Part C

Local Dimension Theory

In Part A, we began the study of an independence relation for elements of models of stable theories. In Chapter II we noticed that, taking dependence to be the negation of independence, our notion does not in general satisfy the transitivity axiom for dependence. In Part B, we sought a global remedy for this problem. That is, we considered several notions of dependence which are transitive and thus provide a notion of the 'closure' of a set. Since none of these are the negation of the independence relation, they do not immediately yield a notion of dimension. In this part, we return to the study of nonforking and study those types, p, such that forking is transitive when restricted to realizations of p. We will be able to assign a well defined dimension, even in the finite case, to the set of realizations of such a 'regular' type.

There are, however, several complications. The most important is that we can not single out the 'regular' types with one definition which is appropriate for any theory T. Rather, we must define a notion of K-regularity where K is the set of $\mathbf{I}$-saturated models for an isolation relation $\mathbf{I}$ and vary the choice of $\mathbf{I}$ according to the stability class of T.

Our goal is to provide a structure theory for the models of a theory T. As a test problem we try to calculate the spectrum function of T. In general, this problem is too difficult to be attacked directly. We can directly attack the problem of calculating the number of $\mathbf{S}$-models of a superstable theory and the number of models of an ω-stable theory. The methods of attack on these two problems in [Shelah 1978], [Shelah 1982], and [Harrington & Makkai 1985] are parallel. Thus, in this book we have attempted to develop a single theory which specializes to the two cases. For this, we define in Chapter XI the notion of an acceptable class K. In so far as is possible, we provide a common development of the theory for any such class. In fact, the class of $\mathbf{S}$-saturated models plays a distinguished role which prevents a completely uniform treatment. The formulation of the notion of acceptable class K given here and the general treatment of the concepts of regularity are only a first attempt at a theory which we hope can be extended eventually, e.g. to the infinitary case.

The bulk of Part C is devoted to the development of what is sometimes referred to as 'intermediate' stability theory. In Part D, we begin the study of 'advanced' stability theory, the use of the tools developed here to analyze models. The building blocks for models in this analysis will be the regular and weight one types defined in Chapters XII and XIII. The mortar between these blocks is the compulsion notion which is introduced in Chapter XI and studied throughout this part of the book.

Chapter XI

Acceptable Classes

Section 1 of this chapter describes the framework for the rest of this book. Our primary concerns are with the class of all models of an ω-stable theory and the class of $\mathbf{S}_{\aleph_0}$-saturated models of a superstable theory. However, the notion of acceptable class will encompass a number of other classes such as the class of $\mathbf{AT}_\lambda$-saturated models of T if $\lambda \geq \kappa(T)$. Section 2 contains some more detailed information about the class of $\mathbf{S}$-models.

1. Basic Properties of an Acceptable Class K

Two classes K are the primary focus of our analysis:

i) the strongly $\kappa(T)$-saturated models of a superstable theory,
ii) all models of an ω-stable theory.

The notion of an acceptable class is somewhat broader. It includes, for example, the class of $\mathbf{AT}_\lambda$-saturated models of T if $\lambda \geq \kappa(T)$ and the class of strongly $\kappa(T)$-saturated models of a stable theory. We try to isolate some of the properties of an acceptable class which would have to be preserved to extend the theory to include, for example, the infinitary case or the class of pure-injective submodules of models of a theory of modules.

The class of $\mathbf{S}$-models of a superstable theory is the easiest to analyze. In some sense it was defined exactly to make the analysis work. The difficulty is to see what variations in the arguments used in that situation will allow the results to carry over to the other cases. For strongly $\kappa(T)$-saturated models the superstable case is distinguished from the stable case by the abundant supply of regular types (cf. Section XII.2).

1.1 Definition. K is an *acceptable class* of structures for the theory T if

i) K is the class of $\mathbf{I}$-saturated models for an isolation relation $\mathbf{I}$ described in Definition IX.2.2 for which $\lambda(I) \geq \kappa(T)$.
ii) T admits K-prime models over all sets.

We denote by $\lambda_0(K)$ or $\lambda_0(\mathbf{I})$ the least cardinality of a member of K.

We will define a number of notions relative to an acceptable class K. These may be prefixed either by **I** as in **I**-prime or K as in K-strongly regular. We sometimes refer to members of K as K-models.

The following easy exercise lists the most interesting acceptable classes.

1.2 Exercise. Verify that if $\lambda \geq \aleph_1$ and T is stable then the classes of $\mathbf{AT}_\lambda$ and $\mathbf{S}_\lambda$-saturated models are acceptable. Moreover, if T is ω-stable the class of all models is acceptable. Finally, for any superstable T, the class of **S**-saturated models is acceptable.

The properties of the class K described in the following exercise suffice to prove all the results in Section XII.1.

1.3 Exercise. Show that if K is acceptable class it has the following property.

For each $M \in K$ and each set A with $|A| < \lambda(I)$ there is a model $M[A] \in K$ containing $M \cup A$ which is **I**-saturated and satisfies the following conditions.

i) $A \rhd_M M[A]$.
ii) If $M, N \in K$ and $M \cup \bar{a} \subseteq N$ then there is an **I**-constructible $M' \in K$ with $M \cup \bar{a} \subseteq M' \subseteq N$.

Of course, since T admits K-prime models $M[A]$ is just a K-prime model over $M \cup A$. We have explicitly excluded L as an acceptable class. The following exercise explains why.

1.4 Exercise. Show that for any stable theory T, the class of **L**-saturated models satisfies condition i) of the last exercise. Find an example to show ii) may fail.

To simplify the statement of the results on the spectra of the classes K, we introduce the following notation.

1.5 Notation. For any cardinal $\aleph_\alpha$ we denote by K_α the collection of members of K with cardinality $\aleph_\alpha$ and by $K_{\leq\alpha}$ the collection of members of K with cardinality at most $\aleph_\alpha$. We write $I(\aleph_\alpha, K)$ for $|K_\alpha|$ and $I^*(\aleph_\alpha, K)$ for $|K_{\leq\alpha}|$.

Occasionally, we will depart from the $\aleph_\alpha$ notation and write $I(\kappa, K)$ for $|\{M \in K : |M| = \kappa\}|$. The basic notation is defined in terms of the $\aleph_\alpha$ notation because the spectrum is often most easily computed in terms of the index. Similarly, it turns out that the formulas for the cumulative hierarchy $I^*(\aleph_\alpha, K)$ are often easier to state than those for $I(\aleph_\alpha, K)$. Clearly, the two functions are interdefinable.

We now define an extremely powerful and natural partial ordering on types. Intuitively, p compels q if q is realized in every model which realizes p. One of the great strengths of stability theory is the ability to translate this intuitively appealing idea into a practical technical tool.

1.6 Definition. Let $M \in K$ and $p, q \in S(M)$. Then we say p *compels* q over K and write $p \mapsto_K q$ if for every $\bar{a}$ realizing p and every $N \in K$ which contains $M \cup \bar{a}$, q is realized in N. If $p \mapsto_K q$ and $q \mapsto_K p$ we write $p \underset{\sim}{\mapsto}_K q$.

We omit the subscript K when it is clear from context.

1.7 Exercise. Show that for any class K, $\mapsto_K$ is a transitive reflexive relation.

1.8 Exercise. Show that when T admits K-prime models the reference in Definition 1.6 to 'all $N \in K$ which contain $M \cup \bar{a}$' can be replaced by '$M[\bar{a}]$'.

1.9 Definition. Let $p \in S(M)$. If for every $q \in S(M)$ which is not realized in M, $p \mapsto q$ implies $q \mapsto p$ then p is a *K-minimal type.*

1.10 Exercise. Show $\underset{\sim}{\mapsto}$ is an equivalence relation on K-minimal types over M.

The following analogy will make it easier to visualize the properties of $\mapsto$. We will prove a precise form of it in Theorem XIII.3.6. Think of the equivalence classes of types over M mod $\underset{\sim}{\mapsto}$ as natural numbers. The relation $\mapsto$ corresponds to divisibility; the K-minimal types are primes.

1.11 Exercise. Let $M \in K$, $p, q \in S(M)$. Show that if $p \mapsto q$ then $p \not\perp^a q$.

Thus far, $\mapsto$ is only defined relative to a model M. In the next section we will see that if K is the class of strongly $\kappa(T)$-saturated models then this restriction can be removed. For now, we show that for any acceptable class K, $\mapsto_K$ is preserved under nonforking extension. It is unclear whether this relation is preserved downwards (for arbitrary acceptable K).

1.12 Lemma. *If $M, N \in K$, p, $q \in S(M)$ and $p \mapsto_K q$ then if p' and q' are nonforking extensions to $S(N)$, $p' \mapsto_K q'$.*

Proof. Let $\bar{b}$ realize p'. Choose $M[\bar{b}] \prec N[\bar{b}]$ and $\bar{c} \in M[\bar{b}]$ to realize q. By FI_1 (letting $N - M$ play the role of $\bar{a}$) $\bar{c} \downarrow_M N$ and we finish.

1.13 Historical Notes. This notion of compulsion, though not the name, was introduced by Shelah in Definition V.2.1 of [Shelah 1978]. However, his emphasis was not on an ordering between types but on the order between the associated indiscernible sets (built by taking a nonforking sequence of realizations of a stationary type). Lascar (independently) introduced the notion as a relation on types [Lascar 1976]. Since for models of arithmetic the relation can be interpreted as the Rudin-Keisler order on ultrafilters, Lascar dubbed it the RK-order. As this connection is somewhat strained we don't use that term.

If a suitable notion **I** of isolation were defined such that all models of a small superstable theory were I-saturated and Exercise 1.3 held for **I**, this would at least clarify matters and perhaps lead to some new results for those theories. However, as Saffe so aptly put it, "I know (?) a lot of things in this direction which are not true."

2. S-Models

This section is devoted to describing a few further properties of the most useful acceptable class, **S**. Precisely because of its usefulness the models in this class have received a variety of names in the past. At various times this concept has been known as an $F^a_{\kappa(T)}$-saturated model, an $\aleph_\epsilon$-saturated model, an ϵ-saturated model, an a-model, and an **S**-model. We showed in Lemma IX.2.13 that **S**-models are the $\mathbf{S}_{\kappa(T)}$-saturated models. We note now that for any stable theory T, the **S**-models of T form an acceptable class. We will primarily be interested in superstable T so the **S**-models are the strongly $\aleph_0$-saturated models.

2.1 Theorem. *For any stable theory T, the class of $\mathbf{S}_{\kappa(T)}$-models of T is an acceptable class.*

Proof. We showed in Chapter X that $\mathbf{S}_{\kappa(T)}$ is a powerful isolation relation which satifies FI_1, FI_2, and FI_3.

The following lemma spells out the meaning of Theorem IV.3.22 which asserts that every strongly $\kappa(T)$-saturated model is good.

2.2 Lemma. *If T is stable and M is strongly $\kappa(T)$-saturated then for each $p \in S(M)$ there is a subset A of M with $|A| < \kappa(T)$ such that p does not fork over A and $p|A$ is stationary.*

There are two other properties of strongly saturated models which will prove useful later. Theorem X.2.27 demonstrated that our **S**-models are the same as the *a*-models of [Makkai 1984]. Exercise 2.3 i) is the special case of Theorem X.1.10 which shows 'strong' **S**-saturation is the same as **S**-saturation. Note that if $\bar{b}$ is omitted from condition i) of Theorem 2.3, we have the hypothesis of condition ii).

2.3 Exercise. i) If $\bar{a} \downarrow_M \bar{b}$ and M is strongly λ-saturated with $\lambda \geq \kappa(T)$ then for any $A \subseteq M$ with $|A| < \kappa(T)$, $stp(\bar{a}; A \cup \bar{b})$ is realized in M.
ii) If for every $B \subseteq M$ with $|B| < \lambda$ and every $\bar{a}$, $stp(\bar{a}; B)$ is realized in M then M is strongly λ-saturated.

2.4 Theorem. *The model M is* **S***-saturated iff for all $p \in S(M)$ there is a set of indiscernibles $I \subseteq M$, $|I| \geq \kappa(T)$ with* $\mathrm{Av}(I, M) = p$.

Proof. If M is **S**-saturated choose $A \subseteq M$ such that p is based on A and $|A| < \kappa(T)$. It is then routine to choose an independent coherent sequence of elements $\bar{a}_i$ of M such that $\bar{a}_i$ realizes $p|A_i$ and the conclusion is clear.

For the converse, let $A \subseteq M$, $|A| < \kappa(T)$, and $c \in C$. We must show $stp(\bar{c}; A)$ is realized in M. Choose $\bar{c}_0$ such that $stp(\bar{c}_0; A) = stp(\bar{c}; A)$ and $t(\bar{c}_0; M) = p$ does not fork over A. Now, if I is a set of indiscernibles contained in M with $|I| \geq \kappa(T)$ and $\mathrm{Av}(I, M) = p$, by Corollary V.1.19 there is an $I_0 \subseteq I$ with $|I_0| < \kappa(T)$ such that $I - I_0$ is a set of indiscernibles realizing $stp(\bar{c}; A)$. Any element of $I - I_0$ verifies that M is **S**-saturated.

2.5 Theorem. *If T is a countable ω-stable theory then every $\aleph_0$-saturated model is* $\mathbf{S}$*-saturated.*

Proof. By Corollary IV.2.15 for each $p \in S(M)$ there is an $A \subseteq M$ with $|A| < \kappa(T)$ and p strongly based on A. Choose $E = \langle e_i : i < \omega \rangle$ so that e_i realizes $p|A \cup E_i$. Clearly $p = \mathrm{Av}(E, M)$ and the theorem follows from Theorem 2.4.

Just as for domination we can extend the notion of compulsion to a property of global types. But as before there is no general guarantee that if $p \mapsto^e_k q$ then $p \mapsto_K q$. One of the major advantages of S-models is that when K is the class of S-models then if $p \mapsto^e_S q$ holds it holds for any restriction of the global types to S-models. We show this in Lemma 2.9.

2.6 Definition. Let p and q be stationary types. Then $p \mapsto^e_K q$ if for some K-saturated M containing $\operatorname{dom} q \cup \operatorname{dom} p$ and some nonforking extensions p', q' of p and q to M, we have $p' \mapsto_K q'$.

The following lemma shows that this definition depends only on the parallelism class of p and q.

2.7 Lemma. *If p, p', q, q' are stationary, $p \parallel p'$, and $q \parallel q'$ then $p \mapsto_K q$ if and only if $p' \mapsto_K q'$.*

Proof. It suffices to show that $p \mapsto_K q$ implies $p' \mapsto_K q'$. Choose nonforking extensions, p_1, q_1 of p and q respectively to witness $p \mapsto_S q$ according to Definition 2.6. Then $p_1 \rhd q_1$. Let p'', q'' be nonforking extensions to a K-model of, respectively, p_1, p' and q_1, q'. Then if $p'' \not\mapsto_K q''$, by Corollary X.2.5, $p'' \not\rhd q''$. But, by applying Corollary VI.3.13 to p_1 and q_1, we have $p'' \rhd q''$.

Deduce the following exercise from Corollary X.2.5. It shows that $\mapsto_{\mathbf{S}}$ can be defined without reference to the acceptable class $\mathbf{S}$.

2.8 Exercise. Show that $\mapsto^e_S$ is equivalent to eventual dominance (Definition VI.3.14).

In the light of Exercise 2.8 and Definition VI.3.14 we usually write $p \sqsupseteq^e q$ rather than $p \mapsto^e_{\mathbf{S}} q$.

2.9 Lemma. *Let T be superstable, M an* $\mathbf{S}$*-model and $p, q \in \mathbf{S}(M)$. Then, $p \mapsto^e_{\mathbf{S}} q$ if and only if $p \mapsto_{\mathbf{S}} q$*

Proof. Let M' be an $\mathbf{S}$-model and p', q' nonforking extensions of p, q to $S(M')$ realized by $\overline{a}'$ and $\overline{b}'$ respectively. Choose successively $A \subseteq M$ and $C \subseteq M'$ such that $|A|, |C| < \kappa(T)$, both p and q (hence p' and q') are strongly based on A, $t(\overline{a}'{}^\frown\overline{b}'; M')$ strongly based on C and $A \subseteq C$. By the $\kappa(T)$ saturation of M, choose an isomorphism α fixing A and mapping C into M. Choose $\overline{a}$ and $\overline{b}$ independent from M over $\alpha(C)$ and realizing $\alpha p'|C$, $\alpha q'|C$ respectively. Now the choice of A guarantees that $\overline{a}$ and $\overline{b}$ realize p and q. But the choice of C guarantees (by Lemma VI.3.12) that

$\bar{a}' \rhd_C \bar{b}'$; thus, $\bar{a} \rhd_{\alpha(C)} \bar{b}$ and by Corollary VI.3.13 $\bar{a} \rhd_M \bar{b}$. Thus $p \rhd_M q$ and by Corollary X.2.5 $p \mapsto_{\mathbf{S}} q$.

It is becoming customary to think of stationary types as representatives of their parallelism equivalence class. Thus in recent papers authors often write $p \sqsubseteq q$ where p and q are stationary (but with no mention of the exact domain) where we write $p \sqsubseteq^e q$. This practice is somewhat risky. For, if specific representatives p', q' of the global types are chosen whether $p' \rhd q'$ depends on the relation between p' and q'. Lemma 2.9 shows that for safety it suffices to choose the representatives over **S**-models. The key fact to remember is that $\mapsto_{\mathbf{S}}$ and $\rhd$ are preserved upwards by nonforking extension of the individual types but downward only if the type of the pair does not fork.

2.10 Historical Notes. Shelah [Shelah 1978] first discovered the importance of **S**-models. He called them $F^a_{\kappa(T)}$-saturated models and defined them by the first condition of Exercise 2.3. The properties of this notion discussed in this section are scattered in [Shelah 1978]. The equivalence of his notion with strong saturation is from [Baldwin 1984]. The proof of Lemma 2.9 is adapted from [Lascar 1984].