CHAPTER 8

THEORY OF GENERAL VARIFOLDS

Here we describe the theory of general varifolds, essentially following W.K. Allard [AW1].

General varifolds in U (U open in $\mathbb{R}^{n+k}$) are simply Radon measures on $G_n(U) = \{(x,S) : x \in U$ and S is an n-dimensional subspace of $\mathbb{R}^{n+k}\}$. One basic motivating point for our interest in such objects is described as follows:

Suppose $\{T_j\}$ is a sequence of integer multiplicity currents (see §27) such that the corresponding integer multiplicity varifolds (as in Chapter 4) are stationary in U (U open in $\mathbb{R}^{n+k}$), and suppose $\partial T_j = 0$ and there is a mass bound $\sup_{j\geq 1} \underline{M}_W(T_j) < \infty \quad \forall\, W \subset\subset U$. By the compactness theorem 27.3 we can assert that $T_{j'} \rightharpoonup T$ for some integer multiplicity T. However it is *not* clear that T is stationary; the chief difficulty is that it is *not* generally true that the corresponding sequence of measures $\mu_{T_{j'}}$ converge to μ_T. Indeed if $\mu_{T_{j'}}$ converges to μ_T (as they would by 34.5 in case the T_j are minimizing in U) then it is not hard to prove that T is stationary in U. This leads one to consider measure theoretic convergence rather than weak convergence of the currents. However if we take a limit (in the sense of Radon measures) of some sub-sequence $\{\mu_{T_{j'}}\}$ of the $\{\mu_{T_j}\}$ then we get merely an abstract Radon measure on U, and first variation of this does not make sense.

To resolve these difficulties, we associate with each T_j a Radon measure V_j on the Grassmaniann $G_n(U)$ ($G_n(U)$ is naturally equipped with a suitable metric - see below); V_j is in fact defined by

$$V_j(A) = \mu_{T_j}(\pi_j(A)) ,$$

where $\pi_j(A)$ denotes $\{x \in U : (x, \langle\vec{T}_j(x)\rangle) \in A\}$ for any subset A of $G_n(U)$. ($\langle\vec{T}_j(x)\rangle$ denotes the n-dimensional subspace determined by $\vec{T}_j(x)$.) One then uses the compactness theorem 4.4 to give $V_{j'} \to V$ for some subsequence $\{j'\}$ and some Radon measure V on $G_n(U)$. It turns out to be possible to define a notion of *stationarity* for such Radon measures (i.e. varifolds) V on $G_n(U)$ and, for example, in the circumstances above V turns out to correspond to a stationary rectifiable varifold (in the sense of Chapter 4). The reader will see that these claims follow easily from the rectifiability and compactness theorems of §42.

§38. BASICS, FIRST RECTIFIABILITY THEOREM

We let $G(n+k,n)$ denote the collection of all n-dimensional subspaces of $\mathbb{R}^{n+k}$, equipped with the metric $\rho(S,T) = |p_S - p_T| = \left(\sum_{i,j=1}^{n+k} (p_S^{ij} - p_T^{ij})^2\right)^{\frac{1}{2}}$, where p_S, p_T denote the orthogonal projections of $\mathbb{R}^{n+k}$ onto S, T respectively, and $p_S^{ij} = e_i \cdot p_S(e_j)$, $p_T^{ij} = e_i \cdot p_T(e_j)$ are the corresponding matrices with respect to the standard orthonormal basis $e_1, \ldots, e_{n+k}$ for $\mathbb{R}^{n+k}$.

For a subset $A \subset \mathbb{R}^{n+k}$ we define

$$G_n(A) = A \times G(n+k,n) ,$$

equipped with the product metric. Of course then $G_n(K)$ is compact for each compact $K \subset \mathbb{R}^{n+k}$. $G_n(\mathbb{R}^{n+k})$ is locally homeomorphic to a Euclidean space of dimension $n+k + nk$.

By an n-*varifold* we mean simply any Radon measure V on $G_n(\mathbb{R}^{n+k})$. By an n-varifold on U (U open in $\mathbb{R}^{n+k}$) we mean any Radon measure V on $G_n(U)$. Given such an n-varifold V on U, there corresponds a Radon measure $\mu = \mu_V$ on U (called the *weight* of V) defined by

$$\mu(A) = V(\pi^{-1}(A)), \; A \subset U,$$

where, here and subsequently, π is the projection $(x,S) \mapsto x$ of $G_n(U)$ onto U. The *mass* $\underline{\underline{M}}(V)$ of V is defined by

$$\underline{\underline{M}}(V) = \mu_V(U) \qquad (= V(G_n(U))).$$

For any Borel subset $A \subset U$ we use the usual terminology $V \llcorner G_n(A)$ to denote the restriction of V to $G_n(A)$; thus

$$(V \llcorner G_n(A))(B) = V(B \cap G_n(A)), \; B \subset G_n(U).$$

Given an n-rectifiable varifold $\underline{\underline{v}}(M,\theta)$ on U (in the sense of Chapter 4) there is a coresponding n-varifold V (also denoted by $\underline{\underline{v}}(M,\theta)$, or simply $\underline{\underline{v}}(M)$ in case $\theta \equiv 1$ on M), defined by

$$V(A) = \mu(\pi(TM \cap A)), \; A \subset G_n(U),$$

where $\mu = \mathcal{H}^n \llcorner \theta$ and $TM = \{(x,T_xM) : x \in M_*\}$, with M_* the set of $x \in M$ such that M has an approximate tangent space T_xM with respect to θ at x in the sense of 11.4. Evidently V, so defined, has weight measure $\mu_V = \mathcal{H}^n \llcorner \theta = \mu$.

The question of when a general n-varifold actually corresponds to an n-rectifiable varifold in this way is satisfactorily answered in the next theorem. Before stating this we need a definition:

38.1 DEFINITION Given $T \in G(n+k,n)$, $x \in U$, and $\theta \in (0,\infty)$, we say that an n-varifold V on U has tangent space T with multiplicity θ at x if

$$(*) \qquad \lim_{\lambda \downarrow 0} V_{x,\lambda} = \theta \underline{\underline{v}}(T) \, ,$$

where the limit is in the usual sense of Radon measures on $G_n(\mathbb{R}^{n+k})$. In (*) we use the notation that $V_{x,\lambda}$ is the n-varifold defined by

$$V_{x,\lambda}(A) = \lambda^{-n} V(\{(\lambda y + x, S) : (y,S) \in A\} \cap G_n(U))$$

for $A \subset G_n(\mathbb{R}^{n+k})$.

38.2 REMARK Note that 38.1(*) implies that the weight measure μ_V has approximate tangent space T with multiplicity θ at x in the sense of 11.8.

38.3 THEOREM (First Rectifiability Theorem)

Suppose V *is an* n-*varifold on* U *which has a tangent space* T_x *with multiplicity* $\theta(x) \in (0,\infty)$ *for* μ_V-a.e. $x \in U$. *Then* V *is* n-*rectifiable: in fact* $M \equiv \{x \in U : T_x, \theta(x) \text{ exist}\}$ *is* $\mathcal{H}^n$-*measurable, countably* n-*rectifiable,* θ *is locally* $\mathcal{H}^n$-*integrable on* M, *and* $V = \underline{\underline{v}}(M,\theta)$.

In the proof of 38.3 (and also subsequently) we shall need the following technical lemma:

38.4 LEMMA *Let* V *be any* n-*varifold on* U. *Then for* μ_V-a.e. $x \in U$ *there is a Radon measure* $\eta_V^{(x)}$ *on* $G(n+k,n)$ *such that, for any continuous* β *on* $G(n+k,n)$,

$$\int_{G(n+k,n)} \beta(S)\, d\eta_V^{(x)}(S) = \lim_{\rho \downarrow 0} \frac{\int_{G_n(B_\rho(x))} \beta(S)\, dV(y,S)}{\mu_V(B_\rho(x))} \, .$$

Furthermore for any Borel set $A \subset U$,

$$\int_{G_n(A)} \beta(S)\, dV(x,S) = \int_A \int_{G(n+k,n)} \beta(S)\, d\eta_V^{(x)}(S)\, d\mu_V(x)$$

provided $\beta \geq 0$.

Proof The proof is a simple consequence of the differentiation theory for Radon measures and the separability of $K(X,\mathbb{R})$ (notation as in §4) for compact separable metric spaces X . Specifically, write $K = K(G(n+k,n),\mathbb{R})$, $K^+ = \{\beta \in K : \beta \geq 0\}$, and let $\beta_1, \beta_2, \ldots \in K^+$ be dense in K^+ . By the differentiation theorem 4.7 we know that (since there is a Radon measure γ_j on $\mathbb{R}^{n+k}$ characterized by $\gamma_j(B) = \int_{G_n(B)} \beta_j(S)\, dV(y,S)$ for Borel sets $B \subset \mathbb{R}^{n+k}$)

$$e(x,j) = \lim_{\rho \downarrow 0} \frac{\int_{G_n(B_\rho(x))} \beta_j(S)\, dV(y,S)}{\mu_V(B_\rho(x))} \tag{1}$$

exists for each $x \in \mathbb{R}^{n+k} \sim Z_j$, where Z_j is a Borel set with $\mu_V(Z_j) = 0$, and $e(x,j)$ is a μ_V-measurable function of x , with

$$\int_A e(x,j)\, d\mu_V(x) = \int_{G_n(A)} \beta_j(S)\, dV(y,S) \tag{2}$$

for any Borel set $A \subset \mathbb{R}^{n+k}$.

Now let $\varepsilon > 0$, $\beta \in K^+$, $x \in \mathbb{R}^{n+k} \sim \left(\bigcup_{j=1}^{\infty} Z_j \right)$, and choose β_j such that $\sup|\beta - \beta_j| < \varepsilon$. Then for any $\rho > 0$

$$\left| \frac{\int_{G_n(B_\rho(x))} \beta(S)\, dV(y,S)}{\mu_V(B_\rho(x))} - \frac{\int_{G_n(B_\rho(x))} \beta_j(S)\, dV(y,S)}{\mu_V(B_\rho(x))} \right| \tag{3}$$

$$\leq \varepsilon \frac{V(G_n(B_\rho(x)))}{\mu_V(B_\rho(x))} = \varepsilon ,$$

and hence by (1) we conclude that

$$\tilde{\eta}_V^{(x)}(\beta) \equiv \lim_{\rho\downarrow 0} \frac{\int_{G_n(B_\rho(x))} \beta(S)\,dV(y,S)}{\mu_V(B_\rho(x))}$$

exists for all $\beta \in K^+$ and all $x \in \mathbb{R}^{n+k} \sim \left(\bigcup_{j=1}^{\infty} Z_j\right)$. Of course, since $|\tilde{\eta}_V^{(x)}(\beta)| \le \sup|\beta| \quad \forall\, \beta \in K^+$, by the Riesz representation theorem 4.1 we have $\tilde{\eta}_V^{(x)}(\beta) = \int_{G(n+k,n)} \beta(S)\,d\eta_V^{(x)}(S)$, where $\eta_V^{(x)}$ is the total variation measure associated with $\tilde{\eta}_V^{(x)}$.

Finally the last part of the lemma follows directly from (2), (3) if we keep in mind that $e(x,j)$ in (1) is exactly $\tilde{\eta}_V^{(x)}(\beta_j) = \int_{G(n+k,n)} \beta_j(S)\,d\eta_V^{(x)}(S)$.

We are now able to give the proof of Theorem 38.3.

Proof of Theorem 38.3 As mentioned in Remark 38.2, μ_V has approximate tangent space T_x with multiplicity $\theta(x)$ in the sense of 11.8 for μ_V-a.e. $x \in U$. Hence by Theorem 11.8 we have that M is $\mathcal{H}^n$-measurable countably n-rectifiable, θ is locally $\mathcal{H}^n$-integrable on M and in fact $\mu_V = \mathcal{H}^n \llcorner \theta$ in U (if we set $\theta \equiv 0$ in $U \sim M$).

Now if $x \in M$ is one of the μ_V-almost all points such that $\eta_V^{(x)}$ exists, and if β is a non-negative continuous function on $G(n+k,n)$, then we evidently have $\eta_V^{(x)}(\beta) = \theta(x)\beta(T_x)$ and hence by the second part of 38.4 we have

$$\int_{G_n(A)} \beta(S)\,dV(x,S) = \int_{M\cap A} \beta(T_x)\,d\mu_V(x)$$

for any Borel set $A \subset U$. From the arbitrariness of A and β it then easily follows that

$$\int_{G_n(U)} f(x,S)\,dV(x,S) = \int_M f(x,T_x)\,d\mu_V(x)$$

for any non-negative $f \in C_c(G_n(U))$, and hence we have shown $V = \underline{\underline{v}}(M,\theta)$ as required (because $\mu_V = \mathcal{H}^n \llcorner \theta$ as mentioned above).

§39. FIRST VARIATION

We can make sense of first variation for a general varifold V on U. We first need to discuss *mapping* of such a general n-varifold. Suppose $U, \tilde{U}$ open $\subset \mathbb{R}^{n+k}$ and $f : U \to \tilde{U}$ is C^1 with $f|\operatorname{spt}\mu_V \cap U$ *proper*. Then we define the image varifold $f_\# V$ on $\tilde{U}$ by

39.1 $$f_\# V(A) = \int_{F^{-1}(A)} J_S f(x)\,dV(x,S), \quad A \text{ Borel}, \ A \subset G_n(\tilde{U}),$$

where $F : G_n^+(U) \to G_n(\tilde{U})$ is defined by $F(x,S) = (f(x), df_x(S))$ and where

$$J_S f(x) = (\det((df_x|S)^* \circ (df_x|S)))^{\frac{1}{2}}, \quad (x,S) \in G_n(U).$$

$$G_n^+(U) = \{(x,S) \in G_n(U) : J_S f(x) \neq 0\}.$$

(Notice that this agrees with our previous definition given in §15 in case $V = \underline{\underline{v}}(M,\theta)$.)

Now given any n-varifold V on U we define the *first variation* δV, which is a linear functional on $\mathcal{K}(U,\mathbb{R}^{n+k})$ (notation as in §4) by

$$\delta V(X) = \frac{d}{dt} \underline{\underline{M}}(\phi_{t\#} V \llcorner G_n(K))\Big|_{t=0},$$

where $\{\phi_t\}_{-1<t<1}$ is any 1-parameter family as in 9.1 (and K is as in 9.1(3)). Of course we can compute $\delta V(X)$ explicitly by differentiation under the integral in 39.1. This gives (by *exactly* the computations of §9)

39.2 $$\delta V(X) = \int_{G_n(U)} \operatorname{div}_S X(x)\, dV(x,S) \ ,$$

where, for any $S \in G(n+k,n)$,

$$\operatorname{div}_S X = \sum_{i=1}^{n+k} \nabla_i^S X^i$$

$$= \sum_{i=1}^{n} < \tau_i, D_{\tau_i} X > ,$$

where $\tau_1,\ldots,\tau_n$ is an orthonormal basis for S and $\nabla_i^S = e_i \cdot \nabla^S$, with $\nabla^S f(x) = p_S(\operatorname{grad}_{\mathbb{R}^{n+k}} f(x))$, $f \in C^1(U)$. (p_S is the orthogonal projection of $\mathbb{R}^{n+k}$ onto S .)

By analogy with 16.3 we then say that V is *stationary in* U if $\delta V(X) = 0 \quad \forall\, X \in K(U,\mathbb{R}^{n+k})$.

More generally V is said to have *locally bounded first variation in* U if for each $W \subset\subset U$ there is a constant $c < \infty$ such that $|\delta V(X)| \le c \sup_U |X| \quad \forall\, X \in K(U,\mathbb{R}^{n+k})$ with $\operatorname{spt}|X| \subset W$. Evidently, by the general Riesz representation theorem 4.1, this is equivalent to the requirement that there is a Radon measure $\|\delta V\|$ (the total variation measure of δV) on U characterized by

39.3 $$\|\delta V\|(W) = \sup_{\substack{X \in K(U,\mathbb{R}^{n+k}) \\ |X| \le 1,\, \operatorname{spt}|X| \subset W}} |\delta V(X)| \qquad (< \infty)$$

for any open $W \subset\subset U$. Notice that then by Theorem 4.1 we can write

$$\delta V(X) = \int_{G_n(U)} \operatorname{div}_S X(x)\, dV(x,S) \equiv - \int_U \nu \cdot X d\|\delta V\| \ ,$$

where ν is $\|\delta V\|$-measurable with $|\nu| = 1$ $\|\delta V\|$ - a.e. in U . By the differentiation theory of 4.7 we know furthermore that

$$D_{\mu_V}\|\delta V\|(x) \equiv \lim_{\rho\downarrow 0} \frac{\|\delta V\|(B_\rho(x))}{\mu_V(B_\rho(x))}$$

exists μ_V - a.e. and that (writing $\underline{\underline{H}}(x) = D_{\mu_V}\|\delta V\|(x)\nu(x)$)

$$\int_U \nu\cdot X d\|\delta V\| = \int_U \underline{\underline{H}}\cdot X d\mu_V + \int_U \nu\cdot X d\sigma ,$$

with

$$\sigma = \|\delta V\| \llcorner Z , \quad Z = \{x\in U : D_{\mu_V}\|\delta V\|(x) = +\infty\} . \quad (\mu_V(Z) = 0 .)$$

Thus we can write

39.4
$$\delta V(X) = \int_{G_n(U)} \mathrm{div}_S X(x)\, dV(x,S)$$
$$= -\int_U \underline{\underline{H}}\cdot X d\mu_V - \int_Z \nu\cdot X d\sigma$$

for $X \in K(U,\mathbb{R}^{n+k})$.

By analogy with the classical identity 7.6 we call $\underline{\underline{H}}$ *the generalized mean curvature of* V *,* Z *the generalized boundary of* V *,* σ *the generalized boundary measure of* V *, and* $\nu|Z$ *the generalized unit co-normal of* V .

§40. MONOTONICITY AND CONSEQUENCES

In this section we assume that V is an n-varifold in U with locally bounded first variation in U (as in 39.3).

We first consider a point $x \in U$ such that there is $0 < \rho_0 < \mathrm{dist}(x,\partial U)$ and $\Lambda \geq 0$ with

40.1 $$\|\delta V\|(B_\rho(x)) \le \Lambda\mu_V(B_\rho(x)) \ , \ 0<\rho<\rho_0 \ .$$

Subject to 40.1 we can choose (in 39.2) $X_y = \gamma(r)(y-x)$, $r = |y-x|$, $y \in U$ as in §17 and note that (by essentially the same computation as in §17)

$$\operatorname{div}_S X = n\gamma(r) + r\gamma'(r) \sum_{i,j=1}^{n+k} e_S^{ij} \frac{x^i-y^i}{r} \frac{x^j-y^j}{r} ,$$

where (e_S^{ij}) is the matrix of the orthogonal projection p_S of $\mathbb{R}^{n+k}$ onto the n-dimensional subspace S . We can then take $\gamma(r) = \phi(r/\rho)$ (again as in §17) and, noting that $\sum_{i,j=1}^{n+k} e_S^{ij} \frac{x^i-y^i}{r} \frac{x^j-y^j}{r} = 1 - |p_{S^\perp}(\frac{y-x}{r})|^2$, conclude (Cf. 17.6(1) with $\alpha = 1$) that $e^{\Lambda\rho}\rho^{-n}\mu_V(B_\rho(x))$ is increasing in ρ , $0<\rho<\rho_0$, and, for $0<\sigma\le\rho<\rho_0$,

40.2 $$\Theta^n(\mu_V,x) \le e^{\Lambda\sigma}\omega_n^{-1}\sigma^{-n}\mu_V(B_\sigma(x)) \le e^{\Lambda\rho}\omega_n^{-1}\rho^{-n}\mu_V(B_\rho(x)) - \omega_n^{-1}\int_{G_n(B_\rho(x)\sim B_\sigma(x))} r^{-n-2}|p_{S^\perp}(y-x)|^2 dV(y,S) \ .$$

In fact if $\Lambda = 0$ (so that V is stationary in $B_{\rho_0}(x)$) we get the precise identity

40.3 $$\Theta^n(\mu_V,x) = \omega_n^{-1}\rho^{-n}\mu_V(B_\rho(x)) - \omega_n^{-1}\int_{G_n(B_\rho(x))} r^{-n-2}|p_{S^\perp}(y-x)|^2 dV(y,S) \ ,$$

for $0<\rho<\rho_0$.

Using $X_y = h(y)\gamma(r)(y-x)$ $(r = |y-x|)$ in 39.2 we also deduce the following analogue of 18.1:

40.4 $$\frac{d}{d\rho}(\rho^{-n}\tilde{I}(\rho)) = \rho^{-n}\frac{d}{d\rho}\int |p_{S^\perp}(y-x)/r|^2\phi(r/\rho)h(y)\,dV(y,S) + \rho^{-n-1}\left(\delta V(X) + \int (y-x)\cdot\nabla^S h(y)\phi(r/\rho)\,dV(y,S)\right) ,$$

where $\tilde{I}(\rho) = \int \phi(r/\rho)\,h\,d\mu_V$.

40.5 LEMMA *Suppose* V *has locally bounded first variation in* U *. Then, for* μ_V - a.e. $x \in U$ *,* $\Theta^n(\mu_V, x)$ *exists and is real-valued; in fact* $\Theta^n(\mu_V, x)$ *exists whenever there is a constant* $\Lambda(x) < \infty$ *such that*

$$(*) \qquad \|\delta V\|(B_\rho(x)) \le \Lambda(x)\mu_V(B_\rho(x)) , \quad 0 < \rho < \frac{1}{2}\operatorname{dist}(x, \partial U) .$$

(Such a constant $\Lambda(x)$ exists for μ_V - a.e. $x \in U$ by virtue of the differentiation theorem 4.7.)

Furthermore $\Theta^n(\mu_V, x)$ *is a* μ_V*-measurable function of* x .

Proof The first part of the lemma follows directly from the monotonicity formula 40.2. The μ_V-measurability of $\Theta^n(\mu_V, \cdot)$ follows from the fact that $\mu_V(\bar{B}_\rho(x)) \ge \limsup_{y \to x} \mu_V(\bar{B}_\rho(y))$, which guarantees that $\mu_V(B_\rho(x))/(\omega_n\rho^n)$ is Borel measurable and hence μ_V-measurable for each fixed ρ . Since $\Theta^n(\mu_V, x) = \lim_{\rho \downarrow 0} (\omega_n\rho^n)^{-1}\mu_V(B_\rho(x))$ for μ_V - a.e. $x \in U$, we then have μ_V-measurability of $\Theta^n(\mu_V, \cdot)$ as claimed.

40.6 THEOREM (Semi-continuity of Θ^n under varifold convergence.)

Suppose $V_i \to V$ *(as Radon measures in* $G_n(U)$*) and* $\Theta^n(V_i, y) \ge 1$ *except on a set* $B_i \subset U$ *with* $\mu_{V_i}(B_i \cap W) \to 0$ *for each* $W \subset\subset U$ *, and suppose that each* V_i *has locally bounded first variation in* U *with* $\liminf \|\delta V_i\|(W) < \infty$ *for each* $W \subset\subset U$ *. Then* $\|\delta V\|(W) \le \liminf \|\delta V_i\|(W)$ $\forall\, W \subset\subset U$ *and* $\Theta^n(\mu_V, y) \ge 1$ μ_V *- a.e. in* U .

40.7 REMARKS

(1) The fact that $\|\delta V\|(W) \le \liminf \|\delta V_i\|(W)$ is a trivial consequence of the definitions of $\|\delta V\|$, $\|\delta V_i\|$ and the fact that $V_i \to V$, so we have only to prove the last conclusion that $\Theta^n(\mu_V, y) \ge 1$ μ_V - a.e.

(2) The proof that $\Theta^n(\mu_V, y) \ge 1$ μ_V - a.e. to be given below is slightly complicated; the reader should note that if $\|\delta V\| \le \Lambda\mu_V$ in U

(i.e. if V has generalized boundary measure $\sigma = 0$ and bounded $\underline{\underline{H}}$ - see 39.4), then the result is a very easy consequence of the monotonicity formula 40.2.

Proof of Theorem 40.6 Set $\mu_i = \mu_{V_i}$, $\mu = \mu_V$, and take any $W \subset\subset U$ and $\rho_0 \in (0, \operatorname{dist}(W, \partial U))$. For $i,j \ge 1$, consider the set $A_{i,j}$ consisting of all points $y \in W \sim B_i$ such that

(1) $$\|\delta V_i\|(\bar{B}_\rho(y)) \le j\mu_i(\bar{B}_\rho(y)) \;, \quad 0 < \rho < \rho_0 \;,$$

and let $B_{i,j} = W \sim A_{i,j}$. Then if $x \in B_{i,j}$ we have *either* $x \in B_i \cap W$ *or*

(2) $$\mu_i(\bar{B}_\sigma(x)) \le j^{-1}\|\delta V_i\|(\bar{B}_\sigma(x)) \quad \text{for some } \sigma \in (0, \rho_0) \;.$$

Let $\mathcal{B}$ be the collection of balls $\bar{B}_\sigma(x)$ with $x \in B_{i,j}$, $\sigma \in (0,\rho_0)$, and with (2) holding. By the Besicovitch covering lemma 4.6 there are families $\mathcal{B}_1, \ldots, \mathcal{B}_N \subset \mathcal{B}$ with $N = N(n+k)$, with $B_{i,j} \sim B_i \subset \bigcup_{\ell=1}^{N}\left(\bigcup_{B \in \mathcal{B}_\ell} B\right)$ and with each $\mathcal{B}_\ell$ a pairwise disjoint family. Hence if we sum in (2) over balls $B \in \bigcup_{\ell=1}^{N} \mathcal{B}_\ell$, we get

$$\mu_i(B_{i,j}) \le Nj^{-1}\|\delta V_i\|(\tilde{W}) + \mu_i(B_i \cap W)$$

($\tilde{W} = \{x \in U : \operatorname{dist}(x,W) < \rho_0\}$), so

(3) $$\mu_i(B_{i,j}) \le cj^{-1} + \mu_i(B_i \cap W) \;,$$

with c independent of i,j. In particular for each $i,j \ge 1$

(4) $$\mu\left(\text{interior}\left(\bigcap_{\ell=i}^{\infty} B_{\ell,j}\right)\right) \le \liminf_{q\to\infty} \mu_q\left(\text{interior}\left(\bigcap_{\ell=i}^{\infty} B_{\ell,j}\right)\right) \le cj^{-1} \;,$$

since $\mu_q(B_q \cap W) \to 0$ as $q \to \infty$.

Now let $j \in \{1,2,\ldots\}$ and consider the possibility that there is a point $x \in W$ such that $x \in W \sim \text{interior}\left(\bigcap_{q=i}^{\infty} B_{q,j}\right)$ for *each* $i = 1,2,\ldots$. Then we could select, for each $i = 1,2,\ldots$, $y_i \in W \sim \bigcap_{q=i}^{\infty} B_{q,j}$ with $|y_i - x| < 1/i$. Thus there are sequences $y_i \to x$ and $q_i \to \infty$ such that $y_i \notin B_{q_i,j}$ for each $i = 1,2,\ldots$. Then $y_i \in A_{q_i,j}$ and hence (by (1))

$$\|\delta V_{q_i}\|(\bar{B}_\rho(y_i)) \le j\mu_{q_i}(\bar{B}_\rho(y_i)) , \quad 0 < \rho < \rho_0 ,$$

for all $i = 1,2,\ldots$. Then by the monotonicity formula 40.2 (with $\Lambda = j$) together with the fact that $\Theta^n(\mu_{q_i}, y_i) \ge 1$ we have

$$\mu_{q_i}(\bar{B}_\rho(y_i)) \ge e^{-j\rho}\omega_n \rho^n , \quad 0 < \rho < \rho_0 , \; i = 1,2,\ldots ,$$

and hence

$$\mu(\bar{B}_\rho(x)) \ge e^{-j\rho}\omega_n \rho^n , \quad 0 < \rho < \rho_0 ,$$

so that $\Theta^n(\mu,x) \ge 1$ for such an x . Thus we have proved $\Theta^n(\mu,x) \ge 1$ for each x with $x \in W \sim \left(\bigcup_{i=1}^{\infty} \text{interior}\left(\bigcap_{\ell=i}^{\infty} B_{\ell,j}\right)\right)$ for some $j \in \{1,2,\ldots\}$. That is

$$(5) \qquad \Theta^n(\mu,x) \ge 1 \quad \forall\, x \in W \sim \left(\bigcap_{j=1}^{\infty} \bigcup_{i=1}^{\infty} \text{interior}\left(\bigcap_{\ell=i}^{\infty} B_{\ell,j}\right)\right) .$$

However

$$\begin{aligned}
\mu\left(\bigcap_{j=1}^{\infty} \bigcup_{i=1}^{\infty} \text{interior}\left(\bigcap_{\ell=i}^{\infty} B_{\ell,j}\right)\right) &\le \mu\left(\bigcup_{i=1}^{\infty} \text{interior}\left(\bigcap_{\ell=i}^{\infty} B_{\ell,j}\right)\right) \qquad \forall\, j \ge 1 \\
&= \lim_{i\to\infty} \mu\left(\text{interior}\left(\bigcap_{\ell=i}^{\infty} B_{\ell,j}\right)\right) \\
&\le c j^{-1} \quad \text{by (4)} ,
\end{aligned}$$

so $\mu\left(\bigcap_{j=1}^{\infty} \bigcup_{i=1}^{\infty} \text{interior}\left(\bigcap_{\ell=i}^{\infty} B_{\ell,j}\right)\right) = 0$ and the theorem is established (by (5)).

§41. THE CONSTANCY THEOREM

41.1 THEOREM (Constancy Theorem)

Suppose V *is an* n*-varifold in* U *,* V *is stationary in* U *, and* $U \cap \operatorname{spt} \mu_V \subset M$ *, where* M *is a connected* n*-dimensional* C^2 *submanifold of* $\mathbb{R}^{n+k}$ *. Then* $V = \theta_0 \underline{\underline{v}}(M)$ *for some constant* θ_0 *.*

41.2 REMARKS

(1) Notice in particular this implies $(\bar{M} \sim M) \cap U = \emptyset$ (if $V \neq 0$) ; this is not *a-priori* obvious from the assumptions of the theorem.

(2) J. Duggan in his PhD thesis [DJ] has recently extended 41.1 to the case when M is merely Lipschitz.

(3) The reader will see that, with only minor modifications to the proof to be given below, the theorem continues to hold if N is an embedded $(n+k_1)$-dimensional C^2 submanifold of $\mathbb{R}^{n+k}$ and if V is stationary in $U \cap N$ in the sense that $\delta V(X) = 0$ $\forall\, X \in \mathcal{K}(U;\mathbb{R}^{n+k})$ with $X_x \in T_x N$ $\forall\, x \in N$, *provided we are given* $\operatorname{spt} V \subset \{(x,S) : x \in N \text{ and } S \subset T_x N\}$. (This last is equivalent to $\operatorname{spt} \mu_V \subset N$ and $p_{\#} V = V$, where $p : U \to U \cap N$ coincides with the nearest point projection onto $U \cap N$ in some neighbourhood of $U \cap N$.)

Proof of 41.1 We first want to argue that $V = \underline{\underline{v}}(M,\theta)$ for some positive locally $\mathcal{H}^n$-integrable function θ on M .

To do this first take any $f \in C_c^2(U)$ with $M \subset \{x \in U : f(x) = 0\}$ and note that by 39.2

$$\text{(1)} \qquad \delta V(f \operatorname{grad} f) = \int |p_S(\operatorname{grad} f)|^2 dV(x,S) \; ,$$

because (using notation as in 39.2)

$$\operatorname{div}_S(f \operatorname{grad} f) = \nabla^S f \cdot \operatorname{grad} f + f \operatorname{div}_S \operatorname{grad} f$$

$$= |p_S(\operatorname{grad} f)|^2 \quad \text{on } M ,$$

where we used $f \equiv 0$ on M. Since $\delta V = 0$, we conclude from (1) that

(2) $$p_S(\operatorname{grad} f(x)) = 0 \quad \text{for all} \quad (x,S) \in \operatorname{spt} V .$$

Now let $\xi \in M$ be arbitrary. We can find an open $W \subset U$ with $\xi \in W$ and such that there are $C^2_c(U)$ functions $f_1,\ldots,f_k$ with $M \subset \bigcap_{j=1}^{k} \{x : f_j(x) = 0\}$ and with $(T_xM)^\perp$ being exactly the space spanned by $\operatorname{grad} f_1(x),\ldots,\operatorname{grad} f_k(x)$ for each $x \in M \cap W$. (One easily checks that such W and $f_1,\ldots,f_k$ exist.) Then (2) implies that

(3) $$p_S((T_xM)^\perp) = 0 \quad \text{for all} \quad (x,S) \in G_n(W) \cap \operatorname{spt} V .$$

But (3) says exactly that $S = T_xM$ for all $(x,S) \in G_n(W) \cap \operatorname{spt} V$, so that (since ξ was an arbitrary point of M), we have

(4) $$\int f(x,S)\,dV(x,S) = \int_{M\cap U} f(x,T_xM)\,d\mu_V(x) \ , \ f \in C_c(G_n(U)) .$$

On the other hand we know from monotonicity 40.2 that $\theta(x) \equiv \Theta^n(\mu_V,x)$ exists for all $x \in M \cap U$, and hence (since $\Theta^n(\mathcal{H}^n \llcorner M,x) = 1$ for each $x \in M$, by smoothness of M), we can use the differentiation theorem 4.7 to conclude from (4) that in fact

(5) $$\int f(x,S)\,dV(x,S) = \int_{M\cap U} f(x,T_xM)\,\theta(x)\,d\mathcal{H}^n(x) \ , \ f \in C_c(G_n(U)) ,$$

(so that $V = \underline{\underline{v}}(M,\theta)$ as required).

It thus remains only to prove that $\theta = \text{const.}$ on $M \cap U$. Since M is C^2 we can take $X \in \mathcal{K}(U,\mathbb{R}^{n+k})$ such that $X_x \in T_xM \quad \forall\, x \in M \cap U$. Then by (5) and 39.2 $\delta V(X) = 0$ is just the statement that $\int_{M\cap U} \operatorname{div} X \theta d\mathcal{H}^n = 0$, where div X is the classical divergence of $X|M$ in the usual sense of differential geometry. Using local coordinates (in some neighbourhood $\tilde{U} \subset \mathbb{R}^n$) this tells us that

$$\int_{\tilde{U}} \sum_{i=1}^{n} \frac{\partial X_i}{\partial x_i} \tilde{\theta}\, dL^n = 0 \quad \text{if} \quad X_i \in C_c^1(\tilde{U}) \ , \ i = 1,\dots,n \ ,$$

where $\tilde{\theta}$ is θ expressed in terms of the local coordinates. In particular

$$\int_{\tilde{U}} \frac{\partial \zeta}{\partial x_i} \tilde{\theta}\, dL^n = 0 \qquad \forall\ \zeta \in C_c(U) \ , \ i = 1,\dots,n$$

and it is then standard that $\tilde{\theta}$ = constant in $\tilde{U}$. Hence (since M is connected) θ is constant in M.

§42. VARIFOLD TANGENTS AND RECTIFIABILITY THEOREM

Let V be an n-varifold in U and let x be any point of U such that

42.1 $$\Theta^n(\mu_V, x) = \theta_0 \in (0,\infty) \quad \text{and} \quad \lim_{\rho \downarrow 0} \rho^{1-n} \|\delta V\| (B_\rho(x)) = 0 \ .$$

By definition of δV (in §39) and the compactness theorem 4.4 for Radon measures, we can select a sequence $\lambda_j \downarrow 0$ such that $\eta_{x,\lambda_j \#}V$ converges (in the sense of Radon measures) to a varifold C such that

$$C \text{ is stationary in } \mathbb{R}^{n+k}$$

and

(*) $$\frac{\mu_C(B_\rho(x))}{\omega_n \rho^n} \equiv \theta_0 \qquad \forall\ \rho > 0 \ .$$

Since $\delta C = 0$ we can use (*) together with the monotonicity formula 40.3 to conclude

$$\int_{G_n(B_\rho(0))} \frac{|p_{S^\perp}(x)|^2}{|x|^{n+2}}\, dC(x,S) = 0 \qquad \forall\, \rho > 0\,,$$

so that $p_{S^\perp}(x) = 0$ for C-a.e. $(x,S) \in G_n(\mathbb{R}^{n+k})$, and hence $p_{S^\perp}(x) = 0$ for *all* $(x,S) \in \operatorname{spt} C$ by continuity of $p_{S^\perp}(x)$ in (x,S). Then by the same argument as in the proof of 19.3, except that we use 40.4 in place of 18.1, we deduce that μ_C satisfies

42.2 $$\lambda^{-n}\mu_C(\eta_{0,\lambda}(A)) = \mu_C(A)\,, \quad A \subset \mathbb{R}^{n+k}\,, \ \lambda > 0\,.$$

We would *like* to prove the stronger result $\eta_{0,\lambda\#}C = C$ (which of course implies 42.2), but we are only able to do this in case $\Theta^n(\mu_C,x) > 0$ for μ_C-a.e. x (see 42.6 below). Whether or not $\eta_{0,\lambda\#}C = C$ without the additional hypothesis on $\Theta^n(\mu_C,\cdot)$ seems to be an open question.

42.3 DEFINITION Given V and x as in 42.1 we let $\operatorname{Var\,Tan}(V,x)$ ("the varifold tangent of V at x") be the collection of all $C = \lim \eta_{x,\lambda_j\#}V$ obtained as described above.

Notice that by the above discussion any $C \in \operatorname{Var\,Tan}(V,x)$ is stationary in $\mathbb{R}^{n+k}$ and satisfies 42.2.

The following *rectifiability theorem* is a central part of the theory of n-varifolds with locally bounded first variation.

42.4 THEOREM *Suppose* V *has locally bounded first variation in* U *and* $\Theta^n(\mu_V,x) > 0$ *for* μ_V-a.e. $x \in U$. *Then* V *is an* n-*rectifiable varifold.* (Thus $V = \underline{\underline{v}}(M,\theta)$, with M an $\mathcal{H}^n$-measurable countably n-rectifiable subset of U and θ a non-negative locally $\mathcal{H}^n$-integrable function on U.)

42.5 REMARK We are going to use Theorem 38.3. In fact we show that V has a tangent plane (in the sense of 38.1) at any point x where

(i) $\Theta^n(\mu_V,x) > 0$, (ii) $\eta_V^{(x)}$ (as in Lemma 38.4) exists, (iii) $\Theta^n(\mu_V,\cdot)$ is μ_V-approximately continuous at x , and (iv) $\|\delta V\|(B_\rho(x)) \le \Lambda(x)\mu_V(B_\rho(x))$ for $0<\rho<\rho_0 = \min\{1,\mathrm{dist}(x,\partial U)\}$. Since conditions (i)-(iv) all hold μ_V- a.e. in U (notice that (iii) holds μ_V- a.e. by virtue of the μ_V-measurability of $\Theta^n(\mu_V,\cdot)$ proved in 40.5), the required rectifiability of V will then follow from 38.3.

Before beginning the proof of 42.2 we give the following important corollary.

42.6 COROLLARY *Suppose* $x \in U$, 42.1 *holds, and* $\lim_{\rho\downarrow 0} \rho^{-n}\mu_V(\{y\in B_\rho(x) : \Theta^n(\mu_V,y) < 1\}) = 0$. *If* $C \in \mathrm{Var\,Tan}(V,x)$, *then* C *is rectifiable and*

$$(*) \qquad \eta_{0,\lambda\#}C = C \qquad \forall\ \lambda > 0 .$$

Proof From the hypothesis $\rho^{-n}\mu_V(\{y\in B_\rho(x) : \Theta^n(\mu_V,y) < 1\}) \to 0$ and the semi-continuity theorem 40.6, we have $\Theta^n(\mu_C,y) \ge 1$ for μ_C- a.e. $y \in \mathbb{R}^{n+k}$. Hence by Theorem 42.4 we have that C is n-rectifiable. On the other hand, since $\Theta^n(\mu_C,y) = \Theta^n(\mu_C,\lambda y)$ $\forall\ \lambda > 0$ (by 42.2), we can write $C = \underline{\underline{v}}(M,\theta)$ with $\eta_{0,\lambda}(M) = M$ $\forall\ \lambda > 0$ and $\theta(\lambda y) = \theta(y)$ $\forall\ \lambda > 0$, $y \in \mathbb{R}^{n+k}$. (Viz. simply set $\theta(y) = \Theta^n(\mu_C,y)$ and $M = \{y\in\mathbb{R}^{n+k} : \theta(y) > 0\}$.) It then trivially follows that $y \in T_yM$ whenever the approximate tangent space T_yM exists, and hence $\eta_{0,\lambda\#}C = C$ as required.

Proof of Theorem 42.2 Let x be as in 42.5(i)-(iv) and take $C \in \mathrm{Var\,Tan}(V,x)$. (We know $\mathrm{Var\,Tan}(V,x) \neq \emptyset$ because 42.5(i),(iv) imply 42.1.) Then C is stationary in $\mathbb{R}^{n+k}$ and

$$(1) \qquad \frac{\mu_C(B_\rho(0))}{\omega_n\rho^n} \equiv \theta_0 \qquad \forall\ \rho > 0 \qquad (\theta_0 = \Theta^n(\mu_V,x)) .$$

Also for any $y \in \mathbb{R}^{n+k}$ (using (1) and the monotonicity formula 40.2)

$$\frac{\mu_C(B_\rho(y))}{\omega_n\rho^n} \le \frac{\mu_C(B_R(y))}{\omega_n R^n}$$

$$\le \frac{\mu_C(B_{R+|y|}(0))}{\omega_n(R+|y|)^n}\,(1+|y|/R)^n$$

$$= \theta_0(1+|y|/R)^n \to \theta_0 \quad \text{as} \quad R \uparrow \infty .$$

That is (again using the monotonicity formula 40.2),

(2) $$\Theta^n(\mu_C,y) \le \frac{\mu_C(B_\rho(y))}{\omega_n\rho^n} \le \theta_0 \qquad \forall\, y \in \mathbb{R}^{n+k},\ \rho > 0 .$$

Now let $V_j = \eta_{x,\lambda_j\#}V$, where $\lambda_j \downarrow 0$ is such that $\lim \eta_{x,\lambda_j\#}V = C$ and where we are still assuming x is as in 42.5(i)-(iv).

From 42.5(iii) we have (with $\varepsilon(\rho) \downarrow 0$ as $\rho \downarrow 0$)

(3) $$\Theta^n(\mu_V,y) \ge \theta_0 - \varepsilon(\rho) , \quad y \in G \cap B_\rho(x) ,$$

where $G \subset U$ is such that

(4) $$\mu_V(B_\rho(x) \sim G) \le \varepsilon(\rho)\rho^n , \quad \rho \text{ sufficiently small.}$$

Taking $\rho = \lambda_j$ we see that (3), (4) imply

(3)' $$\Theta^n(\mu_{V_j},y) \le \theta_0 - \varepsilon_j , \quad y \in G_j \cap B_1(0)$$

with G_j such that

(4)' $$\mu_{V_j}(B_1(0) \sim G_j) \le \varepsilon_j ,$$

where $\varepsilon_j \to 0$ as $j \to \infty$. Thus, using (3)', (4)' and the semicontinuity result of 40.6, we obtain

(5) $$\Theta^n(\mu_C, y) \geq \theta_0 \quad \text{for} \quad \mu_C\text{-a.e.} \quad y \in \mathbb{R}^{n+k}$$

(and hence for *every* $y \in \operatorname{spt} \mu_C$ by 40.3) . Then by combining (2) and (5) we have

(6) $$\Theta^n(\mu_C, y) \equiv \theta_0 \equiv \frac{\mu_C(B_\rho(y))}{\omega_n \rho^n} \qquad \forall\, y \in \operatorname{spt} \mu_C \,,\ \rho > 0 \,.$$

Then by the monotonicity formula 40.3 (with $V = C$), we have

$$p_{S^\perp}(x-y) = 0 \quad \text{for} \quad C\text{-a.e.} \ (x,S) \in G_n(\mathbb{R}^{n+k}) \,.$$

Thus (using the continuity of $p_{S^\perp}(x-y)$ in (x,S)) we have

(7) $$x-y \in S \qquad \forall\, y \in \operatorname{spt} \mu_C \ \ \textit{and} \ \ \forall (x,S) \in \operatorname{spt} C \,.$$

In particular, choosing T such that $(0,T) \in \operatorname{spt} C$ (such T exists because $0 \in \operatorname{spt} \mu_C = \pi(\operatorname{spt} C)$) , (7) implies $y \in T \quad \forall\, y \in \operatorname{spt} \mu_C$. Thus $\operatorname{spt} \mu_C \subset T$, and hence $C = \theta_0 \underline{\underline{v}}(T)$ by the constancy theorem 41.1.

Thus we have shown that, for $x \in U$ such that 42.5(i), (iii), (iv) hold, each element of $\operatorname{Var\,Tan}(V,x)$ has the form $\theta_0 \underline{\underline{v}}(T)$, where T is an n-dimensional subspace of $\mathbb{R}^{n+k}$. On the other hand, since we are assuming (42.5(ii)) that $\eta_V^{(x)}$ exists, it follows that for continuous β on $G(n+k,n)$

(8) $$\lim_{\rho \downarrow 0} \frac{\int_{G_n(B_\rho(x))} \beta(S)\, dV(y,S)}{\mu_V(B_\rho(x))} = \int_{G(n+k,n)} \beta(S)\, d\eta_V^{(x)}(S) \,.$$

Now let $\theta_0 \underline{\underline{v}}(T)$ be any such element of $\operatorname{Var\,Tan}(V,x)$ and select $\lambda_j \downarrow 0$ so that $\lim \eta_{x,\lambda_j \#} V = \theta_0 \underline{\underline{v}}(T)$. Then in particular

$$\lim_{j \to \infty} \frac{\int_{G_n(B_1(0))} \beta(S)\, dV_j(y,S)}{\mu_{V_j}(B_1(0))} = \beta(T) \,,$$

and hence (8) gives

$$\beta(T) = \int_{G(n+k,n)} \beta(S)\, d\eta_V^{(x)}(S) ,$$

thus showing that $\theta_0 \underline{\underline{v}}(T)$ is the *unique* element of Var Tan(V,x). Thus $\lim_{\lambda \downarrow 0} \eta_{x,\lambda\#} V = \theta_0 \underline{\underline{v}}(T)$, so that T is the tangent space for V at x in the sense of 38.1. This completes the proof.

The following *compactness theorem* for rectifiable varifolds is now a direct consequence of the rectifiability theorem 42.4, the semi-continuity theorem 40.6, and the compactness theorem 4.4 for Radon measures, and its proof is left to the reader.

42.7 THEOREM *Suppose* $\{V_j\}$ *is a sequence of rectifiable* n-*varifolds in* U *which are locally of bounded first variation in* U ,

$$\sup_{j\geq 1} (\mu_{V_j}(W) + \|\delta V_j\|(W)) < \infty \qquad \forall\, W \subset\subset U ,$$

and $\Theta^n(\mu_{V_j}, x) \geq 1$ *on* $U \sim A_j$, *where* $\mu_{V_j}(A_j \cap W) \to 0$ *as* $j \to \infty$ $\forall\, W \subset\subset U$.

Then there is a subsequence $\{V_{j'}\}$ *and a rectifiable varifold* V *of locally bounded first variation in* U , *such that* $V_{j'} \to V$ *(in the sense of Radon measures on* $G_n(U)$) , $\Theta^n(\mu_V, x) \geq 1$ *for* μ_V-a.e. $x \in U$, *and* $\|\delta V\|(W) \leq \liminf_{j\to\infty} \|\delta V_j\|(W)$ *for each* $W \subset\subset U$.

42.8 REMARK An important additional result (also due to Allard [AW1]) is the *integral compactness theorem,* which asserts that if all the V_j in the above theorem are integer multiplicity, then V is also integer multiplicity. (Notice that in this case the hypothesis $\Theta^n(\mu_{V_j}, x) \geq 1$ on $U \sim A_j$ is automatically satisfied with an A_j such that $\mu_{V_j}(A_j) = 0$.)

Proof that V is integer multiplicity if the V_i are:

Let $W \subset\subset U$. We first assert that for μ_V-a.e. $x \in W$ there exists c (depending on x) such that

$$\text{(1)} \qquad \liminf \|\delta V_i\|(\bar{B}_\rho(x)) \le c\mu_V(\bar{B}_\rho(x)) \ , \ \rho < \min\{1, \operatorname{dist}(x, \partial U)\} \ .$$

Indeed otherwise $\exists$ a set $A \subset W$ with $\mu_V(A) > 0$ such that for each $j \ge 1$ and each $x \in A$ there are $\rho_x > 0$, $i_x \ge 1$ such that $\bar{B}_{\rho_x}(x) \subset W$ and

$$\mu_V(\bar{B}_{\rho_x}(x)) \le j^{-1}\|\delta V_i\|(\bar{B}_{\rho_x}(x)) \ , \ i \ge i_x \ .$$

By the Besicovitch covering lemma 4.6 we then have

$$\mu_V(A_i) \le cj^{-1}\|\delta V_\ell\|(W) \ , \ \ell \ge i \ ,$$

where $A_i = \{x \in A : i_x \le i\}$. Thus

$$\mu_V(A_i) \le cj^{-1} \limsup_{\ell\to\infty} \|\delta V_\ell\|(W) \ ,$$

and hence since $A_i \uparrow A$ as $i \uparrow \infty$ we have

$$\mu_V(A) \le cj^{-1}$$

for some c $(< \infty)$ independent of j. That is, $\mu_V(A) = 0$, a contradiction, and hence (1) holds. Since $\Theta^n(\mu_V, x)$ exists μ_V-a.e. $x \in U$, we in fact have from (1) that for μ_V-a.e. $x \in U$ there is a $c = c(x)$ such that

$$\text{(2)} \qquad \liminf \|\delta V_i\|(B_\rho(x)) \le c\rho^n \ , \ 0 < \rho < \min\{1, \operatorname{dist}(x, \partial U)\} \ .$$

Now since $V = \underline{\underline{v}}(M, \theta)$, it is also true that for μ_V-a.e. $\xi \in \operatorname{spt} \mu_V$ we have $\eta_{\xi,\lambda\#}V \to \theta_0 \underline{\underline{v}}(P)$ as $\lambda \downarrow 0$, where $P = T_\xi M$ and $\theta_0 = \theta(\xi)$. Then (because $V_i \to V$, and hence $\eta_{\xi,\lambda\#}V_i \to \eta_{\xi,\lambda\#}V$ for each fixed $\lambda > 0$), it follows that for μ_V-a.e. $\xi \in U$ we can select a sequence $\lambda_i \downarrow 0$ such that, with $W_i = \eta_{\xi,\lambda_i\#}V_i$,

(3) $$W_i \to \theta_0 \underline{\underline{v}}(P)$$

and (by (2)) for each $R > 0$

(4) $$\|\delta W_i\|(B_R(0)) \to 0 \ .$$

We claim that θ_0 must be an integer for any such ξ ; in fact for an arbitrary sequence $\{W_i\}$ of integer multiplicity varifolds in $\mathbb{R}^{n+k}$ satisfying (3), (4), we claim that θ_0 always has to be an ineger.

To see this, take (without loss of generality) $P = \mathbb{R}^n \times \{0\}$, let q be orthogonal projection onto $(\mathbb{R}^n \times \{0\})^{\perp}$, and note first that (3) implies

(5) $$p_{\mathbb{R}^n \#}(W_i \llcorner G_n\{x \in \mathbb{R}^{n+k} : |q(x)| < \varepsilon\}) \to \theta_0 \underline{\underline{v}}(\mathbb{R}^n)$$

for each fixed $\varepsilon > 0$. However by the mapping formula for varifolds (§15), we know that (5) says

(5)' $$\underline{\underline{v}}(\mathbb{R}^n, \psi_i) \to \theta_0 \underline{\underline{v}}(\mathbb{R}^n) \ ,$$

where

(6) $$\psi_i(x) = \sum_{y \in p_{\mathbb{R}^n}^{-1}(x) \cap \{z \in \mathbb{R}^{n+k} : |q(z)| < \varepsilon\}} \theta_i(y)$$

(θ_i = multiplicity function of W_i , so that ψ_i has values in $\mathbb{Z} \cap \{\infty\}$) . Notice that (5)' implies in particular that

(7) $$\int_{\mathbb{R}^n} f\psi_i \, d\mathcal{L}^n \to \theta_0 \int_{\mathbb{R}^n} f d\mathcal{L}^n \qquad \forall f \in C_c^0(\mathbb{R}^n) \ .$$

(i.e. measure-theoretic convergence of ψ_i to θ_0 .)

Now we claim that there are sets $A_i \subset B_1(0)$ such that

(8) $$\psi_i(x) \le \theta_0 + \varepsilon_i \quad \forall x \in B_1(0) \sim A_i \ , \ \mathcal{L}^n(A_i) \to 0 \ , \quad \varepsilon_i \downarrow 0 \ ;$$

this will of course (when used in combination with (7)) imply that for any integer $N > \theta_0$, $\max\{\psi_i, N\}$ converges in $L^1(B_1(0))$ to θ_0, and, since $\max\{\psi_i, N\}$ is integer-valued, it then follows that θ_0 is an integer.

On the other hand (8) evidently follows by setting $W = W_i$ in the following lemma, so the proof is complete.

In this lemma, p, q denote orthogonal projection of $\mathbb{R}^{n+k}$ onto $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+k}$ and $\{0\} \times \mathbb{R}^k \subset \mathbb{R}^{n+k}$ respectively.

42.9 LEMMA *For each* $\delta \in (0,1)$, $\Lambda \geq 1$, *there is* $\varepsilon = \varepsilon(\delta, \Lambda, n) \in (0, \delta^2)$ *such that if* W *is an integer multiplicity varifold in* $B_3(0)$ *with*

$$(*) \qquad \mu_W(B_3(0)) \leq \Lambda\,, \quad \|\delta W\|(B_3(0)) < \varepsilon^2\,, \quad \int_{B_3(0)} \|p_S - p\|\, dW(y,S) < \varepsilon^2\,,$$

then there is a set $A \subset B_1^n(0)$ *such that* $L^n(A) < \delta$ *and*, $\forall\, x \in B_1(0) \sim A$,

$$\sum_{y \in p^{-1}(x) \cap \operatorname{spt}\mu_W \cap \{z : |q(z)| < \varepsilon\}} \Theta^n(\mu_W, y) \leq (1+\delta)\, \frac{\mu_W(B_2(x))}{\omega_n 2^n} + \delta\,.$$

42.10 REMARK It suffices to prove that for each fixed N there is $\delta_0 = \delta_0(N) \in (0,1)$ such that if $\delta \in (0, \delta_0)$ then $\exists\ \varepsilon = \varepsilon(n, \Lambda, N, \delta) \in (0, \delta^2)$ such that $(*)$ implies the existence of $A \subset B_1^n(0)$ with $L^n(A) < \delta$ and, for $x \in B_1^n(0) \sim A$ and distinct $y_1, \ldots, y_N \in p^{-1}(x) \cap \operatorname{spt}\mu_W \cap \{z : |q(z)| < \varepsilon\}$,

$$(**) \qquad \sum_{j=1}^{N} \Theta^n(\mu_W, y_j) \leq (1+\delta)\, \frac{\mu_W(B_2(x))}{\omega_n 2^n} + \delta\,.$$

Because this firstly implies an *a-priori* bound, depending only on n, k, Λ, on the number N of possible points y_j, and hence the lemma, as originally stated, then follows. (Notice that of course the validity of the lemma for small δ implies its validity for any larger δ.)

Proof of 42.9 By virtue of the above Remark, we need only prove (**). Let $\mu = \mu_W$, and consider the possibility that $y \in B_1(0)$ satisfies

(1) $$\|\delta W\|(B_\rho(y)) \le \varepsilon\mu(B_\rho(y)) \ , \ 0 < \rho < 1 \ ,$$

(2) $$\int_{B_\rho(y)} \|p_S - p\| dW(z,S) \le \varepsilon\rho^n \ , \ 0 < \rho < 1 \ .$$

Let

$$A_1 = \{y \in B_2(0) \cap \operatorname{spt} W : (1) \text{ fails for some } \rho \in (0,1)\}$$

$$A_2 = \{y \in B_2(0) \cap \operatorname{spt} W : (2) \text{ fails for some } \rho \in (0,1)\}.$$

Evidently if $y \in \operatorname{spt} \mu_W \cap B_2(0) \sim A_1$ then by the monotonicity formula 40.2 we have

(3) $$\frac{\mu(B_\rho(y))}{\omega_n\rho^n} \le e^\varepsilon \frac{\mu(B_1(y))}{\omega_n} \le c \ , \ 0 < \rho < 1 \ ,$$

$(c = c(\Lambda,n))$, while if $y \in A_2 \sim A_1$ we have (using (3))

(4) $$\int_{B_\rho(y)} \|p_S - p\| dW(z,S) \ge \varepsilon\rho_y^n \ge c\varepsilon\mu(B_{\rho_y}(y))$$

for some $\rho_y \in (0,1)$. If $y \in A_1$ then

(5) $$\mu(B_{\rho_y}(y)) \le \varepsilon^{-1}\|\delta W\|(B_{\rho_y}(y))$$

for some $\rho_y \in (0,1)$.

Since then $\{B_{\rho_y}(y)\}_{y\in A_1\cup A_2}$ covers $A_1 \cup A_2$ we deduce from (4), (5) and the Besicovitch covering lemma 4.6 that

$$(6) \qquad \mu(A_1 \cup A_2) \le c\varepsilon^{-1}\left(\int_{B_3(0)} \|p_S - p\| dW(a,S) + \|\delta W\|(B_3(0))\right)$$

$$\le c\varepsilon$$

by the hypotheses on W .

Our aim now is to show (**) holds whenever $x \in B_1^n(0) \sim p(A_1 \cup A_2)$. In view of (6) this will establish the required result (with $A = p(A_1 \cup A_2)$) . So let $x \in B_1^n(0) \sim p(A_1 \cup A_2)$. In view of the monotonicity formula 40.2 it evidently suffices (by translating and changing scale by a factor of 3/2) to assume that $x = 0 \in B_1^n(0) \sim p(A_1 \cup A_2))$. We shall subsequently assume this.

We first want to establish the two formulae, for $y \in B_1(0) \sim (A_1 \cup A_2)$ and $\tau > 0$:

$$(7) \qquad \Theta^n(\mu,y) \le e^{\varepsilon\sigma} \frac{\mu(U_\sigma^{2\tau}(y))}{\omega_n \sigma^n} + c\varepsilon\sigma/\tau , \quad 0 < \sigma < 1 ,$$

and

$$(8) \qquad \frac{\mu(U_\sigma^\tau(y))}{\omega_n \sigma^n} \le e^{\varepsilon\rho} \frac{\mu(U_\rho^{2\tau}(y))}{\omega_n \rho^n} + c\varepsilon\rho/\tau , \quad 0 < \sigma < \rho \le 1 ,$$

where

$$U_\sigma^\tau(y) = B_\sigma(y) \cap \{z \in \mathbb{R}^{n+k}: |q(z-y)| < \tau\} .$$

Indeed these two inequalities follow directly from 40.2 and 40.4. For example to establish (7) we note first that 40.2 gives (7) directly if $\tau \ge \sigma$, while if $\tau < \sigma$ then we first use 40.2 to give $\Theta^n(\mu,y) \le e^{\varepsilon\tau} \frac{\mu(B_\tau(y))}{\omega_n \tau^n}$ and then use 40.4 with h of the form $h(z) = f(|q(z-y)|)$, $f(t) \equiv 1$ for $t < \tau$ and $f(t) \equiv 0$ for $t > 2\tau$.

Since $|\nabla^S f(|q(z-y)|)| \le f'(|q(z-y)|)|p_S - p|$ (Cf. the computation in 19.5) we then deduce (by integrating in 40.4 from τ to σ and using (3))

$$\frac{\mu(B_\tau(y))}{\omega_n \tau^n} \le \frac{\mu(U_\sigma^{2\tau}(y))}{\omega_n \sigma^n} + c\varepsilon\sigma/\tau .$$

(8) is proved by simply integrating in 40.4 from σ to ρ (and using (3)).

Our aim now is to use (7) and (8) to establish

$$(9) \qquad \sum_{j=1}^{N} \frac{\mu(U_\sigma^\tau(y_j))}{\omega_n \sigma^n} \le (1+c\delta^2)\frac{\mu(B_2(0))}{\omega_n 2^n} + c\delta^2$$

with $c = c(n,k,N,\Lambda)$, provided $2\delta^2\sigma \le \tau \le \frac{1}{4}\min_{j\ne\ell}|y_j - y_\ell|$, $y_j \in \operatorname{spt}\mu \cap p^{-1}(0) \cap \{z : |q(z)| < \varepsilon\}$, $0 \notin p(A_1 \cup A_2)$. (In view of (7) this will prove the required result (**) for suitable $\delta_0(N)$.)

We proceed by induction on N . $N = 1$ trivially follows from (8) by noting that $U_\rho^{2\tau}(y_1) \subset B_\rho(y_1)$ (by definition of $U_\rho^{2\tau}(y_1)$) and then using the monotonicity 40.2 together with the fact that $|y_1| < \varepsilon$. Thus assume $N \ge 2$ and that (9) has been established with any $M < N$ in place of N .

Let $y_1,\ldots,y_N$ be as in (9), and choose $\rho \in [\sigma,1)$ such that $\min_{j\ne\ell}|q(y_j) - q(y_\ell)| \left(= \min_{j\ne\ell}|y_j - y_\ell|\right) = 4\delta^2\rho$, and set $\tilde\tau = 2\delta^2\rho$ $(\ge 2\tau)$. Then

$$\begin{aligned}\frac{\mu(U_\sigma^\tau(y_j))}{\sigma^n} &\le \frac{\mu(U_\sigma^{\frac{1}{2}\tilde\tau}(y_j))}{\sigma^n} \\ &\le e^{\varepsilon\rho}\frac{\mu(U_\rho^{\tilde\tau}(y_j))}{\rho^n} + c\varepsilon \quad \text{(by (8))} ,\end{aligned}$$

$c = c(n,k,\delta)$. Now since $\tilde\tau = \frac{1}{2}\min_{j\ne\ell}|q(y_j) - q(y_\ell)|$ we can select $\{z_1,\ldots,z_Q\} \subset \{y_1,\ldots,y_N\}$ $(Q \le N-1)$ and $\hat\tau \le c\tilde\tau$ such that $\hat\tau \ge 3\delta^2\rho$ and

$$\bigcup_{j=1}^{N} U_{\rho}^{\tilde{\tau}}(y_j) \subset \bigcup_{\ell=1}^{Q} U^{\hat{\tau}}_{\rho(1+c\delta^2)}(z_\ell) ,$$

where $c = c(N)$, and such that $\hat{\tau} \le \frac{1}{4} \min_{i \ne j} |z_i - z_j|$.

Since $c\delta^2 < 1/2$ for $\delta < \delta_0(N)$ (if $\delta_0(N)$ is chosen suitably) we then have $\hat{\tau} \ge 2\delta^2\tilde{\rho}$ and

$$\sum_{j=1}^{N} \frac{\mu(U_{\rho}^{\tilde{\tau}}(y_j))}{\rho^n} \le (1+c\delta^2) \sum_{j=1}^{Q} \frac{\mu(U_{\tilde{\rho}}^{\hat{\tau}}(z_j))}{\tilde{\rho}^n} ,$$

where $\tilde{\rho} = (1+c\delta^2)\rho$ and $c = c(N)$. Since $Q \le N-1$, the required result then follows by induction (choosing ε appropriately).