Chapter VIII

Normalization and T^{eq}

In this chapter we give a brief introduction to some important topics in stability theory which do not play a major role in the part of stability theory emphasized in this book. These notions are extremely significant in the study of ω-stable, ω-categorical theories and their importance for the fine structure of the spectrum of models is becoming evident.

The first of these is the introduction of 'imaginary elements' to represent classes of equivalence relations which are definable in a theory T. This notion could have been used far more systematically in this book. Instead, we have placed a greater reliance on strong types. The second section is concerned with normalization, a topic which is closely related to the first and which leads to the introduction of geometric notions into the theory. We briefly discuss the role of these notions in the study of totally categorical theories in Section 3.

1. T^{eq}

The discussion of finite equivalence relations in Chapter IV suggests the desirability of giving special attention to the classes of definable equivalence relations. This idea is also suggested by the usual practice of forming quotient structures in algebra. Accordingly, we introduce an expanded language in which the equivalence classes of a definable equivalence relation can be considered as points. Some authors [Cherlin, Harrington, & Lachlan 1985] have proposed adding such equivalence relations piecemeal as they are needed. We follow Shelah in describing one large expansion, L^{eq}, which will encompass all the constructions we need to make. However, following a suggestion of Makkai we treat the expanded structure as a many-sorted model.

The properties of theories in this expanded language are 'conservative' for the non-technical results in stability theory. That is, T^{eq} provides a shortcut to prove theorems which would be much more difficult to conceptualize or prove without this notion. Shelah called the elements in the expanded models imaginary elements in analogy with the same conservative role the imaginary numbers play with respect to the reals. This analogy is fairly precise; just as there is an interpretation of the theory of the complex numbers into that of the reals, the theory T^{eq} can be interpreted into T.

We work below with many-sorted languages. In a many-sorted language each constant symbol and variable has a fixed sort. The relations (functions) are required to relate (respectively, map between) objects of specified sorts. The quantifiers are restricted to specific sorts. Each model is a disjoint union of sets of objects of the various sorts. The usual theorems of model theory, especially compactness and completeness, can be easily translated into the many-sorted situation (cf. [Enderton 1972]).

1.1 Definition. i) Let T be a theory in a first order language L. Then L^{eq} denotes the many sorted language which has a sort S_E for each equivalence relation (of arbitrarily many variables) which is definable without parameters in T. The symbols of L are considered as relations and functions on objects of sort $S_=$. For each n-ary equivalence relation E there is a function symbol F_E which maps n-tuples of sort $S_=$ to elements of sort S_E.

ii) Each L-structure M is expanded to an L^{eq}-structure denoted M^{eq} by taking the elements of M as the elements of sort $S_=$ and retaining the interpretations of the relations in L. The objects of sort S_E are the equivalence classes of the equivalence relation E. The interpretation of F_E maps a sequence $\overline{a} \in M$ to $\overline{a}/E$.

iii) T^{eq} denotes the theory of M^{eq}.

1.2 Exercise. Show the last part of Definition 1.1 is permissible; that is, if $M \equiv N$ then $M^{\mathrm{eq}} \equiv N^{\mathrm{eq}}$ (and conversely).

1.3 Exercise. Prove the map $^{\mathrm{eq}}$ (from M to M^{eq}) is a functor from the category of models of T and elementary embeddings to the category of models of T^{eq} and elementary embeddings.

Although L^{eq} is a proper expansion of L, the formulas of L^{eq} which involve the new sorts can be easily translated into the original language. The following lemma, which is proved by induction on formulas, accomplishes this translation.

1.4 Lemma. *Let $\phi(x_0,\ldots,x_n)$ be an L^{eq}-formula where each x_i has sort S_{E_i}. There is an L-formula $\phi^*(\overline{x}_0,\ldots,\overline{x}_n)$ where each $\overline{x}_i$ has length m if E_i is an equivalence relation on m-tuples such that:*

$$\mathcal{M}^{\mathrm{eq}} \models \phi(\overline{a}_0/E_0,\ldots,\overline{a}_n/E_n) \leftrightarrow \phi^*(\overline{a}_0,\ldots,\overline{a}_n).$$

Note that if ϕ is an L-formula then ϕ^* is ϕ. The formulas ϕ^* are invariant in the sense that

$$\mathcal{M} \models \phi^*(\overline{a}_0,\ldots,\overline{a}_n) \leftrightarrow \phi^*(\overline{b}_0,\ldots,\overline{b}_n)$$

if $E_i(\overline{a}_i,\overline{b}_i)$ for each $1 \le i \le n$. With this result in hand, it is easy to see that most of the properties discussed in this book transfer from T to T^{eq}.

1.5 Exercise. Show that T is superstable, stable, ω-stable, etc. if and only if T^{eq} is.

The preservation of categoricity from T to T^{eq} is a more subtle matter. Because T^{eq} has infinitely many sorts, when one makes the natural translation of the many sorted language into a single-sorted one there will be elements in some models which realize 'non-standard' sorts. Naturally, such elements wreak havoc with categoricity. In the many-sorted version, these elements may be ignored. Formally, one can consider the reducts of models of T^{eq} to the standard sorts. This approach to many sorted structures is worked out in a different context in [Baldwin & Berman 1981]. Such fine notions as forking are also preserved. To see this we must consider the relation between types in T and types in T^{eq}.

1.6 Definition. Let $p \in S(A)$ and suppose that $\bar{b} \in \mathcal{M}$ realizes p. Then $p^{\mathrm{eq}} = t^{\mathrm{eq}}(\bar{b}; A)$ denotes the L^{eq}-type over A realized by $\bar{b}$.

Similarly we can define stp^{eq}. When we write $t^{\mathrm{eq}}(\bar{b}; \mathrm{cl}(A))$, the algebraic closure is taken in T^{eq}. The following lemma justifies this definition, by showing p^{eq} does not depend on the choice of $\bar{b}$, and yields an important corollary.

1.7 Lemma. *For any $A \subseteq \mathcal{M}$ and $p \in S(A)$, $p \vdash p^{\mathrm{eq}}$.*

Proof. If $\phi(\bar{b}; \bar{a}) \in p^{\mathrm{eq}}$ then the L-formula $\phi^*(\bar{b}; \bar{a}) \in p$. Apply Lemma 1.4.

1.8 Corollary. *Let $A \subseteq B \subseteq \mathcal{M}$ and $p \in S(B)$. Then p forks over A if and only if p^{eq} forks over A.*

Proof. Since $p \subseteq p^{\mathrm{eq}}$ one direction is trivial; but from $p \vdash p^{\mathrm{eq}}$ we easily deduce the other.

The following exercise is immediate.

1.9 Exercise. For any type p, p is stationary if and only if p^{eq} is stationary.

All the results we have discussed so far are direct translations between T and T^{eq}. If this state of affairs held for all properties there would be no reason to introduce the notion. We now discuss several of the ways in which the theories differ.

To shorten notation, if we have a sequence $\bar{a}$ containing elements of various sorts, we write $\bar{F}_{\bar{G}}(\overline{w}) = \bar{a}$ rather than the conjunction over specific equivalence relations G', specific subsequences $\overline{w}'$ of $\overline{w}$ and specific $a' \in \bar{a}$ of the formulas $F_{G'}(\overline{w}') = a'$.

1.10 Theorem. i) *Let A be an algebraically closed subset of $\mathcal{M}^{\mathrm{eq}}$. If the formula ϕ is almost over A then ϕ is over A.*
ii) *$stp(\bar{c}; B) \vdash t(\bar{c}; A)$ if and only if $stp^{\mathrm{eq}}(\bar{c}; B) \vdash t^{\mathrm{eq}}(\bar{c}; \mathrm{cl}(A))$.*

Proof. i) Let $E(\bar{x}, \bar{y}; \bar{a})$ be a finite equivalence relation over A. We will show that each equivalence class of E corresponds to an element of $\mathcal{M}^{\mathrm{eq}}$. (This is nontrivial because E is defined with parameters.) Then we will show each of these classes is over A and the theorem follows immediately by considering the finite equivalence relation on which ϕ depends.

Let $E^*(\overline{u},\overline{v};\overline{w})$ be the L-formula Lemma 1.4 associates with $E(\overline{x},\overline{y};\overline{z})$. Now let $\hat{E}$ be the equivalence relation on tuples with length $\lg(\overline{u})+\lg(\overline{w})$ whose classes are as follows. If $E^*(\overline{u},\overline{v};\overline{a})$ is an equivalence relation then for each $\overline{b}$, $\{\overline{c}: E^*(\overline{c},\overline{b},\overline{a})\}\times\overline{a}$ is a class. A final class is the class of all tuples $\{\overline{c}^\frown\overline{a}\}$ for those $\overline{a}$ such that $E^*(\overline{u},\overline{v};\overline{a})$ is not an equivalence relation. This equivalence relation is defined by the formula $\hat{E}(\overline{x}^\frown\overline{w}_1;\overline{y}^\frown\overline{w}_2)$:

$$(\neg E_1(\overline{w}_1)\wedge\neg E_1(\overline{w}_2))\ \vee\ (\overline{w}_1=\overline{w}_2\wedge E^*(\overline{x},\overline{y},\overline{w}_1)\wedge E_1(\overline{w}_1))$$

where $E_1(\overline{a})$ holds if $E^*(\overline{x},\overline{y};\overline{a})$ is an equivalence relation. Let $d_1,\ldots,d_n$ in $\mathcal{M}^{\mathrm{eq}}$ be the equivalence classes of $\hat{E}$ which are associated with the equivalence classes of $E(\overline{x},\overline{y};\overline{c})$. If we find an L^{eq} formula over A with these as its solution we finish. For some $\overline{c}'$, $\overline{c}=\bar{F}_{\bar{G}}(\overline{c}')$, so such a formula is

$$\bigwedge_{1\le i\le n} S_{\hat{E}}(z_i)\wedge(\exists\overline{x})(\exists\overline{w})[\bar{F}_{\bar{G}}(\overline{w})=\overline{c}\wedge F_{\hat{E}}(\overline{x}^\frown\overline{w})=\overline{z}].$$

ii) This is immediate from the definitions of strong type and algebraic closure.

1.11 Corollary. *If A is an algebraically closed subset of $\mathcal{M}^{\mathrm{eq}}$ then every type over A is stationary.*

Proof. Let p and q be complete types over $B\supset A$ which are distinct, do not fork over A, and have a common restriction to A. Then by the finite equivalence relation theorem (Theorem IV.2.2) there is a formula ϕ which is almost over A and is in p but not in q. But by Theorem 1.10, ϕ is over A which contradicts the assumption that p and q have a common restriction to A.

Next we examine some notions which are primarily useful in $\mathcal{M}^{\mathrm{eq}}$.

1.12 Definition. For any set A, the *definable closure* of A, written $dcl(A)$, is the collection of points definable over A, i.e. the solutions of formulas with parameters from A which have unique solutions. If $dcl(A)=A$ we say A is definably closed.

If the global type $\hat{p}$ is strongly based on the set A, then every automorphism of $\mathcal{M}$ which fixes A also fixes $\hat{p}$. In general there are many such A.

1.13 Definition. i) Let $G_{\hat{p}}$ denote the *stabilizer* of $\hat{p}$, the subgroup of all automorphisms of $\mathcal{M}$ which fix $\hat{p}$.

ii) Let $F_{\hat{p}}$ denote the fixed set of $G_{\hat{p}}$, that is, the set of elements of $\mathcal{M}$ which are fixed by every automorphism of $\mathcal{M}$ which fixes $\hat{p}$.

iii) A *canonical base* of $\hat{p}$, written $\mathrm{cb}(\hat{p})$, is the (necessarily unique) set A such that A is definably closed and the automorphism f is in $G_{\hat{p}}$ if and only if f fixes A pointwise.

1.14 Exercise. Show that each $\hat{p}$ has at most one canonical base.

In general canonical bases need not exist. For example, consider the theory of an equivalence relation with two infinite classes. The type which is realized by a 'new' member of one of the classes has no canonical base. It is defined over any element of that class, but any such element can be moved by automorphisms which fix the type. Passing to T^{eq} we see that the canonical base is the class itself. This phenomenon generally holds; in $\mathcal{M}^{\text{eq}}$ every global type has a canonical base. One further notion is necessary to establish this result.

The definition of $\mathcal{M}^{\text{eq}}$ assigns an element to each equivalence class of an equivalence relation which is definable without parameters. In fact, each definable subset of $\mathcal{M}$ becomes a point of $\mathcal{M}^{\text{eq}}$ and if b is the element assigned to $\phi(\overline{x};\overline{a})$ then $b \in \mathcal{M}^{\text{eq}}$ is in the definable closure of $\overline{a}$. More formally, we have

1.15 Theorem. *For every formula $\phi(\overline{x};\overline{y})$ and every $\overline{a} \in M$ there is a point $b_\phi = b_{\phi(\overline{x};\overline{a})} \in \mathcal{M}^{\text{eq}}$ and a function F^ϕ, which is definable without parameters, such that $F^\phi(\overline{a}) = b_{\phi(\overline{x};\overline{a})}$.*

Proof. Define without parameters the equivalence relation R_ϕ by:

$$R_\phi(\overline{y}, \overline{z}) \leftrightarrow [(\forall \overline{x})(\phi(\overline{x};\overline{y}) \leftrightarrow \phi(\overline{x};\overline{z}))].$$

Now $b = \overline{a}/R_\phi$ and F^ϕ can be taken as F_{R_ϕ}.

Note that if two formulas, say $\phi(\overline{x};\overline{a})$ and $\psi(\overline{x};\overline{b})$, define the same set in $\mathcal{M}$ then they give rise to distinct points in $\mathcal{M}^{\text{eq}}$; but all such points are in the definable closure of $\overline{a}$.

The following exercise provides another perspective for the last result.

1.16 Exercise. Show that for every definable (with parameters) subset X of M there is an element $X^* \in \mathcal{M}^{\text{eq}}$ such that an automorphism of M^{eq} leaves X invariant if and only if it fixes X^*.

We apply this exercise to prove each global type has a canonical base.

1.17 Theorem (Canonical Base). *If $\hat{p}$ is a type over $\mathcal{M}^{\text{eq}}$ then* $\text{cb}(\hat{p})$ *exists.*

Proof. Let d define $\hat{p}$. That is, $\phi(\overline{x};\overline{a}) \in \hat{p}$ iff $\models d\phi(\overline{a};\overline{c}_\phi)$. Let A denote $\{\overline{c}_\phi/R_{d\phi} : \phi \in L\}$. Clearly, $F_{\hat{p}} = dcl(A)$ and every automorphism in $G_{\hat{p}}$ fixes A pointwise.

The definition of canonical base easily extends to any stationary type.

1.18 Definition. For any stationary type p, the *canonical base* of p, denoted $\text{cb}(p)$, is the canonical base of the unique nonforking extension of p to a global type.

1.19 Exercise. If $p \parallel q$ then $\text{cb}(p) = \text{cb}(q)$ (in M^{eq}).

1.20 Exercise. There is a $B \subseteq \text{cb}(p)$ with $|B| < \kappa(T)$ such that if $q \parallel p$ then q does not fork over B.

A slightly more involved argument shows that if p does not fork over A then $\text{cb}(p) \subseteq \text{cl}(A)$. Thus p is based on A in the sense of Definition IV.1.17 ii) if and only if $\text{cb}(p) \subseteq A$. In this manner of speaking, p is strongly based on A if $\text{cb}(p) \subseteq A$. A glance at the proof of Theorem 1.17 shows the following.

1.21 Lemma. *For any* $\hat{p}$, $|\text{cb}(\hat{p})| \leq |T|$.

1.22 Exercise. If T is superstable then for every global type $\hat{p}$ there is a finite $A \subseteq \text{cb}(\hat{p})$ such that $\text{cb}(\hat{p}) \subseteq \text{cl}(A)$. ($T^{\text{eq}}$)

The following extension of the notion of canonical base is due to Anand Pillay. It provides a useful tool for the proof of the normalization lemma in Section 2. Note that the proof makes essential use of T^{eq}.

1.23 Definition. Let P be a set of (possibly incomplete) types over $\mathcal{M}$.

i) G_P denotes the group of automorphisms of $\mathcal{M}$ which permute P.
ii) A *canonical base* for P is a set A such that A is definably closed and an automorphism α of $\mathcal{M}$ is in G_P if and only if it fixes A pointwise.

The following theorem is stated for a sufficiently closed subset Δ of formulas. On first reading, Δ can be taken to be all L-formulas.

1.24 Theorem. *Let T be a stable theory and let $\Delta(\overline{x})$ be a set of formulas closed under Boolean combination and substitution for parameter variables. Let P be a closed subset of $S_\Delta(\mathcal{M})$ with $|P| < |\mathcal{M}|$. The following are equivalent:*

i) *P has a canonical base.*
ii) *For each $p \in S_\Delta(\mathcal{M})$ and each $\phi(\overline{x};\overline{y}) \in \Delta$ the orbit of p_ϕ under G_P is finite.*

Proof. Suppose P has a canonical base, A, but for some $\phi \in \Delta$ and some $p \in P$, the orbit of p_ϕ under G_P is infinite. We will show $|P|$ is unbounded.

Let $\beta(\overline{y};\overline{c})$ define p_ϕ. Fix $\kappa > |P|$ and let $\overline{c}_i$ for $i < \kappa$ be new constant symbols. Let Σ assert that the $\overline{c}_i$ are distinct realizations of $t(\overline{c}; A)$ and that if $i \neq j$ then $\beta(\overline{y};\overline{c}_i) \not\leftrightarrow \beta(\overline{y};\overline{c}_j)$. Since the orbit of p_ϕ under G_P is infinite, Σ is consistent. But the formulas $\beta(\overline{y};\overline{c}_i)$ thus define more than κ distinct conjugates of p_ϕ. Since all can be extended to members of P, this contradicts the choice of κ.

Now suppose ii) holds. Let $P = \bigcup_{i \in I} P_i$ where each P_i is an orbit of some $p \in P$ under G_P. Fix $\phi \in \Delta$ and $i \in I$. Let $\beta_1, \ldots, \beta_k$ define the finitely many ϕ-types in P_i. For each $j \leq k$, let $b_j = b_{\beta_j}$ be the point in M^{eq} attached to β_j using Theorem 1.15. Now $B_i = \{b_1, \ldots, b_k\}$ is a finite and thus definable subset of M^{eq}. Let $a_{\phi,i}$ be the point of M^{eq} attached to (some definition of) B_i by Theorem 1.15. An automorphism, f, of M^{eq} fixes $a_{\phi,i}$ if and only if f permutes B_i if and only if f permutes the ϕ-types in P_i.

We now show that $A = dcl(\{a_{\phi,i} : \phi \in \Delta, i \in I\})$ is the canonical base for P. If $f \in G_P$, f permutes the ϕ-types in P_i for each i, so f fixes A pointwise. Conversely, if f fixes A pointwise and $p \in P_i$ then for each ϕ,

$f(p_\phi) = q_\phi$ for some $q \in P_i$. For, p_ϕ is defined by some β_j and $f(\beta_j)$ defines q_ϕ for some $q \in P_i$. This shows $f(p)_\phi$ has an extension in P for each $\phi \in \Delta$. Since P is a closed subset of $S_\Delta(\mathcal{M})$, this implies $f(p) \in P$ as required.

Pillay [Pillay 198?] proves Theorem 1.24 with the hypothesis that P is closed replaced by a weaker but more complicated hypothesis.

1.25 Historical Notes. The advantage of considering equivalence classes as elements in models was seen at least as early as the Rabin-Scott-Ershov decision procedure [Eršov 1974]. Shelah initiated the systematic use of T^{eq} in stability theory in Chapter III.6 of [Shelah 1978] where all the results here (except Theorem 1.24) and more are obtained. The idea of viewing L^{eq} as a multisorted language was expounded by Makkai [Makkai 1984]. It has the advantage that $\mathcal{M}^{\mathrm{eq}}$ is saturated and that categoricity as well as the stability hierarchy is preserved. The disadvantage, which we have chosen to suppress, is that stability theory has not been developed in many-sorted languages and so our use of it in the study of T^{eq} is not formally justified. Shelah's development goes directly to the step one takes in interpreting many-sorted logics into single sorted logic. He just expands L by adding a new unary predicate P_E whose intended interpretation is the set of E-equivalence classes. In this approach the formulas are somewhat more difficult to write since the range of the variables must be explicitly delimited, rather than thought of as built into the name of the variable. Definition 1.23 and Theorem 1.24 are from [Pillay 198?]. Poizat [Poizat 1985], [Poizat 1983a] provides an important view of the role (nonrole?) of T^{eq} in the theory of algebraically closed fields. In addition to the published expositions our treatment benefitted from discussions with Gisela Ahlbrandt and Anand Pillay.

2. Normalization

In this section we consider the normalization of a formula. This notion, which was introduced by Lachlan, is closely connected both with the finite equivalence relation theorem and with the concept of a canonical base discussed in Section 1.

One reason to consider normalization is the following situation. We want to investigate the relation between a uniformly definable family of sets $\langle \phi(\overline{x};\overline{a}) : \overline{a} \in \mathcal{M}\rangle$ and the set defined by the formula $(\exists \overline{y})\phi(\overline{x};\overline{y})$. Often the quantification may be restricted to a definable set. If, for example, T is ω-stable and $\phi(\mathcal{M};\overline{a})$ has Morley degree 1 there is a natural equivalence relation on the parameters given by $\overline{a} \simeq \overline{b}$ if

$$R_M(\phi(\overline{x};\overline{a}) \wedge \phi(\overline{x};\overline{b})) = R_M(\phi(\overline{x};\overline{a})).$$

We would like to find natural representatives for the equivalence classes under this equivalence relation. The application of the Normalization Lemma in this situation is discussed in Theorem 3.4 (cf. [Lachlan 1980]).

The situation is more easily understood from a slightly more abstract viewpoint.

We write $\triangle$ for the symmetric difference of two elements in a Boolean algebra.

2.1 Notation. Fix a stable theory T. Let $A \subseteq \mathcal{M}$ be invariant under automorphisms of $\mathcal{M}$. Let $\mathcal{F} = F_\Gamma(A)$, the set of A-substitution instances of a set Γ of parameter free formulas which is closed under Boolean combinations. Then $\mathcal{F}$ is a subalgebra of the Boolean algebra $F(\mathcal{M})$ which is invariant under automorphisms of $\mathcal{M}$. Let $\mathcal{I}$ be an ideal in $\mathcal{F}$ which is invariant under automorphisms of $\mathcal{M}$.

i) Two formulas, $\phi, \psi \in \mathcal{F}$ are *equivalent* relative to $\mathcal{I}$ if they are in the same congruence class modulo $\mathcal{I}$. That is, if $\phi \triangle \psi \in \mathcal{I}$. We write $\phi \simeq_{\mathcal{I}} \psi$, omitting the subscript if it is clear from context.
ii) Let $\psi(\overline{x}; \overline{a})$ be in Γ. The set of *distinguished $\mathcal{I}$-extensions* of $\psi(\overline{x}; \overline{a})$, denoted by $P_{\psi(\overline{x};\overline{a})}$, is the set of complete Γ types over $\mathcal{M}$ which contain $\psi(\overline{x}; \overline{a})$ and which contain no formula from $\mathcal{I}$. When ψ is clear we write $P_{\overline{a}}$ for $P_{\psi(\overline{x};\overline{a})}$. We will not write $\mathcal{I}$ when $\mathcal{I}$ is fixed in context.

2.2 Examples. i) Let T be a stable theory, let $\mathcal{F} = F(\mathcal{M})$, and let $\mathcal{I}$ be the ideal of formulas which fork over $\emptyset$.

Of course, this example can be generalized by replacing the empty set by a fixed subset A of $\mathcal{M}$.

ii) Let T be an ω-stable theory. Let $\mathcal{F} = \{\phi(\overline{x}; \overline{a}) : R_M(\phi(\overline{x}; \overline{a})) \leq n\}$. Let $\mathcal{I}$ be the ideal of formulas with Morley rank $< n$.

By defining our notion in terms of symmetric difference, we do not need in Example 2.2 ii) the restriction, which is usually made in discussions of the normalization lemma, that $\phi(\overline{x}; \overline{a})$ have Morley degree one.

2.3 Exercise. Show in Example 2.2 ii) that if $R_M(\phi(\overline{x}; \overline{a})) = n$, $\overline{a}'$ is a conjugate of $\overline{a}$, and $\phi(\overline{x}; \overline{a})$ has Morley degree one, then $\phi(\overline{x}; \overline{a}) \simeq \phi(\overline{x}; \overline{a}')$ if and only if $R_M(\phi(\overline{x}; \overline{a}) \wedge \phi(\overline{x}; \overline{a}')) = n$.

2.4 Exercise. Show that $\phi(\overline{x}; \overline{a}) \simeq \phi(\overline{x}; \overline{a}')$ if and only if $P_{\overline{a}} = P_{\overline{a}'}$.

2.5 Exercise. Show that the equivalence classes under $\simeq$ are closed under positive Boolean combinations.

2.6 Definition. A formula $\phi(\overline{x}; \overline{a}) \in \mathcal{F}$ is said to be *normal* with respect to $\mathcal{I}$ if for any conjugate $\phi(\overline{x}; \overline{a}')$ of $\phi(\overline{x}; \overline{a})$, if $\phi(\overline{x}; \overline{a}) \simeq_{\mathcal{I}} \phi(\overline{x}; \overline{a}')$ then $\phi(\mathcal{M}; \overline{a}) = \phi(\mathcal{M}; \overline{a}')$.

With some further conditions on the pair $(\mathcal{F}, \mathcal{I})$, we will show that every formula in $\mathcal{F}$ can be normalized. That is, to each formula ϕ, there corresponds a formula ϕ^* such that $\phi \simeq_{\mathcal{I}} \phi^*$ and ϕ^* is normal. We prove this result here for the context we described in Paragraph 2.1. We distinguish now some conditions which play an important role in our proof. In fact, as we discuss below, varying the context slightly these conditions suffice to prove the normalization Lemma.

2.7 Conditions. Consider the following conditions on $\phi(\overline{x};\overline{a}) \in \mathcal{F}$ and $P_{\overline{a}}$.

N1. For each $q \in P_{\overline{a}}$, $\phi(\overline{x};\overline{a}) \in q$.
N2. $P_{\overline{a}}$ is closed under automorphisms of $\mathcal{M}$ which fix $\overline{a}$.
N3. $|P_{\overline{a}}| < |\mathcal{M}|$.
N4. If $t(\overline{a};\emptyset) = t(\overline{b};\emptyset)$, $P_{\overline{a}} = P_{\overline{b}}$ if and only if for each $q \in P_{\overline{a}}$, $\phi(x;\overline{b}) \in q$ and for each $r \in P_{\overline{b}}$, $\phi(x;\overline{a}) \in r$.

2.8 Exercise. Prove that in the situation of Paragraph 2.1, each of N1, N2, and N4 holds.

Now we can state the major result of this section.

2.9 Theorem (The Normalization Lemma). *Let $\langle \mathcal{F}, I\rangle$ satisfy the conditions of Paragraph 2.1. Suppose in addition that each $|P_{\overline{a}}| < |\mathcal{M}|$ (N3). For each formula $\phi(\overline{x};\overline{a}) \in \mathcal{F} - I$, there exists a formula $\phi^*(\overline{x},\overline{a})$ such that*

i) $\phi^*(\overline{x},\overline{a}) \simeq \phi(\overline{x},\overline{a})$.
ii) *ϕ^* is equivalent to a positive Boolean combination of conjugates of $\phi(\overline{x};\overline{a})$.*
iii) *$\phi^*(\overline{x};\overline{a})$ is normal with respect to I.*

Note that statement ii) of the theorem is made only 'up to equivalence'. There is no claim that the same formula both contains only the parameter $\overline{a}$ and is a Boolean combination of conjugates of $\phi(\overline{x};\overline{a})$. When reading papers in this area the reader should beware of an identification of properties of the set defined by a formula with properties of the formula.

Before proving the theorem we note that our examples satisfy the conditions of 2.7. Of course, in light of the exercise following their statement we need only check N3. We then derive some general properties of $\langle \mathcal{F}, I\rangle$ which satisfy N3.

2.10 Examples. We continue the examples begun in 2.2.

i) Fix a set $A \subseteq \mathcal{M}$. Then $P_{\overline{a}} = \{p \in S(\mathcal{M}) : \phi(x;\overline{a}) \in p$ and p does not fork over $A\}$.
ii) Let $P_{\overline{a}} = \{p \in S(\mathcal{M}) : \phi(x;\overline{a}) \in p$ and $R_M(p) = n\}$.

2.11 Exercise. Show that the two examples satisfy N3.

We now collect some properties which hold whenever Conditions 2.7 do. It is obvious from N2 and N3 that:

2.12 Lemma. *If $q \in P_{\overline{a}}$ then q does not fork over $\overline{a}$.*

Thus, if $q \in P_{\overline{a}}$, q is definable almost over $\overline{a}$. Let $\beta_i(\overline{y})$ list the formulas over $\mathrm{cl}(\overline{a})$ which define the types p_ϕ (the restriction of p to ϕ) for $p \in P_{\overline{a}}$.

2.13 Lemma. *There is a type $s(\overline{u},\overline{v})$ over $\emptyset$ so that $P_{\overline{a}} = P_{\overline{b}}$ if and only if $\models s(\overline{a},\overline{b})$.*

Proof. Since $P_{\overline{a}}$ is closed under automorphisms which fix $\overline{a}$, for each i and for each of the finitely many $\overline{a}$-conjugates β_i' of β_i, $\models \beta_i'(\overline{a})$. Since the conjunction of the β_i' is invariant under $\overline{a}$-automorphisms we may take it as a formula $\gamma_i(\overline{x};\overline{a})$. Now for each i, we have $\models \gamma_i(\overline{a};\overline{a})$ so $s(\overline{u},\overline{v}) = \{\gamma_i(\overline{u};\overline{v}) \wedge \gamma_i(\overline{v};\overline{u}) : i < \omega\}$ is a consistent type over the empty set. Now, $\models s(\overline{a},\overline{b})$ if and only if for each $p_i \in P_{\overline{a}}$, $\models \gamma_i(\overline{a},\overline{b})$ iff and only if $\phi(\overline{x};\overline{a}) \in p_i$. Switching the roles of $\overline{a}$ and $\overline{b}$ and applying N4 twice, we see $\models s(\overline{a},\overline{b})$ if and only if $P_{\overline{a}} = P_{\overline{b}}$. Thus, we have the Lemma.

The following proposition is the key to the proof of the normalization lemma. We defined the canonical base of a set of types in Definition 1.23.

2.14 Proposition. *With the above notation $P_{\overline{a}}$ has a canonical base.*

Proof. Clearly, $P_{\overline{a}}$ is a closed set and its cardinality is properly less than that of $|\mathcal{M}|$. So, it suffices by Theorem 1.24 to show that for each $p \in P_{\overline{a}}$ and each $\phi \in \Gamma$, the orbit of p_ϕ under $G_{P_{\overline{a}}}$ is finite. Let $\beta(\overline{y};\overline{c})$ be a definition of q_ϕ for some nonforking extension, q, of p to $S(\mathcal{M})$. Add new constants $\overline{a}_i, \overline{c}_i$ to the language for $i < \kappa = |P_{\overline{a}}|^+$. Let Σ be the set of sentences which assert:

$$t(\overline{a}_i{}^\frown\overline{c}_i;\emptyset) = t(\overline{a}{}^\frown\overline{c};\emptyset),$$

$$\models s(\overline{a}_i,\overline{a}),$$

$$\beta(\overline{y},\overline{c}_i) \not\leftrightarrow \beta(\overline{y};\overline{c}_j) \text{ if } i < j < \kappa\ .$$

Since $f \in G_{P_{\overline{a}}}$ if and only if $\models s(\overline{a}, f(\overline{a}))$, Σ is consistent. But the interpretations of Σ yield κ distinct conjugates of $p|\phi$ in $P_{\overline{a}}$ contrary to the choice of κ. Thus, by Theorem 1.24, we have the proposition.

With this in hand we can quickly prove the normalization lemma.

Proof of 2. 9. We want to normalize $\phi(\overline{x};\overline{a})$. Choose a canonical base, A, for $P_{\overline{a}}$ by Proposition 2.14. Let $q \in S(\mathcal{M})$ be a nonforking extension of $t(\overline{a};A)$. By the saturation of $\mathcal{M}$ we can find a set of indiscernibles, E, realizing $t(\overline{a};A)$ so that $q = \mathrm{Av}(E,\mathcal{M})$. By Lemma V.2.6, there is a formula $\chi(\overline{e};\overline{y})$ which is a positive Boolean combination of A-conjugates of $\phi(\overline{x};\overline{a})$ such that for any $\overline{c} \in \mathcal{M}$, $\chi(\overline{e};\overline{c})$ if and only if $\phi(\overline{x};\overline{c}) \in q$. Since q does not fork over A, χ is almost over A. Let χ^* be a formula over A which is equivalent to the disjunction of the (finitely many) A-conjugates of χ. Since χ^* is equivalent to a positive Boolean combination of A-conjugates of $\phi(\overline{x};\overline{a})$ and $\simeq$ classes are closed under such combinations, $\chi^* \simeq \phi(\overline{x};\overline{a})$. It remains to show that χ^* is normal. Let χ' be a conjugate of χ by an automorphism f. Now $\chi' \simeq \chi^*$ if and only if $P_{\chi'} = P_{\chi^*}$. But, $P_{\chi^*} = P_{\overline{a}}$. Thus f permutes $P_{\overline{a}}$. But then f fixes A pointwise and so χ' is an A-conjugate of χ^*. Since χ^* is over A, this implies χ^* is normal.

Since A is a canonical base for $P_{\overline{a}}$, each $\overline{a}$-automorphism fixes A pointwise and thus leaves χ^* invariant. Thus, χ^* is equivalent to a formula $\phi^*(\overline{x};\overline{a})$ as required.

Steve Buechler gave another account of this result which is more ad hoc but gives some additional information. One can use the conditions of Proposition 2.7 as the hypotheses without assuming the exact context described in Paragraphs 2.1 through 2.6. In outline his argument proceeds as follows.

He proves that there is a family $\langle E_i : i < \omega\rangle$ of 0-definable (i.e. definable without parameters) equivalence relations satisfying the following condition. If $\overline{b}, \overline{b}'$ are conjugate to $\overline{a}$ then $\models \bigwedge_{i<\omega} E_i(\overline{b}, \overline{b}')$ if and only if $P_{\overline{b}} = P_{\overline{b}'}$. Now, let $A = \{\overline{a}/E_i : i < \omega\} \subseteq M^{\mathrm{eq}}$. Moreover $A \subseteq dcl(\overline{a})$. Note that if $\overline{b}, \overline{b}'$ are conjugates of $\overline{a}$, then $t(\overline{b}; A) = t(\overline{b}'; A)$ if and only if $\overline{b}/E_i = \overline{b}'/E_i$ for each $i < \omega$. One of these implications is obvious. For the other, note that if $\overline{b}/E_i = \overline{b}'/E_i$ for each i, any automorphism which maps $\overline{b}$ to $\overline{b}'$ fixes A pointwise. With this observation we have the following reformulation of Lemma 2.13. $P_{\overline{b}} = P_{\overline{b}'}$ if and only if $t(\overline{b}; A) = t(\overline{b}'; A)$. Now an argument very similar to that for Theorem 2.9 derives a normalizing formula for $\phi(\overline{x};\overline{a})$.

2.15 Historical Notes. The normalization theorem was originally proved by Lachlan [Lachlan 1974] using a complicated rank argument. The more abstract versions were initiated by [Harnik & Harrington 1984]. The particular version chosen here is due to Anand Pillay [Pillay 1983]. Other proofs have been given by [Buechler 1984c], [Saffe 198?a], and, for the forking case, [Vaughn 1985].

3. 'Geometric' Stability Theory

In this section we summarize recent developments in 'geometric' stability theory. By this we mean the investigation of the fine structure of models through the combination of techniques from combinatorial geometry and group theory with those of stability theory.

There are two major sources for this development. Our discussion is primarily organized around the first, Morley's question, "Can a theory which is categorical in all infinite powers and has no finite models (a totally categorical theory) be finitely axiomatizable?" A major step in the investigation of this problem was Zilber's isolation of the class of *strictly minimal* sets. A set is strictly minimal in the theory T if it is not only strongly minimal but admits no nontrivial definable equivalence relation. A geometry can be associated with any strictly minimal set as follows. The elements of the strictly minimal set are the points of the geometry; the algebraic closure of two points is called a line; the algebraic closure of three noncollinear points is called a plane, etc. The following theorem states the basic fact.

3.1 Theorem. *Each $\aleph_0$-categorical strictly minimal set is associated with one of the following geometries.*

i) *An infinite set with no structure (disintegrated).*
ii) *An infinite dimensional projective space over a finite field.*
iii) *An infinite dimensional affine space over a finite field.*

Cherlin [Cherlin, Harrington, & Lachlan 1985] and Mills (unpublished) independently proved this result by an analysis of the automorphism groups of the finite approximations to the strictly minimal set. This analysis depends upon the classification of the finite simple groups. Zilber [Zilber 1981] proved this result directly with no reliance on a high powered technique imported into the field.

Reflecting the modularity of the associated lattice of closed subsets, a strictly minimal set which is either disintegrated or projective is called *modular*. Otherwise, it is called *affine*.

By a variant on the usual construction of projective space, any affine space can be converted into a modular one in a language obtained *by naming a point*. The effect of such an extension by constants in this situation is so critical that it has spawned some further terminology. A set H is A-definable in M if it is defined by a formula with parameters from A. If $A = \emptyset$, H is said to be 0-definable.

If T is $\aleph_0$-categorical we naturally extend the notion of orthogonality to sets by saying the strongly minimal sets H_1 and H_2 are orthogonal if the unique non-algebraic types of elements in H_1 and H_2 are orthogonal. The crucial step in moving from the 'local' behavior of a particular strictly minimal set to the 'global' analysis of a model is

3.2 Theorem. *If T is ω-stable and $\aleph_0$-categorical, H_0 and H_1 are non-orthogonal* modular *strictly minimal sets then there is a unique* 0*-definable bijection between H_1 and H_2.*

This result leads to the two basic structural properties of an ω-stable, $\aleph_0$-categorical theory, the coordinatization theorem and the fundamental rank inequality. To state these we require some more terminology.

Recall that a group G acts n-transitively on a set X if for each pair of distinct n-tuples, $\overline{a}, \overline{b}$ from X there is an element of G which takes $\overline{a}$ to $\overline{b}$. If an $\aleph_0$-categorical countable structure, M, realizes only one 1-type, the automorphism group of M acts 1-transitively on M since M is homogenous. When the automorphism group of a structure acts 1-transitively on it, M is called *transitive.*

Let P and A be 0-definable subsets of an $\aleph_0$-categorical structure M. We say A *coordinatizes* P, if A generates a complete type over $\emptyset$ and for each $x \in P$, $\mathrm{cl}(x) \cap A \neq \emptyset$. Thus, each element of P is coordinatized by the finite set of elements in A which are in the algebraic closure of X. Here is the fundamental result.

3.3 Theorem (Coordinatization Theorem). *Let M be transitive, ω-stable and $\aleph_0$-categorical. There is a finite extension by definitions, M^* of M*

which is 0*-interpretable into* M *in which there is a rank one set* A *which coordinatizes* M.

M^* is just an appropriate approximation to M^{eq} by adding only finitely many sorts. This result leads to an important result whose statement is somewhat confusing. In the following theorem (and throughout this subject) we are able to discuss families of definable sets as though the sets are elements of the model and the family is a subset. Of course, this is accomplished by passing to (a suitable finite reduct of) T^{eq}. In particular, the rank of a family of definable sets is the rank of the associated set in $\mathcal{M}^{\mathrm{eq}}$.

3.4 Theorem (The Fundamental Rank Inequality). *Let* F *be a definable collection of definable subsets of the* $\aleph_0$*-categorical,* ω*-stable structure* N *with* $R_M(N) = n$. *Suppose each element of* F *has the same Morley rank* r *and the intersection of any two has rank less than* r. *Then* $r + R_M(F) \leq n$.

The almost disjoint family F needed to apply Theorem 3.4 is obtained from the Normalization Theorem. The Coordinatization Theorem is first proved under the hypothesis that the rank of T is finite. It then leads to the fundamental rank inequality. Then a clever argument of Cherlin turns the situation on its head to prove

3.5 Theorem. *If* T *is* ω*-stable and* $\aleph_0$*-categorical then* T *has finite Morley rank.*

A second impetus for 'geometric' stability theory came from Lachlan's analyses of finite homogeneous structures [Lachlan 1984], [Lachlan 1986]. This approach coalesces with the results described so far in this section in

3.6 Theorem. *If* T *is* $\aleph_0$*-categorical and* ω*-stable then every sentence which is consistent with* T *has a finite homogeneous model. Thus,* T *is not finitely axiomatizable.*

Noting that all known ω-stable, $\aleph_0$-categorical stuctures are constructed from finite structures in fairly simple ways, e.g. a direct sum of finite abelian groups, leads to the conjecture that this is always true. Given the difficulty of describing a 'simple' construction without having it in hand, the following test question is formulated. Is every totally categorical theory *quasi-finitely axiomatizable*, that is, finitely axiomatizable relative to an axiom schema of infinity? To indicate the progress made on this question we need two more notions. A theory T is *almost strongly minimal* [Baldwin 1972] if every model of some finite inessential extension of T is contained in the algebraic closure of a strongly minimal set. In the early 1970's these theories were thought of as the 'simple' $\aleph_1$-categorical theories. For example, [Makowsky 1984] had proved that no such theory was finitely axiomatizable. Ahlbrandt connected this notion with the local study of strictly minimal sets via the following definition and theorem. The totally categorical theory T is *almost of modular type* if for some finite principal extension T^* of T, every strictly minimal set in ${T^*}^{\mathrm{eq}}$ is modular.

3.7 Theorem. *The totally categorical theory T is almost strongly minimal if and only if it is almost of modular type.*

Finally Ahlbrandt and Ziegler proved

3.8 Theorem. *If the totally categorical theory T is almost of modular type then T is quasi-finitely axiomatizable.*

Another entry of geometry into stability theory came from Lachlan's proof [Lachlan 1974] that the question of the existence of a countable $\aleph_0$-categorical theory which is stable but not ω-stable could be given the following form. A *pseudoplane* is a two sorted structure of points and lines such that i) each line contains infinitely many points (and dually each point is on infinitely many lines) but ii) a pair of lines intersect in only finitely many points (and dually only finitely many lines pass through any fixed pair of points).

3.9 Theorem. *[Lachlan 1974] If there is a countable, $\aleph_0$-categorical and stable but not ω-stable theory then there is such a pseudoplane.*

Pseudoplanes have since showed up in a number of contexts in the theory. For example, Zilber's extension of the classification of strictly minimal sets to the $\aleph_1$ but not $\aleph_0$-categorical case provides the following theorem.

3.10 Theorem (The Trichotomy Theorem). *If M is an $\aleph_1$-categorical structure then exactly one of the following three conditions holds.*

i) *There is a pseudoplane definable in every strongly minimal structure definable in M*
ii) *Every strongly minimal structure definable in M is locally projective.*
iii) *Every strongly minimal structure definable in M is disintegrated.*

Zilber calls a strongly minimal set, D, *locally projective* if the geometry associated with D over a non-algebraic point is a projective geometry over a division ring. Intuitively, this divides $\aleph_1$-categorical theories into those which are field-like, module-like, or disintegrated. In particular, Zilber conjectures every $\aleph_1$-categorical pseudoplane is biinterpretable with an algebraically closed field.

Buechler [Buechler 1985b] made a significant application of the 'geometric' theory by using it to not only compute the spectrum of an ω-stable $\aleph_0$-categorical theory but to assign invariants to models in a more exact way than the general assignment for models of an arbitrary ω-stable theory. There is more detail on this in Chapter XVIII.

When dealing with theories which are not $\aleph_0$-categorical, certain properties of formulas in $\aleph_0$-categorical theories must be replaced by consideration of types. Thus, the notion of a type-interpretable pseudoplane, a pseudoplane whose incidence relation and sets of points and lines are defined by types, arises. Buechler [Buechler 1984] has shown that a trichotomy similar to Zilber's trichotomy holds for 'simple' types in arbitrary superstable theories. That is, if a stationary type contains a formula with continuous rank one then it defines either a strongly minimal, a locally projective, or a disintegrated set. Using this result he generalizes the coordinatization theorem and the fundamental rank inequality to superstable locally modular theories [Buechler 1986] with finite U-rank.

Pillay has defined a class of *weakly normal* theories. He shows that the stable theory T is weakly normal if and only if T does not type-interpret a pseudoplane. Moreover, in [Pillay 1986], he generalizes the Baur-Cherlin-Macintyre theorem that every stable $\aleph_0$-categorical group is Abelian by finite by showing that an ω-stable group which does not type-interpret a pseudoplane is Abelian by finite. In [Hrushovski & Pillay 1986] every weakly normal group is shown to be Abelian by finite.

3.11 Historical Notes. Since this section was a history there is little to say. Key papers include [Cherlin, Harrington, & Lachlan 1985], [Zilber 1981], [Zilber 1980b], [Zilber 1980a],[Zilber 1984],[Zilber 1984a], [Buechler 1985b], and [Ahlbrandt & Ziegler 1986]. Another interesting line is pursued by Srour and Pillay [Pillay 1983], [Pillay & Srour 1984], and [Pillay 198?a]. Another important tack is the investigation of finite homogeneous structures. Lachlan has written a series of papers which are listed in the bibliography. See also [Cherlin & Lachlan 1986] and [Lachlan & Shelah 1984]. In still another direction a detailed analysis of categorical varieties and quasivarieties has been given ([Givant 1973], [Givant 1976], [Palyutin 1973], [Palyutin 1976]). The representation theorems proved there are remarkably similar to Theorem 3.1.