By the transitivity of nonorthogonality on K-strongly regular types, $p^M \not\perp p'$. By Theorem 2.7 $\dim(p^M, M') = \dim(p'^M, M')$ and by the first case $\dim(p, M') = \dim(q, M') = \dim(p', M')$. By Theorem XII.4.4, we have

$$\dim(p, M') = \dim(p, M) + \dim(p^M, M')$$

and

$$\dim(p', M') = \dim(p', M) + \dim(p'^M, M').$$

By Exercise X.1.21, each realization, $\overline{a}$, of p^M in M' satisfies $\overline{a} \not\downarrow_M \overline{c}$. Thus, $\dim(p^M, M') < \kappa(T) = \omega$ so we can subtract to conclude $\dim(p, M) = \dim(p', M)$.

In this situation we can strengthen the criterion for nonorthogonality of Theorem VI.2.22.

2.10 Lemma. *Let $p \in S(A \cup \overline{b})$ be K-strongly regular and suppose $p \not\downarrow A$. For any $\overline{b}'$ realizing $stp(\overline{b}; A)$, $p_{\overline{b}} \not\perp p_{\overline{b}'}$.*

Proof. By Lemma 2.9, for all $M \supseteq A$, $\dim(p_{\overline{b}}, M) = \dim(p_{\overline{b}'}, M)$. By the contrapositive of Theorem 2.1, we have the result.

If we add the requirement that $t(\overline{b}; A)$ is **I**-isolated then we can weaken the requirement in Lemma 2.10 that $\overline{b}'$ realize the same strong type as $\overline{b}$ to the requirement that it realize the same type. We obtain this generalization in Theorem 2.15. The following facts are needed in the argument.

2.11 Exercise. Show that if $t(\overline{a}; A)$ has finite multiplicity there is an E in $FE(A)$ such that $t(\overline{a}; A) \cup E(\overline{x}; \overline{a}) \vdash stp(\overline{a}; A)$.

2.12 Lemma. *Let T be stable and suppose $\overline{a}, \overline{b} \in M \models T$. Assume $t(\overline{a}^\frown\overline{c}; \emptyset)$ has finite multiplicity and $t(\overline{c}; \overline{a})$ is **AT**-isolated. If $stp(\overline{a}; \emptyset) = stp(\overline{b}; \emptyset)$ then there exists $\overline{d} \in M$ with $stp(\overline{a}^\frown\overline{c}; \emptyset) = stp(\overline{b}^\frown\overline{d}; \emptyset)$.*

Proof. Choose $E \in FE(\emptyset)$ such that $t(\overline{a}^\frown\overline{c}; \emptyset) \cup E(\overline{x}, \overline{a}^\frown\overline{c}) \vdash stp(\overline{a}^\frown\overline{c}; \emptyset)$. Let $\phi(\overline{y}; \overline{a}) \vdash t(\overline{c}; \overline{a})$. Since $stp(\overline{a}; \emptyset) = stp(\overline{b}; \emptyset)$, there is an $\overline{e} \in \mathcal{M}$ with $stp(\overline{a}^\frown\overline{c}; \emptyset) = stp(\overline{b}^\frown\overline{e}; \emptyset)$. Thus, $(\exists\overline{y})[\phi(\overline{y}; \overline{b}) \wedge E(\overline{a}^\frown\overline{c}, \overline{b}^\frown\overline{y})]$ is satisfiable. Choose $\overline{d}$ in M to witness this formula.

The following Lemma does not hold for an arbitrary acceptable class and I have not found an abstract formulation of a sufficient condition. Thus, I remark that it holds in the cases we are most interested in.

2.13 Lemma. *Let T be superstable and $K = \mathbf{S}$ or T be ω-stable and $K = \mathbf{AT}$. Suppose p is **I**-isolated over A with $|A| < \lambda(\mathbf{I})$. For any $\overline{b}$ realizing p and any $M \in K$ with $A \subseteq M$, $stp(\overline{b}; A)$ is realized in M.*

Proof. This is obvious for the class of **S**-models. When T is a countable ω-stable theory we can use the fact that every type has finite multiplicity. For, if $p \in S(A)$ is an isolated type with multiplicity n the type of n points which realize p but realize different strong types over A is also isolated.

2.14 Exercise. Show that this result fails for an arbitrary acceptable class by considering the theory REF_ω and letting $K = \mathbf{AT}$.

2.15 Theorem. *Let T be superstable and $K = \mathbf{S}$ or T be ω-stable and $K = \mathbf{AT}$. Let $A \subseteq M \in K$ and suppose $\overline{b}, \overline{b}'$ realize the **I**-isolated type, q, over A. Let $p \in S(A \cup \overline{b})$ be K-strongly regular, let p' denote $p_{\overline{b}'}$, and suppose $p \not\perp p'$. For any $M \supseteq A \cup \{\overline{b}, \overline{b}'\}$ with $M \in K$, $\dim(p, M) = \dim(p', M)$.*

Proof. Let $N \prec M$ be K-prime over $A \cup \bar{b}$ and choose by Lemma 2.13 $\bar{c} \in N$ realizing $stp(\bar{b}'; A)$. Let r denote $p_{\bar{c}}$. By Lemma 2.9 and Lemma 2.10, $\dim(p', M) = \dim(r, M)$ and $p' \not\perp r$. As N is also K-prime over $A \cup \bar{c}$, $\dim(p, N) = \dim(r, N)$. By the transitivity of nonorthogonality on regular types, $p \not\perp r$. Thus, $\dim(p^N, M) = \dim(r^N, M)$. By Theorem XII.4.4

$$\dim(p, M) = \dim(p, N) + \dim(p^N, M)$$

and

$$\dim(r, M) = \dim(r, N) + \dim(r^N, M).$$

So $\dim(p', M) = \dim(r, M) = \dim(p, M)$ as required.

2.16 Historical Notes. These kinds of construction originate in [Shelah 1978]. However, the emphasis on types rather than indiscernible sets is due to Lascar. Thus, the general outline of the first half of this section stems from Lascar via [Makkai 1984]. Theorem 2.7 through Theorem 2.13 comes fairly explicitly from [Bouscaren 1983] and [Bouscaren & Lascar 1983].

3. *Tractable Types*

We combine here the results of Section X.4 on the dimension of arbitrary indiscernible sets in K-prime models with the more refined results for strongly regular types described in Section 2 of this chapter. If a type is orthogonal to the empty set we are able to determine almost at will the dimension of various copies of the type. In this section we make precise 'almost at will'; in the next chapter we begin to apply these constructions. We are trying to develop a general framework to study both **S**-models and **AT**-models. In the second case, nonisolated types play a special role because they can have finite dimension.

3.1 Definition. We say the K-strongly regular stationary type p over A is (μ, K)-*tractable* if for any C with $t(C; A) \perp p$, if M is K-prime over $A \cup C$ then $\dim(p, M) < \mu + |A|$.

This definition is introduced primarily so that the following two situations can be treated uniformly in constructions. Recall from Definition VI.2.1 that a type p is unbounded if it orthogonal to the empty set.

3.2 Lemma. *Let $p \in S(B)$ be stationary. Suppose further that $B \subseteq A$, and $p \dashv \emptyset$. Regarded as a type over A,*

i) *p is $(\lambda(\mathbf{I})^+, K)$-tractable.*
ii) *If p is K-strongly regular and not $\mathbf{I}$-isolated then p is $(\lambda(\mathbf{I}), K)$-tractable.*

Proof. i) Let M be K-prime over $A \cup C$ and E a maximal independent set of realizations in M of p. By V.1.19 there is an $E_0 \subseteq E$ with $|E_0| < \kappa(T) + |A|$ such that, letting E' denote $E - E_0$, $E' \downarrow_B A$ and E' is a set of indiscernibles over A. Now if $\bar{e} \in E'$, $t(\bar{e}; A) \perp t(C; A)$ so, applying Theorem VI.1.19, $t(E'; A) \perp t(C; A)$. In particular, $E' \downarrow_A C$ and so E' is a set of indiscernibles over $A \cup C$. By Theorem X.4.5, $|E'| \leq \lambda(\mathbf{I})$. So $|E| \leq \lambda(\mathbf{I}) + \kappa(T) + |A|$. Since $\kappa(T) \leq \lambda(\mathbf{I})$, we have i).

ii) Note that $\bar{e} \in E'$ implies $\bar{e} \downarrow_B A \cup C$. Thus, by the open mapping theorem, $t(\bar{e}; A \cup C)$ is not $\mathbf{I}$-isolated and so is not realized in M. That is, $E' = \emptyset$. Since $\lambda(\mathbf{I}) \geq \kappa(T)$, $|E| = |E_0| < \lambda(\mathbf{I})$ as required.

The following exercise is an immediate application of the lemma. Its solution is contained in the proof of the more complicated application in Theorem 3.5.

3.3 Exercise. Suppose $p_i \in S(A_i)$ is $\dashv \emptyset$ (and not $\mathbf{I}$-isolated), $\{A_i : i < \kappa\}$ is an independent sequence of sets with each $|A_i| < \lambda(\mathbf{I})$, and M is K-prime over $A = \cup A_i$. Then for each i, $\dim(p_i, M) < \lambda(\mathbf{I})^+$ ($\dim(p_i, M) < \lambda(\mathbf{I})$.) (Hint: Apply Lemma 3.2 to each p_i taking A_i for both A and B and $\bigcup_{j \neq i} A_i$ for C).

3.4 Exercise. Show that if T is a countable ω-stable theory then $p \in S(A)$ is $(\aleph_0, AT)$-tractable if and only if p is nonprincipal.

We will repeatedly appeal to the construction described in the following theorem.

3.5 Theorem. *Suppose $\kappa > max(\lambda_0(\mathbf{I}), \lambda(\mathbf{I}))$. Fix a cardinal $\mu \geq \kappa(T)$. Let A_i for $i < \lambda$ be an independent sequence of sets with $|A_i| < \lambda(\mathbf{I})$. Let S_i be a set of (μ, K)-tractable types over A_i which are each orthogonal to $\emptyset$. Then there is a model N such that:*

i) $|N| = \kappa$.
ii) $\dim(p, N) < \mu$ *if* $p \in S = \bigcup_{i<\lambda} S_i$.
iii) $\dim(q, N) = \kappa$ *if q is over N and q is irrelevant to S.*

Proof. Let M be K-prime over $A = \bigcup_{i<\lambda} A_i$. If A^i denotes $A - A_i$, we have $A^i \downarrow_\emptyset A_i$. This implies by Theorem VI.2.21 that $p \perp t(A^i; A_i)$ for each i and each $p \in S_i$. Thus, by the definition of tractable, for each $p \in S$, $\dim(p, M) < \mu$. Now extend M to N to satisfy i) and iii) while preserving ii) by Theorem 2.4.

The following exercise indicates one way the sequence of independent sets A_i in the hypothesis of Theorem 3.5 can be found when applying that theorem.

3.6 Exercise. Let $p \in S(B)$ and $p \dashv \emptyset$. Suppose $t(C; B) \perp p$, $p \perp q$, $q \in S(C)$, and $q \dashv \emptyset$. Show that if E is an independent set of realizations of q then $p \perp t(E \cup C; B)$ and thus if M is K-prime over $B \cup C \cup E$, $\dim(p, M) \leq \lambda(\mathbf{I})$.

3.7 Historical Notes. Of course these notions originate with Shelah [Shelah 1978]. But this sort of construction is found more explicitly in [Makkai 1984] and [Bouscaren & Lascar 1983].