CHAPTER 3

COUNTABLY n-RECTIFIABLE SETS

The countably n-rectifiable sets, the theory of which we develop in this chapter, provide the appropriate notion of "generalized surface"; they are the sets on which rectifiable currents and varifolds live (see later).

In the first section of this chapter we give some basic definitions, and prove the important result that countably n-rectifiable sets are essentially characterized by the property of having a suitable "approximate tangent space" almost everywhere.

In later sections we show that the area and co-area formula (see §§8,10 of Chapter 2) extend naturally to the case when M is merely countably n-rectifiable rather than a C^1 submanifold, we make a brief discussion of Federer's structure theorem (for the proof we refer to [FH1] or [RM]), and finally we discuss sets of finite perimeter, which play an important role in later developments.

§11. BASIC NOTIONS, TANGENT PROPERTIES

Firstly, a set $M \subset \mathbb{R}^{n+k}$ is said to be countably n-rectifiable if $M \subset M_0 \cup (\bigcup_{j=1}^{\infty} F_j(\mathbb{R}^n))$, where $\mathcal{H}^n(M_0) = 0$ and $F_j : \mathbb{R}^n \to \mathbb{R}^{n+k}$ are Lipschitz functions for $j = 1,2,\ldots$ * . Notice that by the extension theorem 5.1 this is equivalent to saying

$$M = M_0 \cup (\bigcup_{j=1}^{\infty} F_j(A_j))$$

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* Notice that this differs slightly from the terminology of [FH1] in that we allow the set M_0 of $\mathcal{H}^n$-measure zero.

where $\mathcal{H}^n(M_0) = 0$, $F_j : A_j \to \mathbb{R}^{n+k}$ Lipschitz, $A_j \subset \mathbb{R}^n$. More importantly, we have the following lemma.

11.1 LEMMA *M is countably n-rectifiable if and only if $M \subset \bigcup_{j=0}^{\infty} N_j$, where $\mathcal{H}^n(N_0) = 0$ and where each N_j, $j \geq 1$, is an n-dimensional embedded C^1 submanifold of $\mathbb{R}^{n+k}$.*

Proof The "if" part is essentially trivial and is left to the reader. The "only if" part is an easy consequence of Theorem 5.3 as follows. By Theorem 5.3 we can choose C^1 functions $g_1^{(j)}, g_2^{(j)}, \ldots$ such that, if F_j are Lipschitz functions as in the above definition of countably n-rectifiable, then

$$F_j(\mathbb{R}^n) \subset E_j \cup \left(\bigcup_{i=1}^{\infty} g_i^{(j)}(\mathbb{R}^n) \right), \quad j = 1,2\ldots$$

where $\mathcal{H}^n(E_j) = 0$. Then we let

$$N_0 = \left(\bigcup_{j=1}^{\infty} E_j \right) \cup \left(\bigcup_{i,j=1}^{\infty} g_i^{(j)}(C_{ij}) \right),$$

where $C_{ij} = \{x \in \mathbb{R}^n : J\, g_i^{(j)}(x) = 0\}$ and $J\, g_i^{(j)}$ denotes the Jacobian of $g_i^{(j)}$ as in §8. By the area formula (see §8) we have $\mathcal{H}^n\left(\bigcup_{i,j=1}^{\infty} g_i^{(j)}(C_{ij}) \right) = 0$ and hence $\mathcal{H}^n(N_0) = 0$.

Now for each $x \in \mathbb{R}^n \sim C_{ij}$ we let $U_{ij}(x)$ be an open subset of $\mathbb{R}^n \sim C_{ij}$ containing x and such that $g_i^{(j)} | U_{ij}(x)$ is 1:1. Such $U_{ij}(x)$ exists by the inverse function theorem (since $J\, g_i^{(j)}(x) > 0 \Rightarrow d\, g_i^{(j)}(x)$ has rank n), and the inverse function theorem also guarantees that $g_i^{(j)}(U_{ij}(x)) \equiv N_{ij}(x)$, say, is an n-dimensional C^1 submanifold of $\mathbb{R}^{n+k}$ in the sense of §7. We can evidently choose a countable collection $x_1, x_2, \ldots$ of points of $\mathbb{R}^n \sim C_{ij}$ such that $\bigcup_{k=1}^{\infty} U_{ij}(x_k) = \mathbb{R}^n \sim C_{ij}$, hence $\bigcup_{k=1}^{\infty} N_{ij}(x_k) \supset g_i^{(j)}(\mathbb{R}^n \sim C_{ij})$, so we have $F_j(\mathbb{R}^n) \sim N_0 \subset \bigcup_{i,k=1}^{\infty} N_{ij}(x_k)$ for each j. The required result now evidently follows.

We now want to give an important characterization of countably n-rectifiable sets in terms of approximate tangent spaces, which we first define.

11.2 DEFINITION If M is an $\mathcal{H}^n$-measurable subset of $\mathbb{R}^{n+k}$ with $\mathcal{H}^n(M \cap K) < \infty$ $\forall$ compact K, then we say that an n-dimensional subspace P of $\mathbb{R}^{n+k}$ is the approximate tangent space for M at x (x a given point in $\mathbb{R}^{n+k}$) if

$$\lim_{\lambda \downarrow 0} \int_{\eta_{x,\lambda}(M)} f(y)\, d\mathcal{H}^n(y) = \int_P f(y)\, d\mathcal{H}^n(y) \qquad \forall\, f \in C_c^0(\mathbb{R}^{n+k})$$

(Recall $\eta_{x,\lambda} : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ is defined by $\eta_{x,\lambda}(y) = \lambda^{-1}(y-x)$, $x, y \in \mathbb{R}^{n+k}$, $\lambda > 0$.)

11.3 REMARK Of course P is unique if it exists; we shall denote it by $T_x M$.

It is often convenient to be able to relax the condition $\mathcal{H}^n(M \cap K) < \infty$ $\forall$ compact K in 11.2; we can in fact define $T_x M$ in case we merely assume the existence of a positive locally $\mathcal{H}^n$-integrable function θ on M (the existence of such a θ is evidently equivalent to the requirement that M can be expressed as the countable union of $\mathcal{H}^n$-measurable sets with locally finite $\mathcal{H}^n$-measure).

11.4 DEFINITION If M is an $\mathcal{H}^n$-measurable subset of $\mathbb{R}^{n+k}$ and θ is a positive locally $\mathcal{H}^n$-integrable function on M, then we say that a given n-dimensional subspace P of $\mathbb{R}^{n+k}$ is the approximate tangent space for M at x with respect to θ if

$$\lim_{\lambda \downarrow 0} \int_{\eta_{x,\lambda} M} f(y)\, \theta(x+\lambda y)\, d\mathcal{H}^n(y) = \theta(x) \int_P f(y)\, d\mathcal{H}^n(y) \qquad \forall\, f \in C_c^0(\mathbb{R}^{n+k}) .$$

(By change of variable $z = \lambda y + x$, this is equivalent to

$$\lim_{\lambda \downarrow 0} \lambda^{-n} \int_M f(\lambda^{-1}(z-x))\theta(z)\,dH^n(z) = \theta(x) \int_P f(y)\,dH^n(y) \qquad \forall\ f \in C_c^0(\mathbb{R}^{n+k}) .$$

11.5 REMARK Notice that if $\mu = H^n \llcorner \theta$ and if $M_\eta = \{x \in M : \theta(x) > \eta\}$, then $H^n(M_\eta \cap K) < \infty$ for each compact K and $\Theta^{*n}(\mu, M \sim M_\eta, x) = 0$ for H^n-a.e. $x \in M_\eta$ (by 3.5). Hence for H^n-a.e. $x \in M_\eta$ the approximate tangent space for M with respect to θ coincides with $T_x M_\eta$ (as defined in 11.2) if the latter exists. It follows that the approximate tangent spaces of M with respect to two different positive H^n-integrable functions θ, $\tilde{\theta}$ coincide H^n-a.e. in M. For this reason we often still denote the approximate tangent space defined in 11.4 by $T_x M$ (without indicating the dependence on θ).

The following theorem gives the important characterization of countably n-rectifiable sets in terms of existence of approximate tangent spaces.

11.6 THEOREM *Suppose* M *is* H^n*-measurable. Then* M *is countably* n*-rectifiable if and only if there is a positive locally* H^n*-integrable function* θ *on* M *with respect to which the approximate tangent space* $T_x M$ *exists for* H^n*-a.e.* $x \in M$.

11.7 REMARK If M is H^n-measurable, countable n-rectifiable, then we can write M as the disjoint union $\bigcup_{j=0}^{\infty} M_j$, where $H^n(M_0) = 0$, M_j is H^n-measurable, and $M_j \subset N_j$, $j \geq 1$, with N_j an embedded n-dimensional C^1 submanifold of $\mathbb{R}^{n+k}$. (To achieve this, just define the M_j inductively by $M_j = M \cap N_j \sim \bigcup_{i=0}^{j-1} M_i$, $j \geq 1$, where N_j are C^1 submanifolds with $M_0 \equiv M \sim \bigcup_{j=1}^{\infty} N_j$ having H^n-measure zero; such N_j exist by 11.1.) We shall show below (in the proof of the "only if" part of Theorem 11.6) that then

$$(*) \qquad T_x M = T_x N_j\,, \quad H^n\text{-a.e. } x \in M_j .$$

This is a very useful fact.

Proof of "only if" part of Theorem 11.6 As described in 11.7 above, we may write M as the disjoint union $\bigcup_{j=0}^{\infty} M_j$, where $\mathcal{H}^n(M_0) = 0$, $M_j \subset N_j$, $j \geq 1$, N_j embedded C^1 submanifolds of dimension n, and M_j $\mathcal{H}^n$-measurable. Let $\mu = \mathcal{H}^n \llcorner \theta$, where θ is any positive locally $\mathcal{H}^n$-integrable function on M (e.g. put $\theta = 1/2^j$ on M_j, assuming, without loss of generality, that $\mathcal{H}^n(N_j) < \infty$ $\forall$ j).

Now, by 3.5,

$$(1) \qquad \Theta^{*n}(\mu, M \sim M_j, x) = 0 \,, \quad \mathcal{H}^n\text{-a.e. } x \in M_j \,.$$

Also, since N_j is C^1, we have (by the differentiation theorem 4.7)

$$(2) \qquad \Theta^n(\mu, M_j, x) = \lim_{\rho \downarrow 0} \frac{\mu(B_\rho(x) \cap M_j)}{\mathcal{H}^n(B_\rho \cap N_j)} = \theta(x) \,, \quad \mathcal{H}^n\text{-a.e. } x \in M_j$$

From (1), (2) and the fact that N_j is C^1, it now easily follows that the approximate tangent space for M with respect to θ exists for $\mathcal{H}^n$-a.e. $x \in M_j$, and agrees with $T_x N_j$.

Rather than just proving the "if" part of Theorem 11.6, we prove the following slightly more general result. (The "if" part of Theorem 11.6 corresponds to the case $\mu = \mathcal{H}^n \llcorner \theta$ in this more general result - see Remark 11.9 below.)

11.8 THEOREM *Suppose μ is a Radon measure on $\mathbb{R}^{n+k}$, and for $x \in \mathbb{R}^{n+k}$, $\lambda > 0$ let $\mu_{x,\lambda}$ be the measure given by $\mu_{x,\lambda} A = \lambda^{-n}\mu(x+\lambda A)$. Suppose that for μ-a.e. x there is $\theta(x) \in (0,\infty)$ and an n-dimensional subspace $P \subset \mathbb{R}^{n+k}$ with*

$$(*) \qquad \lim_{\lambda \downarrow 0} \int f(y)\, d\mu_{x,\lambda}(y) = \theta(x) \int_P f(y)\, d\mathcal{H}^n(y) \,.$$

(P is called the approximate tangent space for μ at x, and $\theta(x)$ is called the multiplicity.) *Let* $M = \{x : (*)$ *holds for some* P *and some*

$\theta(x) \in (0,\infty)\}$, *and set* $\theta(x) = 0$, $x \in \mathbb{R}^{n+k} \sim M$.

Then M *is countably* n-*rectifiable,* θ *is* H^n-*measurable on* $\mathbb{R}^{n+k}$, *and* $\mu = H^n \llcorner \theta$.

11.9 REMARK Notice that in case $\mu = H^n \llcorner \theta$, where θ is a non-negative locally H^n-integrable function on $\mathbb{R}^{n+k}$, then

$$\int f \, d\mu_{x,\lambda} = \int_{\eta_{x,\lambda}(M)} f(z)\,\theta(x+\lambda y)\, dH^n(y) \ ,$$

where $M = \{x : \theta(x) > 0\}$, so the approximate tangent space for μ at x is just the approximate tangent space $T_x M$ with respect to θ (in the sense of 11.4). Thus we get the "if" part of Theorem 11.6 in this special case.

Proof of Theorem 11.8 Replacing μ by $\mu \llcorner B_R(0)$ (R chosen so that $\mu(\partial B_R(0)) = 0$) , we may assume that $\mu(\mathbb{R}^{n+k}) < \infty$. First note that (by (*)) we have

$$\text{(1)} \qquad \theta(x) = \lim_{\rho \downarrow 0} \frac{\mu(B_\rho(x))}{\omega_n \rho^n} \quad (\equiv \Theta^n(\mu,x)) \quad \mu\text{-a.e. } x \in \mathbb{R}^{n+k} ,$$

and hence, by Remark 3.1,

$$\text{(2)} \qquad \theta \text{ is } H^n\text{-measurable.}$$

Given any k-dimensional subspace $\pi \subset \mathbb{R}^{n+k}$ and any $\alpha \in (0,1)$ let p_π denote the orthogonal projection of $\mathbb{R}^{n+k}$ onto π and $X_\alpha(\pi,x)$ denote the cone

$$X_\alpha(\pi,x) = \{y \in \mathbb{R}^{n+k} : |p_\pi(y-x)| \geq \alpha|y-x|\} .$$

For k-dimensional subspaces π , π' we define the distance between π , π' , denoted $\operatorname{dist}(\pi,\pi')$, by

$$\operatorname{dist}(\pi,\pi') = \sup_{|x|=1} |p_\pi(x) - p_{\pi'}(x)| .$$

Choose $\theta_0 > 0$ and a Borel-measurable subset $F \subset \mathbb{R}^{n+k}$ such that

(3) $$\mu(\mathbb{R}^{n+k} \sim F) \le \frac{1}{4}\mu(\mathbb{R}^{n+k})$$

and such that for each $x \in F$, μ has an approximate tangent space P_x at x with multiplicity $\theta(x) \ge \theta_0$. Thus in particular for $x \in F$ (by (1) and (*))

(4) $$\lim_{\rho \downarrow 0} \frac{\mu(B_\rho(x))}{\omega_n \rho^n} \ge \theta_0$$

and

(5) $$\lim_{\rho \downarrow 0} \frac{\mu(X_{\frac{1}{2}}(\pi_x, x) \cap B_\rho(x))}{\omega_n \rho^n} = 0 ,$$

where $\pi_x = (P_x)^{\perp}$.

For $k = 1,2,\ldots$ and $x \in F$, define

$$f_k(x) = \inf_{0<\rho<\frac{1}{k}} \frac{\mu(B_\rho(x))}{\omega_n \rho^n}$$

and

$$q_k(x) = \sup_{0<\rho<\frac{1}{k}} \frac{\mu(X_{\frac{1}{2}}(\pi_x, x) \cap B_\rho(x))}{\omega_n \rho^n} .$$

Then

(6) $$\lim f_k(x) \ge \theta_0 \quad \text{and} \quad \lim q_k(x) = 0 \qquad \forall\, x \in F ,$$

and hence by Egoroff's Theorem we can choose a μ-measurable $E \subset F$ with

(7) $$\mu(F \sim E) \le \frac{1}{4}\mu(\mathbb{R}^{n+k})$$

and with (6) holding *uniformly* for $x \in E$. Thus for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$(8) \qquad \frac{\mu(B_\rho(x))}{\omega_n \rho^n} \ge \theta_0 - \varepsilon \ , \quad \frac{\mu(X_{\frac{1}{2}}(\pi_x, x) \cap B_\rho(x))}{\omega_n \rho^n} \le \varepsilon$$

$x \in E$, $0 < \rho < \delta$.

Now choose k-dimensional subspaces $\pi_1, \ldots, \pi_N$ of $\mathbb{R}^{n+k}$ $(N = N(n,k))$ such that for *each* k-dimensional subspace π of $\mathbb{R}^{n+k}$, there is a $j \in \{1, \ldots, N\}$ such that $d(\pi, \pi_j) < \frac{1}{16}$, and let $E_1, \ldots, E_N$ be the subsets of E defined by

$$E_j = \{x \in E : d(\pi_j, \pi_x) < \frac{1}{16}\} .$$

Then $E = \bigcup_{j=1}^{N} E_j$ and we claim that if we take $\varepsilon = \theta_0/16^n$ and let $\delta > 0$ be such that (8) holds, then

$$(9) \qquad X_{\frac{3}{4}}(\pi_j, x) \cap E_j \cap B_{\delta/2}(x) = \{x\} \ , \quad \forall \ x \in E_j \ , \ j = 1, \ldots, N .$$

Indeed otherwise we could find a point $x \in E_j$ and a $y \in X_{\frac{3}{4}}(\pi_j, x) \cap E_j \cap \partial B_\rho(x)$ for some $0 < \rho \le \delta/2$. But since $x \in E$ and $2\rho \le \delta$, we have (by (8))

$$(10) \qquad \mu(X_{\frac{1}{2}}(\pi_x, x) \cap B_{2\rho}(x)) < \varepsilon \ \omega_n (2\rho)^n$$

and (since $B_{\rho/8}(y) \subset X_{\frac{1}{2}}(\pi_j, x) \cap B_{2\rho}(x)$) we have also (again by (8))

$$\begin{aligned} \mu(X_{\frac{1}{2}}(\pi_x, x) \cap B_{2\rho}(x)) &\ge \mu(B_{\rho/8}(y)) \\ &\ge \theta_0 \ \omega_n (\rho/8)^n , \end{aligned}$$

which contradicts (8), since $\varepsilon = \theta_0/16^n$. We have therefore proved (9). Now for any fixed $x_0 \in E_j$ it is easy to check that (9), taken together with the extension theorem 2.1, implies

$$E_j \cap B_{\delta/2}(x_0) \subset q(\text{graph } f)$$

where q is an orthogonal transformation of $\mathbb{R}^{n+k}$ with $q(\pi_j) = \mathbb{R}^k$, and where $f = (f^1,\ldots,f^k)$ is Lipschitz.

Since $j \in \{1,\ldots,N\}$ and $x_0 \in E_j$ are arbitrary, we can then evidently select Lipschitz functions $f_1,\ldots,f_Q : \mathbb{R}^n \to \mathbb{R}^k$ and orthogonal transformations $q_1,\ldots,q_Q$ of $\mathbb{R}^{n+k}$ such that

$$E \subset \bigcup_{j=1}^{Q} q_j(\text{graph } f_j) .$$

Thus by (3), (7) we have

$$\mu(\mathbb{R}^{n+k} \sim \bigcup_{j=1}^{Q} q_j(\text{graph } f_j)) \le \frac{1}{2}\mu(\mathbb{R}^{n+k}) .$$

Since we can now repeat the same argument, starting with $\mu \llcorner (\mathbb{R}^{n+k} \sim \bigcup_{j=1}^{Q} q_j(\text{graph } f_j))$ in place of μ, we thus deduce that there are countably many Lipschitz graphs $F_j = \text{graph } f_j$, $f_j : \mathbb{R}^n \to \mathbb{R}^k$, and that $\mu(\mathbb{R}^{n+k} \sim \bigcup_{j=1}^{\infty} F_j) = 0$. By (1) and 3.2(1) we then deduce $\mathcal{H}^n(M \sim \bigcup_{j=1}^{\infty} F_j) = 0$, so that (by definition) M is countably n-rectifiable. Thus by 11.1 (see in particular Remark 11.7) we can write M as the disjoint union $\bigcup_{j=0}^{\infty} M_j$, where $\mathcal{H}^n(M_0) = 0$, $M_j \subset N_j$, N_j being n-dimensional C^1 submanifolds of $\mathbb{R}^{n+k}$. Then (1) evidently implies that $\lim_{\rho\downarrow 0} \dfrac{\mu(B_\rho(x))}{\mathcal{H}^n(B_\rho(x)\cap N_j)} = \theta(x)$, $\mathcal{H}^n$ a.e. $x \in M_j$; then by the differentiation theorem 4.7 we have $\mu = \mathcal{H}^n \llcorner \theta$ as required.

§12. GRADIENTS, JACOBIANS, AREA, CO-AREA

Throughout this section M is supposed to be $\mathcal{H}^n$-measurable and countably n-rectifiable, so that we can express M as the disjoint union $\bigcup_{j=0}^{\infty} M_j$ (as in

11.7), where $\mathcal{H}^n(M_0) = 0$, M_j is $\mathcal{H}^n$-measurable, $M_j \subset N_j$, $j \geq 1$, where N_j are embedded n-dimensional C^1 submanifolds of $\mathbb{R}^{n+k}$.

Let f be a locally Lipschitz function on U , where U is an open set in $\mathbb{R}^{n+k}$ containing M . Then we can define the gradient of f , $\nabla^M f$, $\mathcal{H}^n$ - a.e. on M according to :

12.1 DEFINITION

$$\nabla^M f(x) = \nabla^{N_j} f(x)$$

whenever $x \in M_j$ and $f|N_j$ is differentiable (which is true $\mathcal{H}^n$ - a.e. $x \in M_j$ by virtue of Rademacher's Theorem 5.2 together with the fact that N_j is C^1).

12.2 REMARK Note that (by 11.7) $\nabla^M f(x) \in T_x M$ for $\mathcal{H}^n$ - a.e. $x \in M$, and is, up to a set of $\mathcal{H}^n$-measure zero in M , independent of the particular decomposition $\bigcup_{j=0}^{\infty} M_j$ used in the definition. (i.e. $\nabla^M f$ is well-defined as an L^1 function on subsets of M with finite $\mathcal{H}^n$-measure). Indeed we can easily check that, at all points x where $f|N_j$ is differentiable, we have $f|L$ is differentiable on the affine space $L = x + T_x N_j$ at the point x , and gradient $f|L\,(x) = \nabla^{N_j} f(x)$. Since $T_x N_j = T_x M$ for $\mathcal{H}^n$ - a.e. $x \in M_j$ (see 11.7), and since $T_x M$ is independent of the particular decomposition $\bigcup_{j=0}^{\infty} M_j$, we thus deduce that $\nabla^M f$ is also independent of the decomposition up to a set of measure zero, as required.

Having defined $\nabla^M f$, we can now define the linear map $d^M f_x : T_x M \to \mathbb{R}$ induced by f by setting

$$d^M f_x(\tau) = \langle \tau, \nabla^M f(x) \rangle , \quad \tau \in T_x M$$

at all points where $T_x M$ and $\nabla^M f(x)$ exist. If $f = (f^1,\ldots,f^N)$ takes values in $\mathbb{R}^N$ (f^j still locally Lipschitz on U , $j = 1,\ldots,N$) , we define $d^M f_x : T_x M \to \mathbb{R}^N$ by

12.3 $$d^M f_x(\tau) = \sum_{j=1}^{N} < \tau, \nabla^M f^j(x) > e_j \,.$$

With such an f, in case $N \geq n$, we define the Jacobian $J_M f(x)$ for H^n-a.e. $x \in M$ by

$$J_M f(x) = \sqrt{\det(d^M f_x)^* \circ (d^M f_x)}$$

(Cf. the smooth case 8.3), where $(d^M f_x)^* : \mathbb{R}^N \to T_x M$ denotes the adjoint of $d^M f_x$. Then we have the general area formula

12.4 $$\int_A J_M f \, dH^n = \int_{\mathbb{R}^N} H^0(A \cap f^{-1}(y)) \, dH^N(y)$$

for any H^n-measurable set $A \subset M$. The proof of this is as follows: We may suppose (decomposing H^n-almost all M_j as a countable union if necessary and using the C^1 approximation theorem 5.3) that $f|M_j = g_j|M_j$, where g_j is C^1 on $\mathbb{R}^{n+k}$, $j \geq 1$.

By virtue of the definition 12.1, 12.3, we then have

$$J_M f(x) = J_{N_j} g_j(x) \,, \quad H^n\text{-a.e.} \quad x \in M_j \,.$$

Thus $J_M f$ is H^n-measurable, and by the smooth case 8.4 of the area formula (with N_j in place of M, $A \cap M_j$ in place of A and g_j in place of f), we have

$$\int_{A \cap M_j} J_M f \, dH^n = \int_{\mathbb{R}^N} H^0(A \cap M_j \cap f^{-1}(y)) \, dH^N \,.$$

We now conclude 12.4 by summing over $j \geq 1$ and using the (easily checked) fact that if $\psi : U \to \mathbb{R}^N$ is locally Lipschitz and B has H^n-measure zero, then $H^n(\psi(B)) = 0$.

We note also that if g is any non-negative $\mathcal{H}^n$-measurable function on M, then, by approximation of g by simple functions, 12.4 implies the more general formula

$$\int_M g\, J_M f\, d\mathcal{H}^n = \int_{\mathbb{R}^N} \left(\int_{f^{-1}(y)} g\, d\mathcal{H}^0 \right) d\mathcal{H}^N(y) .$$

In case f is 1:1 on M, this becomes

12.5 $$\int_M g\, J_M f\, d\mathcal{H}^n = \int_{\mathbb{R}^N} g \circ f^{-1}\, d\mathcal{H}^N .$$

There is also a version of the co-area formula in case M is merely $\mathcal{H}^n$-measurable, countably n-rectifiable and $f : U \to \mathbb{R}^N$ is locally Lipschitz with $N < n$. (U open, $M \subset U$ as before).

In fact we can define (Cf. the smooth case described in §10)

$$J_M^* f(x) = \sqrt{\det(d^M f_x) \circ (d^M f_x)^*}$$

with $d^M f_x$ as in 12.3 and $(d^M f_x)^* =$ adjoint of $d^M f_x$. Then, for any $\mathcal{H}^n$-measurable set $A \subset M$,

12.6 $$\int_A J_M^* f\, d\mathcal{H}^n = \int_{\mathbb{R}^N} \mathcal{H}^{n-N}(A \cap f^{-1}(y))\, d\mathcal{L}^N(y) .$$

This follows from the C^1 case (see §10) by using the decomposition $M = \bigcup_{j=0}^{\infty} M_j$ and the approximation theorem 5.3 in a similar manner to the procedure used for the discussion of the area formula above.

As for the smooth case, approximating a given non-negative $\mathcal{H}^n$-measurable function g by simple functions, we deduce from 12.6 the more general formula

$$\int_A g\, J_M^* f\, dH^n = \int_{\mathbb{R}^n} \int_{f^{-1}(y)\cap M} g\, dH^{n-N}\, dL^N(y) \quad . \tag{12.7}$$

12.8 REMARKS

(1) Note that the remarks 10.7 carry over without change to this setting.

(2) The "slices" $M \cap f^{-1}(y)$ are countably (n-N)-rectifiable subsets of $\mathbb{R}^{n+k}$ for L^N-a.e. $y \in \mathbb{R}^N$. This follows directly from the decomposition $M = \bigcup_{j=0}^{\infty} M_j$, together with the C^1 Sard-type theorem 10.4 and the approximation theorem 5.3.

§13 THE STRUCTURE THEOREM

Notice that an arbitrary subset A of $\mathbb{R}^{n+k}$ which can be written as the countable union $\bigcup_{j=1}^{\infty} A_j$ of sets of finite measure, is always decomposible into a disjoint union

$$A = R \cup P \ , \tag{13.1}$$

where R is countably n-rectifiable and P is *purely n-unrectifiable;* that is P contains no countably n-rectifiable subsets of positive H^n-measure.

To prove 13.1, we simply let R be a *maximal* element of the collection of all countably n-rectifiable subsets of A (ordered by inclusion); such R exists by the Hausdorff maximal principal.

A very non-trivial theorem (the Structure Theorem) due to Besicovitch [B] in case $n = k = 1$ and Federer [FH2] in general, says that the purely unrectifiable sets Q of $\mathbb{R}^{n+k}$ which (like the subset P in 13.1) can be written as the countable union of sets of finite H^n-measure, are characterized by the fact that they have H^n-null projection via almost all orthogonal projections onto n-dimensional subspaces of $\mathbb{R}^{n+k}$. More precisely:

13.2 THEOREM *Suppose* Q *is a purely* n*-unrectifiable subset of* $\mathbb{R}^{n+k}$ *with* $Q = \bigcup_{j=1}^{\infty} Q_j$ *,* $\mathcal{H}^n(Q_j) < \infty\ \forall j$ *. Then* $\mathcal{H}^n(p(Q)) = 0$ *for* σ*-almost all* $p \in O(n+k,n)$ *. Here* σ *is Haar measure for* $O(n+k,n)$ *, the orthogonal projections of* $\mathbb{R}^{n+k}$ *onto* n*-dimensional subspaces of* $\mathbb{R}^{n+k}$ *.*

For the proof of this theorem see [FH1] or [RM].

13.3 REMARK Of course only the purely n-unrectifiable subsets could possibly have the null projection property described in 13.2. Indeed (by 11.1) if Q were not purely n-unrectifiable then there would be an n-dimensional C^1 submanifold M of $\mathbb{R}^{n+k}$ such that $\mathcal{H}^n(M \cap Q) > 0$. It is then an easy matter to check that if we select any $x \in M$ with $\Theta^{*n}(\mathcal{H}^n, M \cap Q, x) > 0$, then $\mathcal{H}^n(p(M \cap Q)) > 0$ for all orthogonal projections p of $\mathbb{R}^{n+k}$ onto an n-dimensional subspace S which is not orthogonal to $T_x M$.

Notice that, by combining 13.1 and 13.2, we get the following useful *rectifiability theorem:*

13.4 THEOREM *If* A *is an arbitrary subset of* $\mathbb{R}^{n+k}$ *which can be written as a countable union* $\bigcup_{j=1}^{\infty} A_j$ *with* $\mathcal{H}^n(A_j) < \infty\ \forall j$ *, and if every subset* $B \subset A$ *with positive* $\mathcal{H}^n$*-measure has the property that* $\mathcal{H}^n(p(B)) > 0$ *for a set of* $p \in O(n+k,n)$ *with* σ*-measure* > 0 *, then* A *is countably* n*-rectifiable.*

§14 SETS OF LOCALLY FINITE PERIMETER

An important class of countably n-rectifiable sets in $\mathbb{R}^{n+1}$ comes from the sets of locally finite perimeter. (Or Caccipoli sets - see De Giorgi [DG], Giusti [G].) First we need some definitions.

If $U \subset \mathbb{R}^{n+1}$ is open and if E is an $\mathcal{L}^{n+1}$-measurable subset of $\mathbb{R}^{n+1}$, we say that E has *locally finite perimeter* in U if the characteristic function χ_E of E is in $BV_{loc}(U)$. (See §6.)

Thus E has locally finite perimeter in U if there is a Radon measure μ_E ($= |D\chi_E|$ in the notation of §6) on U and a μ_E-measurable function $\nu = (\nu^1, \ldots, \nu^{n+1})$ with $|\nu| = 1$ μ_E-a.e. in U, such that

14.1 $$\int_{E\cap U} \operatorname{div} g \, d\mathcal{L}^{n+1} = -\int_U g\cdot\nu \, d\mu_E$$

for each $g = (g^1, \ldots, g^{n+1})$ with $g^j \in C_c^1(U)$, $j = 1, \ldots, n+1$. Notice that if E is open and $\partial E \cap U$ is an n-dimensional embedded C^1 submanifold of $\mathbb{R}^{n+1}$, then the divergence theorem tells us that 14.1 holds with $\mu_E = \mathcal{H}^n \llcorner (\partial E \cap U)$ and with $\nu =$ the inward pointing unit normal to ∂E. Thus in general we interpret μ_E as a "generalized boundary measure" and ν as a "generalized inward unit normal". It turns out (see Theorem 14.3 below) that in fact this interpretation is quite generally correct in a rather precise (and concrete) sense.

We now want to define the *reduced boundary* ∂^*E of a set E of finite perimeter by

14.2 $$\partial^*E = \left\{x \in U : \lim_{\rho\downarrow 0} \frac{\int_{B_\rho(x)} \nu \, d\mu_E}{\mu_E(B_\rho(x))} \text{ exists and has length } 1\right\}.$$

Since $|\nu| = 1$ μ_E-a.e. in U, by virtue of the differentiation theorem 4.7 we have $\mu_E(U \sim \partial^*E) = 0$, so that $\mu_E = \mu_E \llcorner \partial^*E$. We in fact claim much more :

14.3 THEOREM (De Giorgi) *Suppose* E *has locally finite perimeter in* U. *Then* ∂^*E *is countably* n-*rectifiable and* $\mu_E = \mathcal{H}^n \llcorner \partial^*E$. *In fact at each point* $x \in \partial^*E$ *the approximate tangent space* T_x *of* μ_E *exists, has multiplicity* 1, *and is given by*

(1) $$T_x = \{y \in \mathbb{R}^{n+1} : y \cdot \nu_E(x) = 0\} ,$$

where $\nu_E(x) = \lim_{\rho \downarrow 0} \dfrac{\int_{B_\rho(x)} \nu \, d\mu_E}{\mu_E(B_\rho(x))}$ *(so that* $|\nu_E(x)| = 1$ *by* 14.2). *Furthermore at any such point* $x \in \partial^* E$ *we have that* $\nu_E(x)$ *is the "inward pointing unit normal for* E*" in the sense that*

(2) $$E_{x,\lambda} \equiv \{\lambda^{-1}(y-x) : y \in E\} \to \{y \in \mathbb{R}^{n+1} : y \cdot \nu_E(x) > 0\}$$

in the $L^1_{loc}(\mathbb{R}^{n+1})$ *sense.*

Proof By 11.6 and 3.5, the first part of the theorem follows from (1), which we now establish. (2) will also appear as a "by product" of the proof of (1). Assume without loss of generality $\nu \equiv \nu_E$ on $\partial^* E$.

Take any $y \in \partial^* E$. For convenience of notation we suppose that $y = 0$ and $\nu(0) = (0,\ldots,0,1)$. Then we have

(1) $$\lim_{\rho \downarrow 0} \frac{\int_{B_\rho(0)} \nu_{n+1} \, d\mu_E}{\mu_E(B_\rho(0))} = 1$$

and hence (since $|\nu| = 1$ μ_E - a.e.)

(2) $$\lim_{\rho \downarrow 0} \frac{\int_{B_\rho(0)} |\nu_i| \, d\mu_E}{\mu_E(B_\rho(0))} = 0 , \quad i = 1,\ldots,n .$$

Further if $\zeta \in C^1_0(U)$ has support in $B_\rho(0) \subset U$, then by 14.1

(3) $$\int_U \nu_{n+1} \, \zeta \, d\mu_E = - \int_U \chi_E \, D_{n+1} \, \zeta \, dL^{n+1}$$
$$\le \int_E |D\zeta| \, dL^{n+1} .$$

Now (taking $B_\rho(0)$ to be the closed ball) replace ζ by a decreasing sequence $\{\zeta_k\}$ converging pointwise to the characteristic function of $B_\rho(0)$ and satisfying

(4) $$\lim_{k\to\infty} \int_E |D\zeta_k| = \frac{d}{d\rho} \mathcal{L}^{n+1}(E\cap B_\rho(0)) \ .$$

(Notice that this can be done whenever the right side exists, which is $\mathcal{L}^1$-a.e. ρ.) Then (3) gives

(5) $$\int_{B_\rho(0)} \nu_{n+1} \, d\mu_E \le \frac{d}{d\rho} \mathcal{L}^{n+1}(E\cap B_\rho(0))$$

for $\mathcal{L}^1$-a.e. $\rho \in (0,\rho_0)$, $\rho_0 = \operatorname{dist}(0,\partial U)$. Then by (1) we have, for suitable $\rho_1 \in (0,\rho_0)$,

(6) $$\begin{aligned}\mu_E(B_\rho(0)) &\le 2 \frac{d}{d\rho} \mathcal{L}^{n+1}(E\cap B_\rho(0)) \equiv 2\, \mathcal{H}^n(E\cap\partial B_\rho(0)) \\ &\le 2(n+1)\omega_{n+1}\, \rho^n\end{aligned}$$

for $\mathcal{L}^1$-a.e. $\rho \in (0,\rho_1)$.

Then by the compactness theorem 6.3, it follows that we can select a sequence $\rho_k \downarrow 0$ so that $\chi_{\rho_k^{-1}E} \to \chi_F$ in $L^1_{loc}(\mathbb{R}^{n+1})$, where F is a set of locally finite perimeter in $\mathbb{R}^{n+1}$. Hence in particular for any non-negative $\zeta \in C_0^1(\mathbb{R}^{n+1})$

(7) $$\lim_{k\to\infty} \int_{\rho_k^{-1}E} D_i\, \zeta \, d\mathcal{L}^{n+1} = \int_F D_i\, \zeta \, d\mathcal{L}^{n+1} \ .$$

Now write $\zeta_k(x) = \zeta(\rho_k^{-1}x)$ and change variable $x \to \rho_k x$; then

$$\int_{\rho_k^{-1}E} D_i\, \zeta \, d\mathcal{L}^{n+1} = \rho_k^{-n} \int_E D_i\, \zeta_k \, d\mathcal{L}^{n+1} \equiv -\rho_k^{-n} \int_U \zeta_k\, \nu_i \, d\mu_E$$

(by 14.1), so that $\int_{\rho_k^{-1}E} D_i\, \zeta \, d\mathcal{L}^{n+1} \to 0$ by (2) for $i = 1,\ldots,n$. Thus (7) gives

$$\int_F D_i \zeta \, dL^{n+1} = 0 \quad \forall \zeta \in C_0^1(\mathbb{R}^{n+1}) , \quad i = 1,\ldots,n ,$$

and it follows that $F = \mathbb{R}^n \times H$ for some L^1-measurable subset H of $\mathbb{R}$.

On the other hand by 14.1 with $g = \zeta_k e_{n+1}$ and by (1) we have, for k sufficiently large and $\zeta \geq 0$,

$$0 \leq \rho_k^{-n} \int_U \zeta_k \nu_{n+1} \, d\mu_E = \int_{\rho_k^{-1}E} D_{n+1} \zeta$$

$$\to \int_F D_{n+1} \zeta \equiv \int_{\mathbb{R}^n} \left(\int_H \frac{\partial \zeta}{\partial x_{n+1}} (x', x_{n+1}) \, dx_{n+1} \right) dx'$$

as $k \to \infty$, so that χ_H is *non-decreasing* on $\mathbb{R}$, hence

$$(8) \qquad F = \{x \in \mathbb{R}^{n+1} : x_{n+1} < \lambda\}$$

for some λ. We have next to show that $\lambda = 0$. To check this we use the Sobolev inequality (see e.g. [GT]) to deduce that, if $\zeta \geq 0$, spt $\zeta \subset U$ and $\sigma < \operatorname{dist}(\operatorname{spt} \zeta, \partial U)$, then

$$\left(\int_U (\zeta \, \phi_\sigma * \chi_E)^{\frac{n+1}{n}} \, dL^{n+1} \right)^{\frac{n}{n+1}} \leq c \int_U |D(\zeta \, \phi_\sigma * \chi_E)| \, dL^{n+1}$$

$$\leq c \left(\int_U \zeta \, |D(\phi_\sigma * \chi_E)| \, dL^{n+1} + \int_U \phi_\sigma * \chi_E |D\zeta| \, dL^{n+1} \right) .$$

Then by 6.2 it follows that

$$\left(\int_E \zeta^{\frac{n+1}{n}} \, dL^{n+1} \right)^{\frac{n}{n+1}} \leq c \left(\int_U \zeta \, d\mu_E + \int_E |D\zeta| \, dL^{n+1} \right) ,$$

and replacing ζ by a sequence ζ_k as in (4), we get for a.e. $\rho \in (0, \rho_1)$

$$\left(\mathcal{L}^{n+1}(E\cap B_\rho(0))\right)^{\frac{n}{n+1}} \le c\left(\mu_E(B_\rho(0)) + \frac{d}{d\rho}\mathcal{L}^{n+1}(E\cap B_\rho(0))\right) ,$$

which by (6) gives

$$\left(\mathcal{L}^{n+1}(E\cap B_\rho(0))\right)^{\frac{n}{n+1}} \le c\,\frac{d}{d\rho}\mathcal{L}^{n+1}(E\cap B_\rho(0)) \quad \text{a.e. } \rho \in (0,\rho_1) .$$

Integration (using the fact that $\mathcal{L}^{n+1}(E\cap B_\rho(0))$ is non-decreasing) then implies

(9) $$\mathcal{L}^{n+1}(E\cap B_\rho(0)) \ge c\,\rho^{n+1}$$

for all sufficiently small ρ . Repeating the same argument with $U \sim E$ in place of E , we also deduce

(10) $$\mathcal{L}^{n+1}(B_\rho(0)\sim E) \ge c\,\rho^{n+1}$$

for all sufficiently small ρ . (9) and (10) evidently tell us that $\lambda = 0$ in (8).

Now given any sequence $\rho_k \downarrow 0$, the argument above guarantees we can select a subsequence $\rho_{k'}$ such that $\chi_{\rho_{k'}^{-1}E} \to \chi_{\{x\in\mathbb{R}^{n+1}:\, x_{n+1}<0\}}$ in $L^1_{loc}(\mathbb{R}^{n+1})$. Hence $\chi_{\rho^{-1}E} \to \chi_{\{x\in\mathbb{R}^{n+1}:\, x_{n+1}<0\}}$ and (2) of the theorem is established. Then by 14.1, (1) and (2) we have $\mu_{\rho^{-1}E} \to \mu_{\{x\in\mathbb{R}^{n+1}:\, x_{n+1}<0\}} \equiv \mathcal{H}^n \llcorner \{x\in\mathbb{R}^{n+1} : x_{n+1} = 0\}$ as $\rho \downarrow 0$ and the proof is complete.