Chapter VII

Rank

Stability theory arose from Morley's investigation of $\aleph_1$-categorical theories, where he introduced the notion of the rank of a type in an ω-stable theory. From this beginning, Shelah and others developed a number of different rank notions to investigate more complicated (e.g. stable and superstable) theories. In this chapter we will investigate these various rank functions and their relationships to nonforking. It will become clear that our entire machine could have been constructed on the basis of rank. There are two reasons that we did not do this. In the first place rank is actually a finer tool than nonforking. We will show that for a rank function R satisfying certain axioms (in a superstable or ω-stable theory), p is a nonforking extension of q exactly if $R(p) = R(q)$. However, the rank codes additional information that is lost by the dichotomy 'forks or doesn't fork.' That is, if p is a forking extension of q we can ask how much less is $R(p)$ than $R(q)$. This additional information plays an important role in some branches of stability theory (e.g. [Cherlin, Harrington, & Lachlan 1985]). But, in the basic study of free extensions the additional information obscures the issues. More importantly, no equally simple definition of nonforking in terms of rank holds for stable theories. Rank also provides more information because it allows one to compare two types neither of which extends the other.

In Section 1 we exhibit some axioms for rank and show that any rank which satisfies these axioms induces the nonforking relation. In Section 2 we describe a number of the important rank functions. In Section 3 we prove some theorems illustrating the greater power of the rank notion.

1. Ranks and Forking

We describe here the properties that the notion of free extension satisfies when we define $p \mathcal{F} q$ if $q \subseteq p$ and $R(p) = R(q)$, where R satisfies the axioms given below. We show that these properties of the resulting freeness relation completely characterize the nonforking relation on stable theories.

1.1 Axioms For Rank. Let R be a map which associates with each p in $S(A)$ (for A a subset of $\mathcal{M}$) an ordinal $R(p)$ which is called its *rank.*

i) If f is an elementary map then $R(p) = R(fp)$.
ii) If $p \subseteq q$ then $R(q) \leq R(p)$.
iii) For each B containing $\operatorname{dom} p$ there is at least one $q \in S(B)$ with $R(q) = R(p)$.
iv) If $p \in S(A)$ there is a finite subset $A_0 \subseteq \operatorname{dom} p$ with $R(p) = R(p|A_0)$.
v) There is a cardinal λ such that any type p has at most λ mutually contradictory extensions with the same rank as p.

1.2 Notation. If R is a rank satisfying the axioms of 1.1, we denote by $\mathcal{F}_R$ the notion of freeness given by: p is a free extension of q if p extends q and $R(p) = R(q)$.

$\mathcal{F}_R$ is defined on complete types but we extend it to incomplete types by saying q is free over A if for some complete q' extending q, $q' \mathcal{F}_R q'|A$. It is now easy to verify the following lemma.

1.3 Lemma. *If R satisfies the axioms in 1.1 then $\mathcal{F}_R$ is a freeness relation obeying the isomorphism convention (II.1.2), the first monotonicity axiom (II.1.5 M_1), the extension axiom (II.1.14 E_1), and the full local character of freeness axiom (II.1.21). Moreover, there is a cardinal λ such that no p has more than λ mutually contradictory free extensions.*

Note that the condition on the number of free extensions is implied by the axiom asserting that types over models are stationary (in the presence of the second monotonicity axiom).

Now we show that the only relation satisfying these axioms is the relation of nonforking for stable theories. We have shown that nonforking satisfies these axioms (and others) in Sections III.3 and III.4.

1.4 Theorem. *If $\mathcal{F}$ satisfies the properties listed in Lemma 1.3 then T is superstable and $\mathcal{F}$ is the nonforking relation.*

Proof. Clearly the bound on the number of extensions in rank and the local character of freeness imply that T is stable in μ for all $\mu \geq \lambda + |T|$ (by counting the types over A). Now suppose $q\mathcal{F}p$ with $p \in S(A)$. Extend q to a global type $\hat{q}$ with $\hat{q}\mathcal{F}p$. Since all conjugates over A of $\hat{q}$ are free over p there at most λ of them. By Lemma IV.1.14, $\hat{q}$ and, a fortiori, q do not fork over A.

Conversely, suppose q is a nonforking extension of $p \in S(A)$. Extend q to $\hat{q}$ a global type which does not fork over A. Also extend p to $\hat{p}$ a global type with $\hat{p}\mathcal{F}p$. By the first part of the proof, $\hat{p}$ does not fork over A. Thus by the conjugacy lemma $\hat{p}$ and $\hat{q}$ are conjugate over A. Thus $\hat{q}\mathcal{F}p$ and by monotonicity $q\mathcal{F}A$.

Note that the symmetry and transitivity properties now follow from the axioms listed in Lemma 1.3 since we know they hold of the only relation which satisfies those axioms.

There is no ordinal valued rank function which is defined on all types exactly if T is a stable theory. However, a slight relaxation of the conditions in Lemma 1.3 describes the properties of a freeness relation which imply that the underlying theory is stable and the relation is nonforking.

1.5 Exercise. Show that if iv) of 1.1 is weakened by requiring the cardinality of the set A to be less than $|T|$, in Theorem 1.4 we can still conclude that T is stable and $\mathcal{F}$ is the nonforking relation. (rng R is a linear order.)

1.6 Historical Notes. The first axiomatic approaches to rank were in [Baldwin & Blass 1974] and [Lascar 1976]. Theorem 1.4 is derived from [Lascar 1976]. Another useful account is [Harnik & Harrington 1984].

2. A Plethora of Ranks

This section is devoted to the definitions of a number of rank functions and the verification that they satisfy the axioms discussed in Section VII.1. There are a number of minor variants of the definitions but we forgo the tedious discussion of equivalences among these variants. For details see [Shelah 1978].

The most important rank functions are Morley rank, U-rank, and continuous rank (Shelah-degree). The Morley rank of each type in a countable theory is defined if and only if the theory is ω-stable. The U-rank and continuous rank are defined for all complete types exactly if T is superstable. We describe below the distinctions among these three ranks. In short, continuous ranks and Morley ranks are defined on formulas; the rank can be extended to complete types. U-rank is defined only on complete types but it enjoys an additivity property which facilitates calculating the rank of the type of a pair of elements. One reason for the more satisfactory analyses of theories which are categorical in some infinite power is that on such theories Morley rank and U-rank coincide.

Most of the ranks we discuss can be defined either directly on types or on formulas. We will define the rank on formulas when possible and then extend to types by setting $R(p) = \inf\{R(\phi) : \phi \in p\}$. This automatically guarantees that each type has a finite subtype of the same rank, thus establishing the local character of dependence for the associated freeness relation.

Morley originally defined rank for a complete type over a set by an extension of the Cantor-Bendixson rank of the Stone space. In fact, if M is ω-saturated and $p \in S(M)$ then $R_M(p)$ is exactly the Cantor-Bendixson rank of p in $S(M)$. However, we prefer to define the rank directly on formulas. We give first a definition which specializes to both the original Morley rank and to the important local version considered in Section III.1. We require some further notation.

2.1 Notation. Let Δ be a collection of formulas of the form $\phi(\overline{x};\overline{y})$. We denote by $C(\Delta)$ the collection of finite conjunctions of substitution instances $\phi(\overline{x},\overline{a})$ of formulas in Δ.

The two most important uses of this notation occur when Δ is the collection of all n-ary formulas for some n and when $\Delta = \{\phi(\overline{x};\overline{y}), \neg\phi(\overline{x};\overline{y})\}$ for some single formula ϕ. By Exercise III.1.3 this second case is the same as considering an arbitrary finite set of formulas (instead of just one).

2.2 Definition ((Δ,μ)-Rank). Let $\phi(\overline{x};\overline{a})$ be a formula, μ a cardinal, Δ a set of formulas and α an ordinal. The (Δ,μ)-*rank*, R_Δ, of a formula $\phi(\overline{x};\overline{a})$ is defined as follows.

- $R_\Delta(\phi(\overline{x};\overline{a})) = -1$ iff $\models \neg(\exists\overline{x})\phi(\overline{x};\overline{a})$.
- $R_\Delta(\phi(\overline{x};\overline{a})) \geq 0$ iff $\models (\exists\overline{x})\phi(\overline{x};\overline{a})$
- $R_\Delta(\phi(\overline{x};\overline{a}) \geq \alpha + 1$ iff for each $k < \mu$, $*_{(\alpha,k)}$ holds:

  There exists a sequence of formulas $\theta_i(\overline{x};\overline{a}_i)$ for $i \leq k$ with $\theta_i(\overline{x};\overline{a}_i) \in C(\Delta)$ such that $R_\Delta(\phi(\overline{x};\overline{a}) \wedge \theta_i(\overline{x};\overline{a}_i)) \geq \alpha$ and where the formulas $\{\phi(\overline{x};\overline{a}) \wedge \theta_i(\overline{x};\overline{a}_i) : i < k\}$ are pairwise contradictory. $\quad *^{(\alpha,k)}$

- $R_\Delta(\phi(\overline{x};\overline{a})) \geq \delta$ where δ is a limit ordinal iff $R_\Delta(\phi(\overline{x};\overline{a})) \geq \beta$ for each $\beta < \delta$.
- $R_\Delta(\phi(\overline{x};\overline{a})) = \gamma$ if γ is the least ordinal β with $R_\Delta(\phi(\overline{x};\overline{a})) \not\geq \beta$.

Note that if $R_\Delta(\phi(\overline{x};\overline{a})) = \alpha$ then there is a least $m < \mu$ so that there is no sequence of formulas $\theta_i(\overline{x};\overline{a}_i)$ for $i < m+1$ satisfying $^*(\alpha, m+1)$. Such an m is called the *degree* of $\phi(\overline{x};\overline{a})$ and denoted $D_\Delta(\phi(\overline{x};\overline{a}))$. This notion is denoted $R^n(p,\Delta,\mu)$ in [Shelah 1978], where the n indicates that only formulas with n free variables are ranked. We have suppressed the explicit mention of μ assuming that it will be clear in context.

2.3 Definition. i) (Morley rank) If Δ is the collection of all formulas and $\mu = \omega$ we call the resulting rank *Morley rank* and denote it by $R_M(\phi(\overline{x};\overline{a}))$.

ii) (local rank) If Δ is a finite set of formulas and μ is 2, R_Δ is the local rank R studied in Section III.1. We often code Δ by a single formula ϕ and speak of ϕ-rank.

iii) If Δ is a finite set of formulas and μ is ω, R_Δ is called simply (Δ,ω) rank.

In this chapter for finite Δ, R_Δ denotes local rank unless we explicitly say otherwise.

It is now straightforward to verify the following result by induction.

2.4 Lemma. *Let T be stable. For any finite Δ and any p, $R_\Delta(p)$ is defined. If T is ω-stable then $R_M(p)$ is defined for any p. If μ is infinite and condition v) is restricted to Δ-types then each R_Δ satisfies conditions i) through v) of Axiom 1.1.*

The next exercise asks for an example showing the necessity of assuming μ is infinite to satisfy condition iii). Exercise 2.6 illustrates another aspect of the same point.

2.5 Exercise. Show that condition iii) may fail for local rank.

2.6 Exercise. Show that $R_M(\phi \vee \psi) = \max(R_M(\phi), R_M(\psi))$.

2.7 Exercise. Compute $R_M(x{=}x)$ for the three theories REF_ω, CEF_ω, and ACF_0.

Exercise 2.8 provides a naive illustration of the difficulty in satisfying Axiom 1.1 v) for all types with a Δ-rank for a Δ which does not contain all formulas.

2.8 Exercise. Show that if T is the theory of a single equivalence relation with infinitely many infinite classes, p is the unique 1-type over the empty set, and Δ is $\{x = y\}$ then p has unboundedly many incompatible extensions q_i with $R_\Delta(q_i) = R_\Delta(p)$.

The following definition and theorem provide a rank for stable theories. Note however that the values of the rank function are sequences of integers rather than ordinals.

2.9 Notation. If $p \subseteq q$, q is a free extension of p for finite sets of formulas (written $q\mathcal{F}_f p$) if for every finite Δ, $R_\Delta(q) = R_\Delta(p)$.

The proof of the following theorem approximates that of Theorem 1.4. We must replace Axiom 1.1 iv) by the observation that if we define

$$R_f(p) = \langle R_\Delta(p) : \Delta \text{ a finite subset of } F(L)\rangle$$

then every type has a subtype with the same rank which is over a set with cardinality at most $|T|$.

2.10 Theorem. *Let T be a stable theory. If $p \subseteq q$ then q does not fork over* $\operatorname{dom} p$ *iff $q\mathcal{F}_f p$.*

Proof. By Lemma IV.1.14 it suffices to show that for some λ, p has at most λ extensions q_i with $q_i\mathcal{F}_f p$. Since T is stable, for each finite Δ there are only finitely many Δ-types of extensions with the same Δ-rank. Thus there are at most $2^{|T|} \times |A|^{|T|}$ extensions of p to complete types with the same rank.

Now we consider a measure of exactly how much a type can be made to fork. This rank was introduced by Lascar who originally defined it as the least connected (see below) rank function. It differs significantly from the other ranks we discuss because although every type has a subtype over a finite set with the same rank, there may not be a single formula with the same rank. Accordingly, we define this notion for complete types over some subset of the monster model.

2.11 Definition. For $p \in S(A)$ we define the *U-rank* (also called the Lascar rank) of a type p, $U(p)$, by induction.

- $U(p) \geq 0$ iff p is consistent.
- $U(p) \geq \beta + 1$ iff there exists B containing $\operatorname{dom} p$ and $q \in S(B)$ such that $p \subseteq q$ and $U(q) \geq \beta$ and q forks over A.
- $U(p) \geq \delta$ for a limit ordinal δ iff $U(p) \geq \beta$ for each $\beta < \delta$.
- $U(p) = \alpha$ if α is the least ordinal γ such that $U(p) \not\geq \gamma$.

2.12 Exercise. Show that if T is superstable the U-rank of each type is defined.

The following notions distinguish Morley and Lascar rank. We say an ordinal-valued rank function is connected if the range of its restriction to complete types forms a connected subset of the ordinals under the order topology. Thus, we have

2.13 Definition. i) The rank function R is *connected* if for every type p with $R(p) > \alpha$, there is a complete type, p', extending p with $R(p') = \alpha$.

ii) The rank function R is *continuous* if for every set A and ordinal α, $\{p \in S(A) : R(p) \geq \alpha\}$ is a closed subset of $S(A)$.

2.14 Exercise. i) Show Morley rank is continuous and U-rank is connected.

ii) Show that Morley rank may not be connected and U-rank may not be continuous. (Hint: Example 2.17 is germane.)

Now we define another notion of rank in a completely different way from U-rank. These ranks share the property that if T is stable they are defined for all types precisely when T is superstable. The next notion was called *infinity rank* in [Shelah 1978] and *degree* (not to be confused with the Morley degree) when Shelah introduced it in [Shelah 1971]. The crucial fact about this continuous rank is that it assigns a rank to formulas rather than to types. The advantage of U-rank is the additivity properties discussed below.

2.15 Definition. The *continuous rank*, R_C, of a formula $\phi(\overline{x};\overline{a})$ is defined by induction as follows.

- $R_C(\phi(\overline{x};\overline{a})) \geq 0$ iff $(\exists \overline{x})\phi(\overline{x};\overline{a})$.
- $R_C(\phi(\overline{x};\overline{a})) \geq \beta + 1$ iff there exists a formula $\psi(\overline{x};\overline{y})$ and sequences $\overline{c}_i$ for $i < (2^{|T|})^+$ such that $R_C(\phi(\overline{x};\overline{a}) \wedge \psi(\overline{x};\overline{c}_i)) \geq \beta$ for each i and the set of formulas $\{\psi(\overline{x};\overline{c}_i)\}$ is n-inconsistent for some n.
- $R_C(\phi(\overline{x};\overline{a})) \geq \delta$ for a limit ordinal δ just if $R_C(\phi(\overline{x};\overline{a})) \geq \beta$ for each $\beta < \delta$.

$R_C(\phi(\overline{x};\overline{a})) = \gamma$ if γ is the least ordinal β such that $R_C(\phi(\overline{x};\overline{a})) \not\geq \beta$.

Shelah actually makes the bound in ii) to be $|T|^+$. The present definition yields the same result with fewer combinatorial difficulties. The delicate relation between various versions of this notion is discussed in Chapter II of [Shelah 1978]. The next theorem indicates the relative value of R_C and U. In one direction we are restricted to finite ordinals.

2.16 Theorem. *For each formula $\phi(\overline{x};\overline{y})$, each $\overline{a}$, and each natural number n, $R_C(\phi(\overline{x};\overline{a})) \geq n$ iff there exists a $p \in S(\overline{a})$ with $\phi(\overline{x};\overline{a}) \in p$ and $U(p) \geq n$. For any ordinal α, if there exists such a p with $\phi(\overline{x};\overline{a}) \in p$ and $U(p) \geq \alpha$ then $R_C(\phi(\overline{x};\overline{a})) \geq \alpha$.*

Proof. Suppose $R_C(\phi(\overline{x};\overline{a})) \geq n+1$. Then there are a formula $\psi(\overline{x};\overline{y})$ and sequences $\overline{c}_i$ for $i < (2^{|T|})^+$ such that $R_C(\phi(\overline{x};\overline{a}) \wedge \psi(\overline{x};\overline{c}_i)) \geq n$ and with $\{\psi(\overline{x};\overline{c}_i) : i < (2^{|T|})^+\}$ n-inconsistent. By the pigeonhole principle there is an infinite subset of the $\overline{c}_i$ which realize the same type over $\overline{a}$. That is, for some $\overline{c}_i$ (wlog $\overline{c}_0$)$\psi(\overline{x};\overline{c}_0)$ divides over $\overline{a}$. But then if q is any completion of $\{\phi(\overline{x},\overline{a}) \wedge \psi(\overline{x};\overline{c}_0)\}$ to a complete type over $\overline{a} \cup \overline{c}_0$, q forks over $\overline{a}$ and by induction $U(q) \geq n$. Thus $q|\overline{a}$ is the required member of $S(\overline{a})$ with $U(q|\overline{a}) \geq n+1$.

Since we have shown in Lemma V.3.9 that a complete type which forks contains a formula which divides the converse is easy.

This lemma is best possible in a sense which illustrates the difference between U being based on complete types and R_C being based on formulas.

2.17 Example. For any $\alpha < \aleph_1$ there is a theory T_α with $R_C(x = x) = \alpha$ but for every p, $U(p)$ is finite. We construct the T_α by induction on α. If α is finite there is nothing to prove. If we have T_α, let $T_{\alpha+1}$ be the theory of an equivalence relation with infinitely many infinite classes, each of which is a model of T_α. If δ is a limit ordinal let T_δ be the theory of the disjoint union of models of the T_α for $\alpha < \delta$.

The next theorem describes two important properties of continuous rank which we will exploit in the next section.

2.18 Theorem. i) *R_C is the least continuous rank function.*
ii) *If $[p] \geq [q]$ (in the fundamental order) then $R_C(p) \geq R_C(q)$.*

Theorem 2.18 i) is proved by a routine induction; Theorem 2.18 ii) follows directly from the definition (using Section V.3).

2.19 Exercise. Show that if R is a continuous rank function and T is superstable (in particular, if $R = R_C$) then there is a complete type with maximal R-rank.

As we have seen on many occasions one of the major problems in this subject is to relate the behavior of singletons to the behavior of pairs. This problem arises again for rank. The behavior of Morley rank in this situation is extremely complicated ([Lachlan 1980]). Lascar rank is better behaved.

2.20 Theorem. *For any $\overline{b}, \overline{c}$ in a model of a superstable theory*

$$U(t(\overline{c}; A \cup \overline{b}) + U(t(\overline{b}; A)) \leq U(t(\overline{b}^\frown\overline{c}; A)) \leq U(t(\overline{c}; A \cup \overline{b})) \oplus U(t(\overline{b}; A)).$$

Here $\oplus$ denotes the natural sum of ordinals. That is, to add two ordinals put them in Cantor normal form and add them as polynomials in ω. Note that natural sum agrees with ordinary arithmetic on finite ordinals. If $\overline{b} \downarrow_A \overline{c}$

then the second inequality becomes an equality. For the proof of this theorem see [Lascar 1976]. This inequality has been extensively exploited. See, for example, [Berline & Lascar 1986] and [Cherlin, Harrington, & Lachlan 1985].

2.21 Historical Notes. For further information see [Lascar 1976], [Lachlan 1980], [Shelah 1978], or [Pillay 1983a]. Pillay's book has a more detailed introductory survey which includes a fuller proof of Theorem 2.16. Morley rank was introduced in [Morley 1965]. If T is categorical in either $\aleph_1$ or $\aleph_0$, Morley rank and U-rank agree [Lascar 1976]. Moreover, in such theories the rank of every type is finite. For $\aleph_1$-categorical theories this was first proved in [Baldwin 1973]. Later, more accessible, proofs appear in [Poizat 1978] and [Zilber 1974]. The proof for $\aleph_0$-categorical theories is in [Cherlin, Harrington, & Lachlan 1985]. In fact, if T is unidimensional then R_C and U are equal and finite [Saffe 1984].

3. Ranks and Stable Groups

The major goal of this section is to prove the theorem of Cherlin, Shelah, and Macintyre that a superstable field is algebraically closed. The proof given here rests ultimately on the ability to compare the ranks of two types over the same set. Thus, we exploit one of the advantages of rank over forking which we mentioned at the beginning of this chapter. We will somewhat disguise this fact by expounding Poizat's notion of the stratified order which provides a nice framework for formulating the result.

The proof of the main theorem requires three major steps. We already know from Section III.5 that the additive group of a superstable division ring is connected. The model theoretic task in this section will be to show that this implies that the multiplicative group of the division ring is connected. Knowing that both groups are connected, an algebraic proof due to Macintyre yields the theorem.

The fundamental insight is that the action of a group on itself by (left) multiplication preserves rank. To make this idea precise we define the translate and the inverse of a type. As in Section III.5, by a 'group' we mean any structure which includes a group operation among its definable operations.

3.1 Definition. Let M be a group, $a \in M$, and let $c \in \mathcal{M}$ realize the type $p \in S(M)$. Then ap, the *left translate of p by a* is $t(ac; M)$. That is, $ap = \{\phi(a^{-1}x; \overline{m}) : \phi(x; \overline{m}) \in p\}$. Similarly, $p^{-1} = t(c^{-1}; M)$.

Clearly, the operation of left translation by elements of M induces an equivalence relation on $S^1(M)$ which preserves any of the rank functions we have discussed (except the local ranks). But the relation is too fine; we also want to identify types which are only potentially translates of each other. The following notion captures this idea.

3.2 Definition. If $M, N \models T, p \in S^1(M), q \in S^1(N)$ then the *stratum* of p is at least that of q, written $s(p) \geq s(q)$, if for every formula of the form $\phi(yx; \overline{z})$, whenever ϕ is represented in $p(x)$ then ϕ is represented in $q(x)$. That is, if for some a and $\overline{b}$, $\phi(ax; \overline{b}) \in p$ then for some a' and $\overline{b}'$, $\phi(a'x; \overline{b}') \in q$.

The partial order which results by identifying types p and q if $s(p) \geq s(q)$ and $s(q) \geq s(p)$ is called the *stratified order.* We call the equivalence class of p under this relation the *stratum* of p.

3.3 Exercise. Show that if $q \leq p$ in the sense of the fundamental order then $s(q) \leq s(p)$. Show the converse fails.

3.4 Exercise. Show that if $\phi(x; \overline{y})$ is represented in p and $s(p) \geq s(q)$ then $\phi(zx; \overline{y})$ is represented in q. (Remember $\phi(x; \overline{y})$ is $\phi(1x; \overline{y})$.) In particular, all translates of a type are in the same stratum.

3.5 Exercise. Show that there are at most $2^{|T|}$ elements in the stratified order.

3.6 Theorem. *Suppose* $p \in S^1(M), q \in S^1(N)$ *and* $s(p) \geq s(q)$. *Then there is an* $\alpha \in \mathrm{Aut}(\mathcal{M})$ *and an* $a \in \mathcal{M}$ *such that* $p \subseteq a\alpha(q^{\mathcal{M}})$.

Proof. Extend L by adding a unary predicate P, names for the elements of M and new constants a and c. We first show that there is a type q_1 such that $q_1 \simeq q$ and $p \geq aq_1$. Let Γ be the following set of sentences:

$$\begin{aligned} &T \cup T|P \cup \{P(a)\} \\ &\quad \cup \{(\exists \overline{y})[P(\overline{y}) \wedge \phi(ac; \overline{y})] : \phi(x; \overline{y}) \text{ is represented in } p\} \\ &\quad \cup \{(\exists \overline{y})[P(\overline{y}) \wedge \phi(c; \overline{y})] : \phi(x; \overline{y}) \text{ is represented in } q\} \\ &\quad \cup \{(\forall \overline{y})[P(\overline{y}) \rightarrow \neg\phi(c; \overline{y})] : \text{if } \phi(x; \overline{y}) \text{ is not represented in } q\}. \end{aligned}$$

Note that if $\phi(x; \overline{y})$ is represented in p then $\phi(zx; \overline{w})$ is represented in q. With this in mind it is easy to see that any finite subset of Γ is satisfiable by interpreting P as the universe of N, a as an appropriate element of N, and c as a realization of q. Then the required q_1 is $t(c; P(M_1))$ if $M_1 \models \Gamma$.

A similar compactness argument shows that for any r and p, if $p \geq r$ there is an extension r_1 of p with $r_1 \simeq r$. Apply this argument to p and aq_1 to get a model $N_1 \supseteq M$ and $r_2 \in S(N_1)$ such that $r_2 \simeq aq_1$ and $p \subseteq r_2$. Let c realize $a^{-1}r_2$ and let $q_2 = t(c; N_1)$. Then $q_2 \simeq q$.

Let $\hat{p}$ be a nonforking extension of p to $\mathcal{M}$, $\hat{q}$ a nonforking extension of q to $\mathcal{M}$, and $\hat{q}_2$ a nonforking extension of q_2 to $\mathcal{M}$. Then $\hat{q} \simeq \hat{q}_2$ so by Theorem III.2.36 there is an automorphism α of $\mathcal{M}$ with $\alpha(\hat{q}) = \hat{q}_2$ and so $p \subseteq a\alpha(q^{\mathcal{M}})$ as required.

The following exercise restates the omitted compactness argument in the last proof.

3.7 Exercise. Show that $p \geq q$ implies that p is contained in a conjugate of a nonforking extension of q.

We deduce the following crucial property from this result.

3.8 Corollary. *Suppose $s(p) = s(q)$; then if R is any one of R_M, U, or R_C, $R(p) = R(q)$. Moreover, $R(q) = R(q^{-1})$.*

Proof. All of these ranks are preserved by left translation.

Note that this result does not hold for all local ranks because ap_ϕ is not always a ϕ-type. Our aim now is to identify types of maximal stratum with types of maximal rank. This will have a number of useful corollaries, the first being the existence of types with maximal U-rank. To save space we give the most direct proofs for superstable theories. Some of the results hold for the stable case.

3.9 Lemma. *Let G be a group and suppose $T = \mathrm{Th}(G)$ is superstable. There is exactly one maximal stratum.*

Proof. Fix a strongly ω-saturated model of T. Since T is superstable, there is a $p \in S_1(M)$ with maximal continuous rank. As $s(q) > s(p)$ implies $q > p$ (in the fundamental order), which implies $R_C(q) > R_C(p)$ by Corollary 2.18, we have that the stratum of p is maximal. We want to show that if $R_C(q) = R_C(p)$ then $s(q) = s(p)$. We first show that if $R_C(q) = R_C(p)$ then $s(q^{-1}) = s(p)$. Applying this observation with $p = q$ yields $s(q) = s(q^{-1})$ and then applying it to p and q^{-1} yields the theorem.

To see that if $R_C(q) = R_C(p)$ then $s(q^{-1}) = s(p)$, let a realize p and b realize q with $a \downarrow_M b$. We first claim that $a \downarrow_M ab$. For this, choose $N \supset M \cup a$ with $b \downarrow_{M \cup a} N$. Now $R_C(t(ab; N)) = R_C(t(b; N))$ since ab is definable from $N \cup b$. But $R_C(t(b; N))$ is maximal so $ab \downarrow_M N$. In particular, $ab \downarrow_M a$ as claimed. This allows us to choose $M' \supset M \cup ab$ with $M' \downarrow_M a$. Denote $t(a; M')$ by p_1. Now, $b^{-1}a^{-1} \in M'$ and $b^{-1}a^{-1}p_1 = t(b^{-1}; M')$. Thus $s(t(b^{-1}; M')) = s(p_1) = s(p)$ which is maximal. But $q^{-1} \subseteq t(b^{-1}; M')$ implies $s(t(b^{-1}; M')) \leq s(q^{-1})$ implies $s(q^{-1}) = s(p)$ and we finish.

By emulating the proof of the existence of maximal elements in the fundamental order, Lemma III.3.11, and then refining the previous argument one can prove the following stronger result of Poizat.

3.10 Exercise. Show that if T is the theory of a stable group then there is a unique maximal stratum.

3.11 Exercise. Show that if T is the theory of a superstable group then the types in the maximal stratum do not fork over the empty set.

3.12 Corollary. *Let G be a group and suppose $T = \mathrm{Th}(G)$ is superstable. There are types of maximal U-rank.*

Proof. The types in the maximal stratum must have maximal U-rank. For, $s(p) \leq s(q)$ implies $U(p) \leq U(q)$.

Our next step is to link the connectivity property discussed in Section III.5 with the number of types of maximal stratum. The following exercise leads to the easy half of the connection.

3.13 Exercise. Show that if $\phi(x; \overline{m})$ defines a subgroup of M with finite index then $\phi(zx; \overline{y})$ is represented in every type over M.

3.14 Lemma. *If M contains a proper definable subgroup of finite index then there is more than one type in the maximal stratum.*

Proof. Let $\phi(x;\overline{m})$ define a proper subgroup H with finite index in M. Now, if e denotes the identity of the group and $a \in M - H$, $\phi(ax;\overline{m})$ and $\phi(ex;\overline{m})$ occur in distinct types of maximal stratum.

To show the converse of this lemma we require a little more notation.

3.15 Definition. i) Let $M \models T$ where T is a theory of groups and let p be a type (not necessarily complete) over M. The *stabilizer of p* is $F_p = \{a \in M : ap = p\}$.

ii) For each formula $\phi(x;\overline{y})$, let $\hat{\phi}(x,z,\overline{y})$ denote $\phi(zx;\overline{y})$.

3.16 Lemma. *For any $p \in S(M)$ and any formula ϕ, $F_{p_{\hat{\phi}}}$ is a definable subgroup of M. If p has maximal stratum then $[M : F_{p_{\hat{\phi}}}] < \omega$.*

Proof. The required definition is:

$$\chi(u) = (\forall\overline{y})(\forall z)[d\hat{\phi}(z;\overline{y}) \leftrightarrow d\hat{\phi}(uz;\overline{y})].$$

Now, $[M : F_{p_{\hat{\phi}}}] = |\{ap_{\hat{\phi}} : a \in M\}|$. But since p and thus ap is in a maximal strata, each $ap_{\hat{\phi}}$ extends to a global type which does not fork over the empty set. Each of the types is definable over every model of T. Thus, the number of translates of p is bounded by $2^{|T|}$. But then by compactness the number of translates of $p_{\hat{\phi}}$ must be finite.

3.17 Corollary. *If M is connected there is a unique type of maximal stratum.*

Proof. By Lemma 3.16, if p has maximal stratum, then for each ϕ, $F_{p_{\hat{\phi}}} = M$. Thus all types of maximal stratum are equal (since we can extend to a model where they are translates of each other).

3.18 Exercise. Show that if $p \in S(M)$, then F_p is defined by an infinite family of formulas. (That is, by an intersection of infinitely many definable sets.)

The concept of an infinitely definable subgroup has been exploited extensively ([Berline & Lascar 1986] and [Hrushovski 1986]).

3.19 Theorem. *If T is the theory of a superstable structure which admits two group structures then one is connected if and only if the other is connected.*

Proof. We have shown that the group is connected if and only if there is only one type in the associated maximal stratum. But by Corollary 3.17 the types in the maximal stratum for the two operations are the same, namely, the types of maximal U-rank.

We showed in III.5 that the additive group of a superstable division ring is connected. Now we can conclude:

3.20 Corollary. *If M is a superstable division ring, both the additive and multiplicative groups of M are connected.*

We turn now to the algebraic information necessary to conclude the proof of the main theorem. The following theorem can be found, for example, in [Lang 1971] or Chapter 23 of [Adamson 1982].

3.21 Theorem. *Let K be an extension of prime degree, q, of the field F such that F is the fixed field of the automorphism group of K and $x^q - 1$ splits in K. Let p be the characteristic of K.*

i) *If $q = p$ then K is generated over F by the solution of an equation $x^p - x = a$ for some $a \in F$.*
ii) *If $q \neq p$ then K is generated over F by the solution of an equation $x^q = a$ for some $a \in F$.*

In the first case K is called an *Artin-Schreier extension* of F and in the second a *Kummer* extension. Recall that a field F with characteristic $p \neq 0$ is *perfect* if every element of F has a pth root in F and a *Galois extension* of a field is an algebraic extension which is both normal and separable.

3.22 Lemma. *If F is a superstable field then F is perfect and has no Artin-Schreier or Kummer extensions.*

Proof. Let $h(x)$ be either of the following two maps.

i) $x \mapsto x^p - x$ (if the characteristic of F is $p > 0$).
ii) $x \mapsto x^n$ for $x \neq 0$ an element of F where $n \geq 1$ is an arbitrary natural number.

Now, h is a definable endomorphism of the additive group of F in the first case, and of the multiplicative group of F in the second. In each case the kernel of h is finite. Since we have just shown that both groups are connected, Theorem III.5.25 shows that in each case h is surjective. The surjectivity of the first map implies that F has no Artin-Schreier extension and the surjectivity of the second that F is perfect and has no Kummer extension. This yields the lemma.

3.23 Theorem. *If F is a superstable field then F is algebraically closed or finite.*

Proof. Assume for contradiction that F is an infinite superstable field which is not algebraically closed. By the previous lemma, F is perfect and so it has a Galois extension of some finite degree n. Consider all pairs of fields (F, K) such that (*) F is an infinite superstable field and K is a proper finite dimensional Galois extension of F. Choose such a pair with the degree, q, of K over F minimal.

We claim q is prime and $x^q - 1$ splits over F. If q is not prime, choose an r which divides q and let F_1 be the fixed field of an element of order r in $\mathrm{Gal}(K/F)$. Since F_1 is a finite dimensional extension of F, F_1 is also superstable and so the pair (F_1, K) contradicts the minimal choice of q.

Thus q is prime. Now, let K_1 be the splitting extension of $x^q - 1$ over F. Then the degree of K_1 over F divides $q-1$, so by the minimality of q, $K_1 = K$, as required.

Now Lemmas 3.21 and 3.22 yield a contradiction unless $q = 1$ and we finish.

Theorem 3.23 has been strengthened by Cherlin to remove the hypothesis that F is commutative. We will not give the full argument for this result here. The basic theme of the proof is to show that the division ring D is finite dimensional over a subfield, F. One algebraic result and an application of Theorem 3.23 then yield the result. Many of the basic ideas in finding a large Abelian subgroup of a stable group are contained in the following theorem of Reineke.

We rely in the next proof on two standard group theoretic facts. If all elements of a group have order two then it is Abelian. There is a 1-1 correspondence between the conjugates of an element a and the cosets of $C_G(a)$. We write $[a,b]$ for the commutator, $a^{-1}b^{-1}ab$ of a and b.

3.24 Theorem. *If the theory of the group G is strongly minimal then G is Abelian.*

Proof. We will apply at various points in the proof the following strong consequences of assuming that G is strongly minimal. First, G can have no infinite definable proper subgroup, H (as H and any coset of H would be disjoint infinite sets). In particular, G is connected. Since G is superstable, we conclude from Exercise III.5.23 that any finite normal subgroup of G is contained in $Z(G)$. Since there are no infinite definable subgroups, we have, in fact, that every definable normal subgroup of G is contained in $Z(G)$. Thus, $Z^2(G) = \{a : \forall b[a,b] \in Z(G)\} = Z(G)$. It is now easy to see that $\bar{G} = G/Z(G)$ is centerless and strongly minimal.

Let $a \neq 1 \in \bar{G}$. Then, $C_{\bar{G}}(a)$ is finite. This easily implies that a has finite order (since all powers of a commute with a) and that C_a, the conjugacy class of $a \in \bar{G}$ is infinite (by the second remark before the theorem). Since $\{C_a : a \neq 1\}$ partition $\bar{G} - \{1\}$, all nonidentity elements of $\bar{G}$ are conjugate. This implies that they all have the same order, l, which is prime. For otherwise, we have elements of different orders that are conjugate. Since we finish if l is two, we may assume that l is odd.

For some b, $bab^{-1} = a^{-1}$. Thus, $(bab^{-1})^{-1} = ba^{-1}b^{-1} = a$. It is now easy using this fact to show by induction that b^kab^{-k} equals a if k is even and a^{-1} if k is odd. Thus, $b^lab^{-l} = a^{-1}$. Since, the order of b is l, we conclude that $a = a^{-1}$ and thus the theorem.

We omit the proofs of the long string of generalizations of this result, [Baur, Cherlin & Macintyre 1979], [Cherlin 1978], [Poizat 1981], and just state the strongest result which is due to Lascar and Berline [Berline & Lascar 1986].

3.25 Theorem. *If G is a superstable group and $U(G) \geq \omega^\alpha$, then G contains a definable Abelian subgroup H with $U(H) \geq \omega^\alpha$.*

Assuming this result we finish the proof of Cherlin's commutativity theorem.

3.26 Theorem. *An infinite superstable division ring D is an algebraically closed field.*

Proof. Without loss of generality, we may assume D is $\aleph_1$-saturated. Let $U(D) = \omega^\alpha k + \beta$ with $\beta < \omega^\alpha$. Let H be a definable Abelian subgroup of D^* with $U(H) \geq \omega^\alpha$ whose existence is guaranteed by Theorem 3.25. Let $D_1 = C_D(H)$ and let F be the center of D_1.

We next show that D is finite dimensional over F. Since $H \subseteq F^*$, $U(F) \geq \omega^\alpha$. We claim that if $d_1, \ldots, d_m$ are F-linearly independent elements of D then the U-rank of $d_1F + \cdots + d_mF$ is $U(F)m$ which is at least $\omega^\alpha m$. Thus, m is at most k. Let D_0 be any model containing $\{d_1, \ldots, d_m\}$. For any m-tuple $\bar{a} = \langle a_1, \ldots, a_m \rangle$ from F, $\bar{a}$ is algebraic over $D_0 \cup \{d_1a_1 + \cdots + d_ma_m\}$ and $d_1a_1 + \cdots + d_ma_m$ is algebraic over $D_0 \cup \bar{a}$. So the types of $d_1a_1 + \ldots d_ma_m$ and $\bar{a}$ over D_0 have the same U-rank. But

$$U(t(\bar{a}; D_0)) \leq \sum_i U(t(a_i; D_0)) \leq m \cdot U(F).$$

Moreover, if the a_i are independent realizations of the type of maximal U-rank ($= U(F)$) then equality holds and we conclude the claim.

To conclude the argument we require two facts from algebra. The first is an elementary property of tensor products; the second appears in [Herstein 1975].

i) Let $Z \subseteq F \subseteq D$ be division rings (with Z a field). As vector spaces, $\dim_Z D = \dim_F(D \otimes_Z F)$. (cf. [Lang 1971])

ii) If the division ring D is finite dimensional over the maximal subfield F and $Z = Z(D)$ then $F \otimes_Z D \approx M_n(F)$ for some n. (This follows from the proof of the Corollary to Theorem 4.2.1 in [Herstein 1975].)

From the claim, we have that D is finite dimensional over F. Without loss of generality, we may assume that F is a maximal subfield of D. By ii), we have $D \otimes_Z F \approx M_n(F)$ is finite dimensional over F. But then, by i), and since $D \approx D \otimes_Z Z$, $\dim_Z D$ is finite. Thus Z is infinite. Since Z is a definable subfield of D, Z inherits the superstability of D. This implies Z is algebraically closed by Theorem 3.23. Now any element $a \in D - Z$ satisfies a polynomial over Z and since a centralizes Z, Z and a generate a subfield of D. As Z has no finite algebraic extensions we have the theorem.

3.27 Exercise. Show that for any of the rank functions considered if $a \downarrow_G b$ then $R(t(a; G)) \leq R(t(ab; G))$.

3.28 Historical Notes. This section comes primarily from [Cherlin & Shelah 1980] and [Poizat 1981]. Macintyre [Macintyre 1971a] had earlier proved that every ω-stable field is algebraically closed. Further important developments in this direction can be found in [Berline 1983] and [Berline & Lascar 1986] The account of Cherlin's result that a superstable division ring is commutative, is taken largely from [Berline & Lascar 1986]