JONES POLYNOMIALS AND 3-MANIFOLDS

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Given the relationship between exactly-solved models in statistical mechanics on 2-dimensional lattices at criticality, and conformal theories in 2-dimensional quantum field theory as arising in Witten's generalization of Jones polynomials, it is appropriate to briefly describe how these models give rise, via the braid group, to polynomial invariants of classical links. Subsequently we mention some of the properties of these invariants, still in the classical context. Finally we mention some of the rôles knots and links play in the representation and construction of closed, orientable 3-manifolds, finishing with some remarks on Thurston's geometrization conjectures. Hopefully this talk will be a useful supplement for non-topologists interested in Witten's recent preprint [Wi].

Some excellent detailed surveys of different aspects of Jones polynomials (pre-Witten) now exist, which we recommend for further reading and references. These are Connes [Co], de la Harpe, Kervaire and Weber [HKW], Kauffman [Kau2], Lehrer [Le] and Lickorish [Li2]. Since the subject is evolving rapidly, some questions raised here may be resolved in the very near future.

1 INTRODUCTION

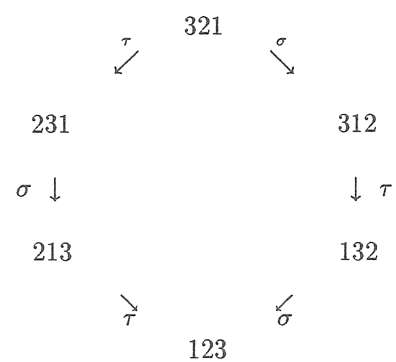
As regards Jones' polynomials [Jo1] and 3-manifolds, the dust has not yet settled after Witten's most recent contribution [Wi]. Since Jones introduced his polynomial, there have been several emergent themes, certain salient points gradually becoming clear.

A fundamental problem has been to ascertain the extent to which Jones-type invariants have their origin in 2-dimensions, within the realm of 3-manifold topology proper, (epitomized by a hexagon and cube respectively as in Figure 1), or merely as a bizarre legacy of combinatorics.

Keeping track of this evolution of understanding has been a little like partaking in the old 'pea in the shell' game—three shells, the hidden pea— where one attempts to keep track of the pea.

The basis of this 'game', aside from possible sleight of hand, is that the same permutation of 3 objects can be represented as a product of 3 transpositions in several different ways, but which are easily remembered.

Given three objects in a row, reverse their order by nearest-neighbour transpositions. Let τ and σ denote transpositions of the first two and last two respectively. We obtain the following commuting *hexagon* of transpositions:



This can be depicted using a cube. Shade three adjacent faces black, the others red. Each set of three faces forms a hemisphere, with common boundary a hexagon (Figure 2). Label and orient all edges according to some choice of parallel coordinate axes. The hexagon has two oriented hemispheres labelled 321 and 123.

The faces then can be interpreted as *transpositions*, by drawing crossing strings as in Figure 3. This which will be familiar to physicists as a representation of the factorisation of the S-matrix in field theory, and of the Star-Triangle relation or Yang-Baxter equation [Ba1].

Observe that we may slice the cube symmetrically by a regular hexagon, as in Figure 4. The edges of the hexagon are alternatively coloured red and black, which can be interpreted as the alternation of σ and τ. The same fundamental relation of permutations manifests in a variety of ways. We will return briefly to the cube and hexagon at the conclusion of this paper.

The Star-Triangle relation plays a crucial role in the *solvability by transfer matrices* of many statistical mechanical models defined on a 2-dimensional lattice.

The *Ising model* on the standard $m \times n$ lattice (ignoring boundary conditions) concerns an array of 'particles' arranged on vertices of the lattice, each interacting with nearest neighbours, and each being in one of two possible states, often thought of as charge ± 1, or spin up or down.

More generally, the *q-state Potts model*, allows q possible states at each vertex, . For $q = 3$ we can spuriously and suggestively depict these possible 'spins' or 'states' at each lattice site as edges of the lattice passing over, under or repelling each other, as in Figure 5. The result is what is generally called a *braid*, and is suggestive of some connection between statistical mechanics and the theory of knots and links, via the theory of braids.

2 BRAIDS, KNOTS AND LINKS

Good recent references for knots and links are the books by Rolfsen [Rol], Burde and Zieschang [BZ] and for braids, that of Birman [Bi].

A *braid on n strings* is obtained by suspending n strings in 3-space, and introducing crossings σ_i between the i^{th} and $(i+1)^{\text{st}}$ strands. This gives rise to Artin's braid group $\mathcal{B}_n$ with presentation:

$$\text{generators} : \sigma_1, \ldots, \sigma_{n-1} \tag{1}$$

$$\text{relations} : \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| > 1, \tag{2}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}. \tag{3}$$

Given $\omega \in \mathcal{B}_n$, a word in the σ_i, construct a braid by reading ω from left to right, building the braid from the top down. (Other conventions can be used.) The relations are easily verified (Figure 6), corresponding to simple situations for which the resulting topological configurations of strings are equivalent to each other via *isotopy*, which in this context allows motion of the strands so that they remain disjointly embedded smooth curves with tangents nowhere horizontal, and endpoints fixed.

Any n-string braid can be expressed as a word in the generators σ_i, two words representing the same element of $\mathcal{B}_n$ iff the corresponding braids are topologically equivalent (isotopic).

Given $\omega \in \mathcal{B}_n$, obtain its *closure* $\bar{\omega}$ by joining the top and bottom strands, thereby obtaining an *oriented link* $\mathcal{L}_\omega$: a collection of disjointly embedded oriented circles in R^3 (or in $S^3 = R^3 \cup \infty$). All of the strands wind around a common axis in R^3, and the orientation is induced from the braid strings.

Example 1 The figure-8 knot, 4_1 in Rolfsen's tables, is the closure of $\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}$, as in Figure 7.

Jones defined his polynomial invariant in terms of a braid word whose closure is the desired link. For this to be well-defined, we need two results:

Theorem 2 (Alexander) *Every oriented link can be represented as some* closed *braid.*

There are many distinct words whose closures give the same oriented link. It is necessary to understand the relationship between two words $\omega \in \mathcal{B}_m$ and $\omega' \in \mathcal{B}_n$, for which $\bar{\omega} = \bar{\omega'}$. There are two obvious topological moves one can perform on a braid representation of a closed, oriented link, referred to as *Markov moves of type I and II.* These are depicted in Figure 8, and are *equivalent* to the algebraic relations

1. If $\omega,\ \sigma_i \in \mathcal{B}_n$, then $\bar{\omega} = \overline{\sigma_i^{\pm 1}\omega\sigma_i^{\mp 1}}$.
2. If $\omega \in \mathcal{B}_n$, then $\bar{\omega} = \overline{\omega\sigma_n^{\mp 1}}$, $\omega\sigma_n^{\mp 1} \in \mathcal{B}_{n+1}$.

The first corresponds to conjugation in the braid group. Set $\mathcal{B}_1 = \{1\}$, and define $\mathcal{B}_\infty = \coprod_{n\geq 1} \mathcal{B}_n$, the disjoint union of all braid groups.

Theorem 3 (Markov) *Two braid representations of the same oriented link differ by a finite sequence of Markov moves and their inverses.*

A very nice proof of this has been recently given by Morton [Mo].

Markov moves generate an equivalence relation on $\mathcal{B}_\infty$, equivalent words corresponding to different braid representations of the same oriented link. An equivalence class is called a *Markov class.* The Jones polynomial can be defined for each element of the braid group, and is constant on equivalence classes. It thus defines an invariant of oriented links. Its definition is via a trace on the *Temperley-Lieb algebra* [TL] arising in the study of Potts models.

Alexander's theorem provides an easy proof of an important property of oriented links.

Proposition 4 *Every oriented link bounds an orientable surface in R^3.*

The simplest way of constructing such a surface is to apply *Seifert's algorithm* to a planar representation: at each crossing, replace according to the scheme of Figure 9. Since no crossings remain, we obtain an embedded closed 1-manifold in the plane – a collection of disjointly embedded, possibly nested, oriented circles. These bound discs, which can be moved slightly in R^3 to be disjoint. For nested circles, consider smaller discs as lying *above* larger ones. At each crossing, we add a half-twisted band to obtain both the original oriented link and an oriented surface bounded by it.

The simplest applications of this algorithm occur when applied to a braid representation of a link. For an n-string closed braid $\hat{\omega}$, we obtain n discs stacked on top of each other, with one half-twisted band for each σ_i occurring in ω. In this case it is easier to see the discs if each is shrunk in the horizontal direction, as in the last picture.

This procedure is similar to that giving rise to line representations of ice-type models in statistical mechanics (Baxter [Ba] p.128).

3 STATISTICAL MECHANICS AND THE STAR-TRIANGLE RELATION

Consider the q-state Potts model. A *configuration* ρ of the system assigns a 'spin' value ρ_i to each i^{th} site). The model specifies the *Hamiltonian* $E(\rho)$ of the configuration: the interaction energy is postulated as $-J\delta_{\rho_i,\rho_j}$ between *adjacent* sites i and j, giving total energy on a lattice with m rows and n columns

$$E(\rho) = -J\sum_{i,j}\delta_{\rho_i,\rho_j},$$

the sum being over adjacent sites. The *partition function* is defined as

$$\mathcal{Z}_{m,n} = \sum_{\rho} exp\{-E(\rho)/kT\} = \sum_{\rho} exp\{K\sum_{i,j}\delta_{\rho_i,\rho_j}\}.$$

Here T is the temperature, and k is Boltzmann's constant. The problem is to describe $\mathcal{Z}_{m,n}$, in terms of m, n, q, and T, in a form from which *macroscopic* properties of the system can be derived, such as the derivative with respect to T of appropriate weighted limits as m, n become large. The introductory chapters of Baxter's book [Ba] provide further details.

Now apply the *transfer-matrix* approach, first introduced by Onsager in his solution of the $q = 2$ (Ising) model. The partition function can be written as a *trace* of a product of matrices V, W, giving (Baxter [Ba])

$$\mathcal{Z} = \xi^t(VWVW\ldots VWV)\xi,$$

where there are m V's, $m-1$ W's and

1. ξ is the q^n-dimensional column vector, all of whose entries are 1,

2. V is the q^n by q^n matrix with indices $\rho = \{\rho_1, \ldots, \rho_n\}$ and $\rho' = \{\rho'_1, \ldots, \rho'_n\}$ (each ρ_i being one of q possible values), and matrix elements

$$V_{\rho,\rho'} = \exp\{K\sum_{j=1}^{n-1}\delta_{\rho_j,\rho'_{j+1}}\}\prod_{j=1}^{n}\delta_{\rho_j,\rho'_j}.$$

3. W is the q^n by q^n matrix with indices $\rho = \{\rho_1, \ldots, \rho_n\}$ and $\rho' = \{\rho'_1, \ldots, \rho'_n\}$, and elements
$$W_{\rho,\rho'} = \exp\{K \sum_{j=1}^{n} \delta_{\rho_j,\rho'_j}\}.$$

Heuristically, by considering a configuration as built up by assigning states to successive rows of a rectangular lattice, that there are only nearest-neighbour interactions suggests the partition function arises by adding the contributions coming from new vertical, and new horizontal interactions. Hence the factorization by transfer matrices.

For the q-state Pott's model, define matrices $U_1, \ldots, U_{2n-1}$ by

$$(U_{2i-1})_{\rho,\rho'} = q^{-1/2} \prod_{j \neq i}^{n} \delta_{\rho_j,\rho'_j} \tag{4}$$

$$(U_{2i})_{\rho,\rho'} = q^{1/2} \delta_{\rho_i,\rho_{i+1}} \prod_{j=1}^{n} \delta_{\rho_j,\rho'_j}. \tag{5}$$

Then setting $v = \exp K - 1$, we obtain

$$V = \prod_{j=1}^{n-1} \{\mathcal{I} + q^{-1/2} v U_{2j}\} \tag{6}$$

$$W = \prod_{j=1}^{n} \{v\mathcal{I} + q^{1/2} U_{2j-1}\}. \tag{7}$$

The matrices U_i define the *Temperley-Lieb algebra*, satisfying

$$U_i^2 = q^{1/2} U_i \tag{8}$$

$$U_i U_{i\pm1} U_i = U_i \tag{9}$$

$$U_i U_j = U_j U_i \qquad \text{for } |i-j| \geq 2. \tag{10}$$

Jones rediscovered the Temperley-Lieb algebra, as an algebra of projection operators in C^*-algebras [Jo1].

Define t by

$$q = t + 2 + t^{-1}, \quad \sqrt{q} = (1+t)/\sqrt{t} = (t^{1/2} + t^{-1/2}).$$

Take new generators
$$\mathcal{T}_i = t^{1/2} U_i - \mathcal{I}.$$

It is a simple matter to verify that these generators satisfy the following

$$\text{relations} : \mathcal{T}_i^2 = (t-1)\mathcal{T}_i + t, \tag{11}$$

$$\mathcal{T}_i \mathcal{T}_{i+1} \mathcal{T}_i = \mathcal{T}_{i+1} \mathcal{T}_i \mathcal{T}_{i+1}, \tag{12}$$

$$\mathcal{T}_i \mathcal{T}_j = \mathcal{T}_j \mathcal{T}_i \qquad \text{if} \qquad |i-j| \geq 2. \tag{13}$$

Note that these relations are weaker than those satisfied by the U_i above. See [Jo2].

At this stage, we can entirely forget the specific matrices $\mathcal{T}_i$ we started with, and think of the abstract algebra defined by these generators and relations just given. The

Hecke algebra $\mathcal{H}_{n,t}$ over a field $F \ni t$. $\mathcal{H}_{n,t}$ has

$$\begin{aligned} \text{generators} &: \; 1, T_1, \ldots, T_{n-1} && (14)\\ \text{relations} &: \; T_i^2 = (t-1)T_i + t, && (15)\\ & \; T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, && (16)\\ & \; T_iT_j = T_jT_i \quad \text{if} \quad |i-j| \geq 2. && (17) \end{aligned}$$

In this way, we sacrifice certain special relations within the algebra of matrices, but enough structure remains to define the 2-variable Jones polynomial — we can now define a specific *representation of the braid group* into the Hecke algebra $\mathcal{H}_{n,t}$. Hecke algebra approaches to the Jones polynomial ([Co], [Jo2], [Le]) make contact with the Kazdahn-Lustig polynomials of reductive group theory, and thus with Kac-Moody algebras and the conformal group.

The existence of a *trace* on the Temperley-Lieb algebra, and Hecke algebra, is important in both statistical mechanics and C^*-algebras, and enables us to define the polynomial invariants of oriented link complements. In statistical mechanics, with q integral, $\mathcal{Z}_{m,n}$ can be determined exactly: Set

$$R = U_1U_3 \ldots U_{2n-3}U_{2n-1} = q^{-1/2}\xi\xi^t.$$

Then for any q^n by q^n matrix X of the matrix algebra generated by the U_i and the identity $\mathcal{I}$, define $\tau(X)$ by $RXR = q^{-n/2}\xi^t X\xi \equiv \tau(X)R$. Thus

$$\mathcal{Z}_{m,n} = q^{n/2}\tau(VWVW\ldots V).$$

This will be some *polynomial* in $q^{\pm 1/2}$. The reader is referred to Baxter's book for further remarks, and to [Jo2].

Now consider the representation

$$\begin{aligned} \psi_t : \mathcal{B}_\infty &\longrightarrow \mathcal{H}_{\infty,t} && (18)\\ \sigma_i &\mapsto T_i, && (19) \end{aligned}$$

where the latter is the limit of Hecke algebras.

A *trace* is a linear functional ϕ satisfying $\phi(AB) = \phi(BA)$. Usually we normalize by setting $\phi(I) = 1$. It is called *Markov trace* if it also satisfies the *Markov property:*

$$\phi(wT_i) = \lambda\phi(w), \quad \lambda = \phi(T_i) \;\; \forall i$$

$$\phi(wT_i^{-1}) = \bar{\lambda}\phi(w), \quad \bar{\lambda} = \phi(T_i^{-1}) \;\; \forall i,$$

for any w in the algebra generated by 1, $T_1, \ldots, T_{i-1}$. Jones verified the Markov property for traces considered in [Jo1].

Polynomial invariants of oriented links arise as follows: For each braid word $\omega \in \mathcal{B}_n$, set

$$V_\omega(t) = (\lambda\bar{\lambda})^{-(n-1)/2}(\bar{\lambda}/\lambda)^{e/2}\phi(\psi_t(\omega)),$$

where e is the exponent sum of the σ_i's in ω. Setting

$$\lambda = -q^{-1/2} = -t^{1/2}/(1+t) = -(t^{1/2}+t^{-1/2})^{-1} = \bar{\lambda},$$

we obtain

Theorem 5 (Jones) *For each n, the assignment $\sigma_i \mapsto T_i$ defines a homomorphism ψ_q from $\mathcal{B}_n$ to $\mathcal{H}_{n,t}$. The assignment $\omega \mapsto V_\omega(t)$ defines a topological invariant for the oriented link $\overline{\omega}$ in S^3, for each $t \in \mathbf{C}$.*

Remark 6 This associates a Laurent polynomial in $u = t^{1/2}$ to every oriented link in S^3. Equivalently, this can be considered a Laurent polynomial in $q^{1/2}$. The reader is warned that in some versions of the Jones polynomial, the roles of t and q are interchanged. Our variable t is traditionally a prime number in the theory of Hecke algebras, and hence denoted q!

The trace above is called a Markov trace, because the polynomial is constant on any Markov class. Note that q *was originally a natural number.*

Schematically, we depict the association of a polynomial to an oriented link as in Figure 10.

There is in fact a *1-parameter family* of Markov traces discovered by Ocneanu, and this gives rise to the *2-variable* Jones polynomial,

$$\omega \quad \mapsto \quad \mathcal{P}_\omega(l,m) \in \mathcal{Z}[l, l^{-1}, m, m^{-1}].$$

Various approaches to this exist [H*], all involving some analogue of the Star-Triangle relation.

The properties of the Markov trace, and algebraic relations, enable us to compute the polynomial of a link by first using a sequence of *crossing changes* to simplify it, and then reconstructing via the recursion or *skein relation*. For the original 1-variable polynomial we find this from the first relation of the Hecke algebra. This can be written as

$$T_i^{+1} + (1-t)T_i^0 + (-t)T_i^{-1} = 0.$$

Correspondingly, for three links $\mathcal{L}_+$, $\mathcal{L}_0$, $\mathcal{L}_-$, which are identical everywhere except at a single crossing, where they differ according to Figure 11, we obtain their polynomials satisfy

$$tV_+ + (\sqrt{t} - 1/\sqrt{t})V_0 - t^{-1}V_- = 0.$$

The 2-variable polynomial can be characterized by such a recursion scheme. Renaming variables, we find

Theorem 7 (H*) *There exists a unique 2-variable Laurent polynomial oriented link invariant satisfying*

1. $l\mathcal{P}(\mathcal{L}_+) + m\mathcal{P}(\mathcal{L}_0) + l^{-1}\mathcal{P}(\mathcal{L}_-) = 0$
2. $\mathcal{P}(\text{trivial knot}) = 1$.

The skein relation allows us to compute $\mathcal{P}(\mathcal{L}_\pm)$ from a knowledge of $\mathcal{P}(\mathcal{L}_\mp)$ and $\mathcal{P}(\mathcal{L}_0)$. The latter two links differ from the original by either the changing of a crossing, or by the removal of a crossing. Removal of all crossings results in a trivial link of unknotted components embedded in the plane, as in Seifert's algorithm. On the other hand, every link can be 'unlinked and unknotted' by judiciously changing some of the crossings. It is no surprise then that a sequence of changes

$$\mathcal{L}_\pm \mapsto \mathcal{L}_\mp \cup \mathcal{L}_0$$

can be chosen resulting in a link embedded in the plane, whose polynomial can be then be determined: using the skein relation allows us to reconstruct $P_{\mathcal{L}_\pm}$.

In Figure 12 we illustrate how the skein relation enables us to compute the 2-variable polynomial of the figure-8 knot. We leave as an exercise for the reader the fact that a trivial link of two components has polynomial $(-1-l^2)/ml$. Thus, with obvious shorthand notation,

$$\begin{aligned} \mathcal{P}_+ &= -l^{-1}\{m\mathcal{P}_0 + l^{-1}\mathcal{P}_-\} & (20)\\ &= -l^{-1}\{m(-l\{l\mathcal{P}_{0+} + m\mathcal{P}_{00}\}) + l^{-1}\mathcal{P}_-\} & (21)\\ &= -l^{-1}\{m(-l\{l(-(1+l^2)/ml) + m\}) + l^{-1}\} & (22)\\ &= -l^{-2} + m^2 - 1 - l^2. & (23) \end{aligned}$$

The 2-variable polynomial reduces to the original 1-variable Jones polynomial, under the substitution

$$l = it, \quad m = i(\sqrt{t} - 1/\sqrt{t}).$$

That Witten's approach recaptures the original Jones polynomial is proved by demonstrating that the skein relation is satisfied. However, Witten's approach gives rise to specific *values* of the polynomial for a given oriented link: for general 3-manifolds it is not yet clear that the numbers obtained derive from an underlying *polynomial*.

Each of the different approaches to the 2-variable Jones polynomial makes contact with different branches of mathematics, each raising intriguing questions. We first describe generalizations arising from statistical mechanics, which at this stage has clearer relationships with Witten's generalization.

Kauffman [Kau1] has given a different 2-variable polynomial, which also admits a specialization to the 1-variable polynomial. His approach is based on a skein relation, but also has an interpretation in statistical mechanics, via state-models. Turaev [Tu1] has described the general relationship between exactly solved models in statistical mechanics and polynomial invariants of links. Given a solution of the Yang-Baxter equations, and an appropriate Markov trace, an invariant can be defined just as has been done above. Turaev shows how a class of solutions relating to the simply-laced simple Lie algebras gives rise to the 2-variable Jones and Kauffman polynomials. A more categorical description is given in [Tu2].

Proceeding along similar lines, the Japanese school has independently found an infinite hierarchy of distinct 2-variable polynomials. These invariants correspond to the N-state vertex models, but admit an interpretation closely related to the work of Rehren and Schroer [RS] in 2-dimensional quantum field theory, cited in Witten [Wi]. In [AW], Akutsu, and Wadati describe a procedure for producing new braid-group representations from old, by a process they call *fusion*:

Suppose $\sigma_i \mapsto T_i$ is a representation into the 'Temperley-Lieb' algebra. Construct a new representation

$$\sigma_i \mapsto G_i = P_i P_{i+1} T_{2i} T_{2i-1} T_{2i+1} T_{2i} P_i P_{i+1}$$

where $P_i = (1+t)^{-1}(t + T_{2i-1})$. The original trace gives rise to a Markov trace on this new operator algebra, and gives a new polynomial invariant of oriented links with skein relation

$$\mathcal{P}_{++} = t(1 - t^2 + t^3)\mathcal{P}_+ + t^2(t^2 - t^3 + t^5)\mathcal{P}_0 - t^8\mathcal{P}_-,$$

where the subscripts refer to the nature of the crossings, as done above.

The construction is depicted schematically in Figure 13, and can be interpreted as replacing each strand of a braid by $N-1$ strands, where in this case $N=3$. A similar explicit construction of new representations can be given for each natural number $N \geq 2$. The P_i correspond to projection operators onto the highest-spin states of a tensor product of representations, and the representations themselves correspond to solutions of the Yang-Baxter equation for the N-state vertex models. For each N, they obtain a 2-variable polynomial link invariant, which for $N=2$ is the Jones polynomial. The $N=3$ case is interesting, being *closely related* to the Kauffman polynomial, having a common 1-variable specialization.

Jones' original polynomial arose from a C^*-algebra associated to Hecke algebras of type A_n. It is natural to reverse the process and ask whether Kauffman's polynomial arises via a representation of the braid groups into some *other* C^*-algebra. Birman and Wenzl [BW], and Murakami [Mur], succeed in this, obtaining an algebra with intriguing properties. It is a *deformation* of an algebra investigated by Brauer in the 30's, just as the Hecke algebras are deformations of the group algebra $\mathbb{C}S_n$ of the symmetric group. Moreover, there are two distinct homomorphisms of their algebra onto Jones' C^*-algebra.

4 COMPUTATIONS, INTERPRETATIONS AND APPLICATIONS

The 1-variable Jones polynomial is a specialization of the 2-variable polynomial. Another substitution gives the classical *Alexander polynomial* $\Delta_K(t)$ of a knot K: set

$$l = i, \quad m = i(\sqrt{t} - 1/\sqrt{t}).$$

The original Jones polynomial differs considerably from the Alexander polynomial, but the original 2-variable polynomial is best considered a *generalisation* of both the 1-variable Alexander polynomial, and 1-variable Jones polynomial.

Alexander's invariant has been a major tool over the decades in the understanding and classification of knots. Accordingly we briefly compare some of what is known about the Alexander and Jones polynomials. Good references for the Alexander polynomial are Rolfsen [Rol], and Burde–Zieschang [BZ]. These comparisons are intended to demonstrate the wealth of interesting problems raised by Jones' new polynomials.

1. **Calculation by skein relations:** Clearly both polynomials are on equal footing from his viewpoint. For knots with not too many crossings (less than 20, say!), polynomials can be calculated on a computer using the skein relation. This enables a sharpening of the classification of such knots, but the number of steps in the calculation grows too rapidly to be of great practical use, at least with current understanding.

2. **Direct combinatorial approaches:** Given a projection of an oriented knot, choose an initial point, and label the beginnings of the n over-arcs cyclically, as in Figure 14 for the figure-8 knot. At a crossing, an arc labelled i passes under some k^{th} arc. There are two ways this can occur, depending on the orientation of the k^{th} arc. These are indicated in Figure 15. Encode this data directly as the entries of a matrix M, taking as entries for the i^{th} row

$$M_{ij} \;=\; (1-t^{\epsilon_i}) \qquad \text{if j = i} \tag{24}$$

$$= \quad -1 \qquad \text{if } j = i + 1 \qquad (25)$$
$$= \quad t^{\epsilon_i} \qquad \text{if } j = k \qquad (26)$$
$$= \quad 0 \qquad \text{otherwise.} \qquad (27)$$

Delete any corresponding row and column. The determinant of the resulting matrix is the Alexander polynomial, defined up to a factor $\pm t^s$. No similar interpretation is known for the Jones polynomial.

3. **Module structures:** The matrix just constructed actually gives a *presentation matrix* for the first-homology of the 'infinite cyclic cover' of the knot [Mi], with coefficients over $Z[t, t^{-1}]$. Thus the Alexander polynomial is part of a more complex *module invariant* of a link. No such interpretation is known for the Jones polynomial.

4. **Cyclic covers and torsion invariants:** This module structure has a direct topological interpretation, in terms of the cyclic covers of the complement of the knot. This is one of the standard constructions of algebraic topology. For the Jones polynomial, certain specializations give numbers with a covering space interpretation, but a full understanding is lacking.

5. **Higher dimensional generalizations:** The Alexander invariants can be defined for knotted n-spheres in the $(n+2)$-sphere. No such interpretation is known for the Jones polynomial.

6. **Seifert surfaces:** If we cut open the complement of a knot along a Seifert surface, we can construct the cyclic covers by glueing copies of the resulting space along copies of the surface. This enables the Alexander invariants to be computed from a Seifert surface. The intermediate *Seifert matrix* V is constructed, whose entries are the linking numbers of circles on the surface representing a basis for its first homology group. Then
$$\Delta_K(t) = det\{V - tV^t\}.$$
No such interpretation is known for the Jones polynomial.

7. **Fibred knots:** An n-strand braid can be considered to lie inside a solid torus $S^1 \times D^2$. Each disc is punctured by the braid in n distinct points. If such a punctured disc is moved once around the solid torus, returning to its original position setwise, the resulting homeomorphism is generally non-trivial. This can be considered one of the most concrete examples of a non-trivial fibre-bundle over the circle. A *fibred knot* is a knot whose complement fibres over the circle. (These exist in abundance.) The Alexander polynomial of a fibred knot is the characteristic polynomial of the induced Z-linear map on the first homology of the fibre under the monodromy of the fibration. No such simple interpretation is known for the new polynomials.

8. **Seifert forms and signatures:** By considering the 3-dimensional sphere as the boundary of a 4-dimensional ball, we immediately see that every knot bounds an embedded surface in the 4-ball. (Just push a Seifert surface into the interior of the ball.) If a knot bounds an embedded disc in S^3, it is unknotted, but there are many examples of non-trivial knots bounding an embedded disc in the 4-ball. The simplest obstruction to this is the *signature* of the knot. Robertello [Rob]

has given an expression in the coefficients of the Alexander polynomial, whose mod-2 reduction gives such an obstruction, the Arf invariant. No comprehensive interpretation is known for the Jones polynomial, but the Arf invariant can be derived from Jones' 1-variable polynomial.

9. **Casson's invariant:** We will shortly describe the process of *surgery* on a knot K in S^3, to produce interesting 3-manifolds. Robertello's invariant has an interpretation in this context as the $Z/2Z$-valued *Rohlin invariant* of the homology 3-sphere obtained by +1-surgery on K. Casson has recently shown that by 'counting' the number of flat connections on trivial $SU(2)$-bundles over a homology sphere H^3, one can concoct a Z-valued invariant $\lambda(H^3)$, whose reduction modulo 2 is the Rohlin invariant. Moreover, this can be calculated from the Alexander polynomial in the case where H^3 is obtained by surgery on a knot.

   It is believed there are deep connections between Casson's invariant and the Jones polynomial, particularly since Witten has given an interpretation of Casson's invariant, in terms of topological quantum field theory, Floer homology groups and the Donaldson polynomials of smooth simply-connected 4-manifolds [At].

10. **Satellites and cables:** Torus knots are those which can be drawn as an embedded circle on an unknotted torus. If the torus is now tied in a knot K', the original torus knot forms a *cable knot* K around K'. The Alexander polynomial of K' can be computed in terms of those of the original torus knot and K'. Analogous formulae are not known for Jones polynomials, but a simple result cannot be expected, due to the negative result of Morton and Short [MS], who show that two cable knots obtained by cabling in the same way about two different knots with the same Jones polynomial, can have very different Jones polynomials. Very recent work of Morton shows how the Witten invariants of a cable knot are determined by those of the ingredient knots, but where contributions arise from the various possible representation assignments to constituent components.

11. **Existence results:** There exist knots K with

    - $\Delta_K(t)$ trivial, but $V_K(t)$ non-trivial,
    - different $\Delta_K(t)$, but the same $V_K(t)$,
    - the same $\Delta_K(t)$, but different $V_K(t)$
    - the same $\Delta_K(t)$, the same $V_K(t)$, but different $\mathcal{P}_K(l,m)$
    - the same $\mathcal{P}_{K'}(l,m)$, but whose (2,1)-cables have different $\mathcal{P}_K(l,m)$.
    - It is *unknown* whether a knot exists with either $V_K(t)$ or $\mathcal{P}_K(l,m)$ trivial.

12. **Band-sums:** If $K_1 \# K_2$ is the band-sum of K_1 and K_2, then

$$\mathcal{P}_{K_1 \# K_2}(l,m) = \mathcal{P}_{K_1}(l,m).\mathcal{P}_{K_2}(l,m)$$

13. **Disjoint unions:** If K_1 and K_2 are links contained in disjoint balls, then

$$\mathcal{P}_{K_1 \coprod K_2}(l,m) = -(l+l^{-1})/m.\mathcal{P}_{K_1}(l,m)\mathcal{P}_{K_2}(l,m)$$

14. **Orientation sensitivity:** If orientations are reversed for every component of a link, $\mathcal{P}_K(l, m)$ remains unchanged.

15. **Symmetry properties:** If we reflect an oriented knot L in a mirror, taking the image L^* with its consequent orientation, the Alexander polynomial remains unchanged, but $\mathcal{P}_{L^*}(l, m) = \mathcal{P}_L(l^{-1}, m)$

16. **Braid representations:** The Jones polynomial has been successfully used to give bounds on the *braid-index* of a knot K, which is the least n for which $K = \bar{\omega}$, $\omega \in \mathcal{B}_n$.

17. **Alternating knots. Tait Conjectures:**

    There are some longstanding conjectures of knot theory, made last century, which have finally been resolved affirmatively. This is at this stage perhaps the most significant application of the Jones polynomial. These Tait Conjectures are concerned specifically with *alternating knots and links.* An alternating link is one admitting a projection to the plane with respect to which, following around any component, the crossings are alternatively over/under/over/ etcetera, as in Figure 9. Murasugi (see references in [BZ]) has in the past shown for alternating knots that there are very strong connections between natural Seifert surfaces, the structure of the Alexander polynomial, and whether such knots have complement fibering over the circle. Kauffman, Murasugi and Thistlethwaite have now independently proved that *if K is an alternating knot, then any (reduced) alternating projection has the least number of crossings of any projection.* Their approach involves a careful analysis of the breadth of exponents, and non-vanishing of coefficients, of the Jones polynomials. These in turn depend on the *number of Seifert circles* appearing in the application of Seifert's algorithm in the construction of a Seifert surface, manifesting also in the length of reduction to trivial links by the skein relation. Their results have a flavour similar to those on the braid index.

The last two applications emphasize that the Jones polynomial appears to carry information about 2-dimensional aspects of links: it appears to have bearing on questions related to the existence of *projections of a knot to the plane with some specialized properties.*

5 THE CONSTRUCTION OF 3-MANIFOLDS

We describe standard constructions of 3-manifolds, involving surfaces and links. Rolfsen [Ro] is a good reference.

5.1 Heegard Splittings

A *handlebody* H_n is a 3-dimensional ball with n handles attached, for some $n \geq 0$. Observe that the closed orientable surface of genus n occurs as the boundary ∂H_n.

A useful viewpoint is obtained by considering the effect of adding another handle to H_n. '1-handle addition' involves taking an interval I, thickening to obtain a '1-handle' $I \times D^2$, and then identifying the two discs $\partial I \times D^2 = S^0 \times D^2$ with disjoint discs on ∂H_n. The new surface is obtained by removing $S^0 \times D^2$, and replacing it with the annulus $I \times S^1 = I \times \partial D^2$. (Figure 16.)

Theorem 8 *Every closed orientable 3-manifold M^3 can be obtained as the union of two handlebodies, glued together by a diffeomorphism of their boundaries.*

This is called a *Heegard decomposition*, and the surface is called a *Heegard surface*. The proof is to first take some triangulation of M^3 (such can be proved to exist!), and to take a thickened neighbourhood of the 1-skeleton. This is a finite number of balls at the vertices, together with neighbourhoods of the edges, manifesting as 1-handles attached to these balls. The complementary handlebody arises by taking balls at the centres of tetrahedra, connected by 1-handles running through the centres of common triangular faces. (Figure 17.)

An explicit example of a Heegard decomposition is shown in Figure 18. To set the scene, note that the 2-sphere can be stereographically projected onto the plane, the North pole corresponding to the 'point at infinity'. The unit disc in R^2 corresponds to the Southern hemisphere, and the exterior of the unit disc, together with the point at infinity, corresponds to the Northern hemisphere. In the same way, the 3-sphere S^3 can be viewed as $R^3 \cup \infty$, the two hemispheres corresponding to a ball centred at the origin, and the ball obtained from the exterior by adding a point at infinity.

It is easy to see that a ball with n unknotted holes drilled through it gives a handlebody of genus n. If we drill such holes through the ball at the origin of R^3, the complement of the resulting handlebody is also a handlebody.

5.2 Surgery Presentations

All closed orientable 2-manifolds arise as the boundary of a 3-ball with 1-handles attached. Similarly, we can consider the effect of adding 2-handles to a 4-dimensional ball.

'Attaching a 2-handle' means: take a disc D_1^2, and thicken it to 4-dimensions by taking the product $D_1^2 \times D^2$. The boundary splits as the union $(\partial D_1^2 \times D^2) \cup (D_1^2 \times \partial D^2)$, each of which is a solid donut (compare Figure 18 with only one hole drilled).

Now take an embedded S^1 (a knot K) in $S^3 = \partial B^4$, and identify $\partial D_1^2 \times D^2$ with a solid-donut neighbourhood N of the knot. The result is a 4-manifold, with boundary obtained by *surgery on the knot K*. Observe that N is now in the interior, and $D_1^2 \times \partial D^2$ is part of the new boundary. (Compare Figure 16.)

Attaching a 2-handle to a 4-manifold with boundary requires

- Specifying a knot along which attachment is to occur
- Specifying an integer, defining how the two solid tori $S^1 \times D^2$ are to be identified.

The latter integer specifies how many times one of the solid tori twists relative to the other as we move around the circle. This corresponds to taking a solid donut, cutting it along a disc to obtain a ball with two 2-discs on the boundary, and regluing after some number of twists, as in Figure 19. It can be equivalently characterized in terms of a *framing circle*, lying on the boundary torus of a solid tubular neighbourhood of the knot. A Seifert surface intersects such a torus in an embedded circle, a longitude of the knot, which we take as '0' (Figure 20). The integer n then characterizes the circle $\lambda + n\mu$, in terms of the longitude and meridion.

This construction may be carried out on a *framed link*: take any collection of disjointly embedded circles, each assigned some integer, and attach a 2-handle to each. This procedure, resulting in a new 3-manifold, gives rise to the *surgery representation* of 3-manifolds.

Theorem 9 (Lickorish; Wallace) *Each closed orientable 3-manifold is the boundary of a 4-ball with 2-handles attached. Thus every such 3-manifold admits a representation as surgery on a framed link in S^3.*

This was proved using different methods by Lickorish [Li1] and Wallace [Wa]. Wallace's technique is to observe that every M^3 arises as the boundary of some orientable 4-manifold W^4, and then to modify the interior of W^4 so that the resulting 4-manifold is built by adding thickened disks $D^2 \times D^2$ to the boundary of the 4-ball.

That every closed, orientable 3-manifold bounds – the oriented cobordism group Ω_3 vanishes – was first proved by Rohlin: Whitney has shown that every orientable manifold N^k *immerses* in R^{2k-1}, and so every M^3 immerses in R^5. Generically, the self-intersection set will be a link of double points in R^5 with preimage a link in M^3. Cutting and pasting along the link in R^5 gives rise to an embedded 3-manifold, which can be easily shown to be oriented-cobordant to the original.

Alternative proofs of Rohlin's result exist. Recently, Rourke [Rou] has given an extremely elementary proof based on the existence of a Heegard splitting, and trivial properties of intersections of closed curves on a surface. Lickorish, on the other hand, gives a description of finitely-many generators for the mapping class group of a closed orientable surface. These are Dehn twists on the closed curves a_i, b_j and c_k of Figure 21, where by Dehn twist, we mean *cut the surface open along an embedded circle, give a full rotation through $\pm 2\pi$, and reglue*, as occurs on the boundary in Figure 19. Now start with the standard Heegard splitting of S^3, and observe that a Dehn twist along some curve on the Heegard surface can be achieved by surgery on the curve, viewed as a knot in S^3. A sequence of such surgeries then corresponds to a diffeomorphism of the surface, which effectively alters the way in which the two handlebodies are to be glued together. All 3-manifolds are so obtained, and moreover, the interpretation of integral surgeries as attaching maps for 2-handles attached to the 4-ball proves the vanishing of Ω_3.

The Rohlin invariant of a homology 3-sphere H^3 is defined via a 4-manifold with *spin structure* bounded by H^3.

Theorem 10 *Every closed orientable 3-manifold has a trivial tangent bundle. Moreover, the spin cobordism group $\Omega_3(Spin)$ vanishes.*

That every 3-manifold M^3 with spin structure arises as the boundary of a spin 4-manifold, is due to Milnor, using techniques of algebraic topology. A more geometric proof, using the Kirby-Craggs calculus to modify a given link presentation of M^3 to make all components evenly framed, is due to Kaplan. References and an elementary direct proof, based on Rourke's approach, can be found in Aitchison [Ai1].

5.3 Fibre Bundles over the Circle

Every closed orientable 3-manifold is 'close' to being a surface bundle over the circle. To describe such manifolds, we first consider Thurston's description of diffeomorphisms of surfaces.

For a diffeomorphism $\phi : F \mapsto F$ of a closed surface F, Thurston [Thu1] proves that there is a unique decomposition of F, up to isotopy, by a finite collection of disjointly embedded non-trivial circles C_i, set-wise invariant under ϕ. For simplicity, suppose each circle is actually preserved.

On the complement of the C_i , ϕ is isotopic to a map which is either *periodic*, or *pseudo-Anosov*. In the former case, *every* non-trivial homotopy class of loops in F is preserved under *some* finite iteration of ϕ, whereas pseudo-Anosov maps are characterised by the existence of *no* loop left invariant by *any* non-trivial power of ϕ.

Consider now the mapping torus of ϕ, a closed 3-manifold M^3 fibering over S^1 with fibre F. We construct this by taking the cartesian product $F \times [0,1]$, and identifying $x \times \{0\}$ with $\phi(x) \times \{1\}$. The cylinders $C_i \times [0,1]$ become embedded tori, the components of F on which ϕ is periodic giving rise to *Seifert-fibred* components of M^3, and the pseudo-Anosov components giving rise to the *simple* pieces of M^3. Such a decomposition of a 3-manifold into simple and Seifert-fibred pieces is the prototypical *torus decomposition* of 3-manifolds, known to exist in more general circumstances. Seifert-fibred manifolds are unions of circles, and many simple manifolds admit a metric of constant curvature -1.

Stallings [St] has proved that every alternating braid has complement fibering over the circle, with fibre the Seifert surface obtained by the prescription given earlier. Hence if we perform surgery on an alternating braid, using the framing defined by this surface, we obtain a closed 3-manifold fibering over the circle. We call this *fibred surgery*.

Of course, not every 3-manifold can be obtained by fibred surgery, since many 3-manifolds do not fibre over the circle. However, if we take an alternating n-braid $\bar{\omega}$, with fibre surface F_0, consider $\omega^* = \omega\sigma_n^{\pm 2}$. This defines an $(n+1)$-braid, which is also alternating, with one extra component, a 'meridional circle' to one of the original components. Now frame this new link, using the original braid surface to determine framings for the original components, and assign framing 0 to the new component. Call this *$*$-fibred surgery*. (Figure 22.)

Theorem 11 (Aitchison) *Every closed orientable 3-manifold can be obtained by $*$-fibred surgery on infinitely many distinct alternating braids.*

This is proved [Ai2] by taking any framed link representing M^3, putting it in braid form, threading a new component around the braid (a process initiated by Stallings [St]) to make it alternating, and so that the braid surface meets each component in the correct framing. Finally, add the 0-framed meridional circle linking this new component. That this gives a new surgery description of the same 3-manifold, using elementary notions from the calculus of framed links ([Rol]).

Corollary 12 (Myers; Gonzales-Acuña) *Every 3-manifold can be obtained by surgery on a section of a surface bundle over the circle.*

Motivation for specializing to *alternating* braids is that in addition to the complements of the corresponding links fibering over the circle, *the complements of such links admit a complete metric of constant curvature* -1.

For example, the figure-8 knot can be represented as an alternating braid, with fibre a punctured torus. The monodromy on the torus – the effect on the homology after pushing the fibre once around the fibration – has matrix representation $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. *The characteristic polynomial of the monodromy of a fibred knot is the Alexander polynomial* *of the knot*. So this is perhaps the most graphic, comprehensible interpretation of the Alexander polynomial. What is the interpretation of the Jones' polynomials? No one knows at this stage.

Consider the monodromy of the figure-8 knot. Being a symmetric matrix, there are two orthogonal eigenspaces with eigen-values $\lambda^{\pm 1} = 3/2 \pm \sqrt{5}/2$. This gives two orthogonal foliations of R^2 by families of lines parallel to the eigenvectors. These descend to foliations of the torus, each invariant under the monodromy, but with one dilated by λ, the other by λ^{-1}. The constant λ, the larger of the two eigenvalues, is the *stretching factor* of the monodromy.

More generally, Thurston [Thu2] has shown that a surface bundle over the circle, with fibre of negative Euler characteristic, admits a complete metric of constant curvature -1 iff the monodromy is *pseudo-Anosov*. This means there are two *unique* transverse invariant (measured) foliations, necessarily with singularities if the genus is greater than 1, one of which is dliated by λ, the other by λ^{-1}, for some $\lambda > 1$. Again, this is called the stretching factor. Thurston has proved that in some cases—when there is an 'orientable train-track'—the stretching factor of a fibred hyperbolic knot in S^3 is again the largest positive eigenvalue of the Alexander polynomial. Hence the Alexander polynomial contains delicate *geometric* information. On the other hand, we mention [Ai3,4]:

Theorem 13 (Aitchison) *There is an infinite class of knots* $\{K_i\}$*, satisfying* (i) *All* K_i *have the same Alexander invariants as* $4_1\#4_1$, (ii) *All* K_i *are ribbon knots,* (iii) *All* K_i *are obtained as the intersection of an* unknotted *2-sphere in the 4-sphere with an equatorial 3-sphere,* (iv) *All* K_i *are* symmetric *intersections of the 0-spun* 4_1*-knot* (v) *All* K_i *are fibred knots, of genus two,* (vi) *All* K_i *have monodromy arising from the isotopy of a genus 2 handlebody within the 3-sphere. Hence the monodromy extends over a handlebody,* (vii) *All* K_i *have pseudo-Anosov monodromy, endowing the complements with metrics of constant curvature* -1, **but** (viii) *All* K_i *have different stretching factors.*

These cannot all have orientable train tracks, and the stretching factor is buried more deeply than in the Alexander polynomial alone. Silver and Hitt [HS] have recently shown that some of these examples can be distinguished by their (1-variable) Jones polynomials. We ask: *For a surface bundle over* S^1*, can one extract the stretching factor of a pseudo-Anosov map from the Jones polynomial?*

Although Witten's approach to Jones polynomials gives *topological information*, being defined independently of any metric, we are also obtaining *geometric* information.

Thurston's Geometrization Conjecture: *Every closed orientable 3-manifold can be cut up canonically by a collection of embedded 2-spheres and tori into pieces admitting a metric with geometry modelled on one of the eight 3-dimensional homogeneous structures,* S^3, H^3, E^3, $H^2 \times R$, $S^2 \times R$, $SL(2; R)$, Nil, and Sol. (These are characterised by having transitive isometry groups.)

Hence we expect to find all sorts of geometric information buried in topologically-invariant polynomials, and vice-versa. Canonical pieces M_i with homogeneous metrics arise from discrete representations of the fundamental groups $\pi_1(M_i)$ into the corresponding isometry group of the model structure.

As another remark in this vein, observe that corresponding to a closed hyperbolic surface bundle over S^1, there are numerous natural *links* arising from the fixed points of the monodromy.

Any diffeomorphism can be isotoped so that it fixes an open ball pointwise. For a pseudo-Anosov diffeomorphism ρ, existence of the invariant singular foliations means this pathology does not occur, and in fact, such a map has the least number of fixed points in its isotopy class. These are well-defined by uniqueness of the invariant foliations.

Accordingly, for each integer $n \geq 1$, there is a well-defined set of fixed points of ρ^n. These give rise to canonical families of of links $\mathcal{L}_n$ in M^3. Clearly such links also exist in the periodic case, and correspond to the singular fibres of the Seifert fibration.

It would be interesting to know both how the polynomials of these links relate to those of the closed manifold, and how these give information about geometrical invariants. In this vein, we mention that results of Fried [Fri] and Franks [Fra] show how considerations of the homology of closed orbits in the dynamical system corresponding to the fibration, give rise to the Alexander polynomial.

There are interpretations of Thurston's work on pseudo-Anosov diffeomorphisms in the language of quadratic differentials, and in terms of the geometry of Teichmüller space under the action of the mapping-class group. That relationships with conformal string theories should exist is obvious.

5.4 Branched Covers over the 3-Sphere

There is another role knots and links play in representing all possible closed orientable 3-manifolds, which is in the construction of *branched covers over links in* S^3.

Recall that the 2-dimensional torus T^2 is a 2-fold branched cover over 4 points on the 2-sphere S^2 (Figure 23): symmetrically skewer a donut, and identify points under a rotation by π. The result is the 2-sphere.

Taking the 'product' of this construction with S^1, we obtain the 3-dimensional torus $T^2 \times S^1$ as a branched cover of $S^2 \times S^1$, branched over 4 circles.

Theorem 14 *Every closed orientable 3-manifold can be obtained in infinitely many different ways as*

- [Alexander] *a branched cover of* S^3*, branched over a link,*
- [Montesinos; Hilden] *a 3-fold branched cover of* S^3*, branched over a knot,*
- [Aitchison] *a 3-fold branched cover of* S^3*, branched over an alternating braided knot.*

There is some merit in branching over a fibred link. In this context, we mention that any genus-2 Riemann surface is *hyperelliptic*, i.e. has a symmetry exactly as for the torus above, but with branch set being 6 points in the 2-sphere. It follows that any 3-manifold $M^3 \cong F^2 \times_\phi S^1$ fibering over S^1, with fibre a genus-2 surface with monodromy ϕ is a 2-fold branched cover over a closed 6-braid in $S^2 \times S^1$. Jones has exploited this to obtain polynomial invariants of diffeomorphisms of genus-2 surfaces, via representaions of the braid group of 6 points on S^2. Such invariants give an alternative approach to distinguishing the knots mentioned in the previous section, but more importantly, raise the question: *How do the invariants of Witten and Jones compare for a genus-2 surface bundle over the circle?*

More recently, the existence of *universal* knots and links has emerged for branched-cover constructions [HLM]:

Theorem 15 (Hilden-Lozano-Montesinos) *There are infinitely many knots K in S^3 such that* every *closed orientable 3-manifold is* some *branched cover over K.*

Example of universal knots are the 5_2-knot, and figure-8 knot 4_1. The existence of universal links is again due to Thurston. Note that there is no information given about the possible degree of the cover, or degrees of the branch set. Again we ask, what are Jones polynomials telling us about such constructions? How do we compute?

5.5 The Cube and Singular Geometry

One reason for mentioning this is to bring us back to the humble cube. Recall that a 2-dimensional square, with edges pairwise identified, gives rise to a torus, Klein bottle or projective plane. There is a well-known description of the 3-dimensional torus, as the result of identifying opposite faces of a cube. Other 3-manifolds also arise by different identifications:

Proposition 16 *The knot 5_2 arises from two edges of a* single cube, *folded up to give S^3.*

Thus, in a sense, every 3-manifold arises from a single cube, since 5_2 is universal. In terms of geometry, we mention a piece-wise linear result on curvature and homotopy type [AR1]:

Theorem 17 (Aitchison-Rubinstein) *If M^3 can be obtained as a branched cover over the knot 5_2, with all branching degrees ≥ 4, then any N^3 homotopy equivalent to M^3 is actually homeomorphic to M^3.*

If each cube is given its usual geometric structure, the degree condition guarantees no angle deficiencies along edges or at vertices: such manifolds behave sufficiently like manifolds of negative curvature.

6 A BIZARRE CONCLUSION

Having indicated how a single cube, interpreted as in the last section, gives rise to all possible closed 3-manifolds, it is fitting to return full-circle to the beginning of the talk, and describe how the hexagon also plays a role. We saw that the braid relation corresponds to 2-colouring the edges of a hexagon, adjacent edges having different colours.

Of course the regular hexagonal tiling of the *Euclidean* plane, beloved of statistical mechanics, *cannot* be 2-coloured consistently in this way. However,

Proposition 18 *The hyperbolic plane can be tesselated by regular, right-angled hexagons, with edges consistently 2-coloured.*

The Poincaré disc model $\mathcal{H}^2$ for the hyperbolic plane is the interior of the unit disc in R^2, with geodesics being arcs of circles meeting the unit circle at right-angles. The tesselation obtained is the underlying symmetric pattern of M.C. Escher's *Heaven and Hell* (often known as Circle Limit IV).

Consider the group Γ of symmetries of this pattern, considered as a discrete group of isometries of the hyperbolic plane. All edges of hexagons fit together to produce two families of disjoint geodesics, coloured respectively red and black (say). If we take any orientation preserving, fixed-point free subgroup $H < \Gamma$ of finite index, the quotient surface $F_H = \mathcal{H}^2/H$ inherits two families of disjoint simple closed curves, dividing the surface into 2-coloured hexagons, as in Figure 24. We construct a possibly singular 3-dimensional space: Take $F_H \times [0,1]$, adding thickened discs $D^2 \times I$ along annular neighbourhoods of one family of curves in $F_H \times \{0\}$, and to the other family in $F_H \times \{1\}$. This gives a 3-dimensional manifold with a number of boundary components, each of which we cone off to a different point. The result is a closed 3-manifold M^3_H iff all boundary components are 2-spheres, but otherwise has singular points. Such a 3-manifold will be said to be *covered by Heaven and Hell* ([AR2]).

Theorem 19 (Aitchison & Rubinstein) *Fibre bundles over the circle is covered by Heaven and Hell in infinitely many ways.*

Hence, in a precise sense, 'Heaven and Hell' lies over all possible 3-dimensional manifolds! A number of questions arise from this construction.

- There is a 'canonical' construction of pseudo-Anosov diffeomorphisms arising from the curve data on F_H, giving infinitely many numerical invariants $\mathcal{I}_H$ of this combinatorial structure of M_H, all being stretching factors. If $M_H \cong M_G$, how are $\mathcal{I}_H$ and $\mathcal{I}_G$ related?
- Do these numbers arise as values of some underlying polynomials?
- Do topological invariants of M_H arise in this fashion?
- The symmetries of Heaven and Hell are related to right-angled Coxeter groups, and it is natural to ask: is there is any route to polynomial invariants via any associated Hecke algebra constructions?

Added in proof: V. Turaev and N. Reshetikhin of the Steklov Institute, Leningrad, have found a rigorous description of (some of) Witten's invariants of 3-manifolds, directly from surgery descriptions.

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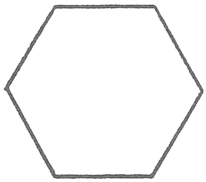
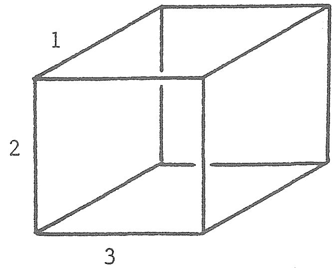
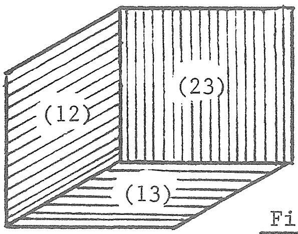
Figure 1.







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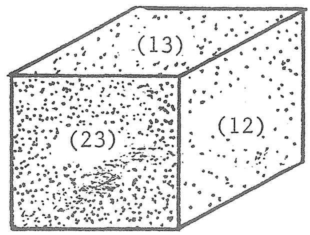

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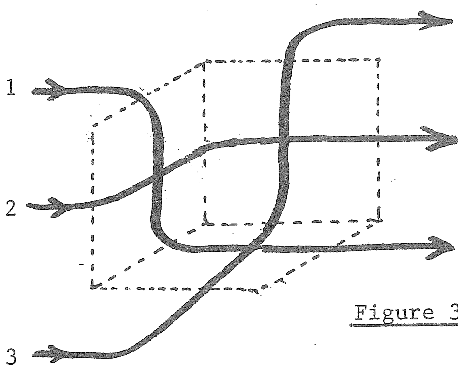

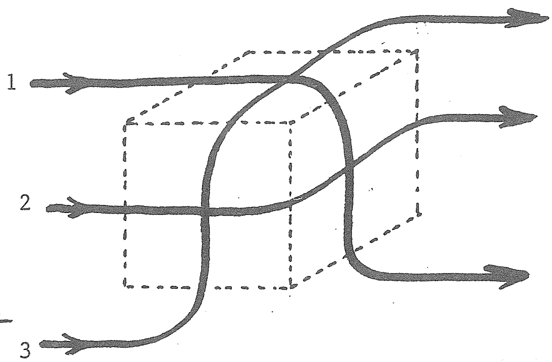

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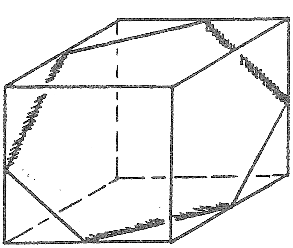

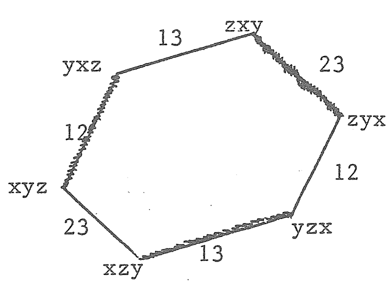

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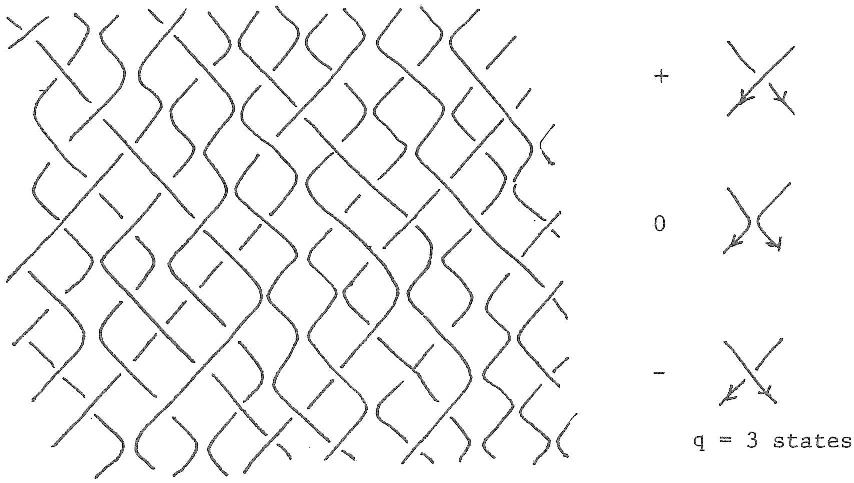

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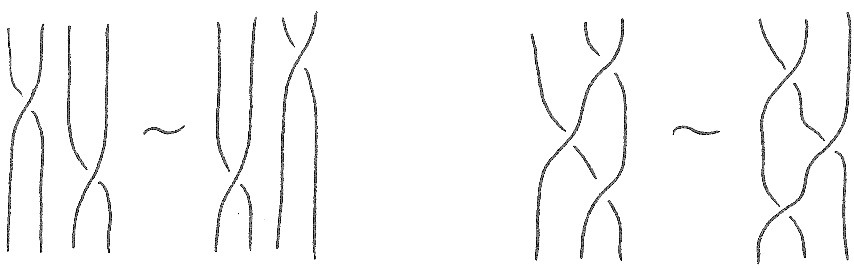

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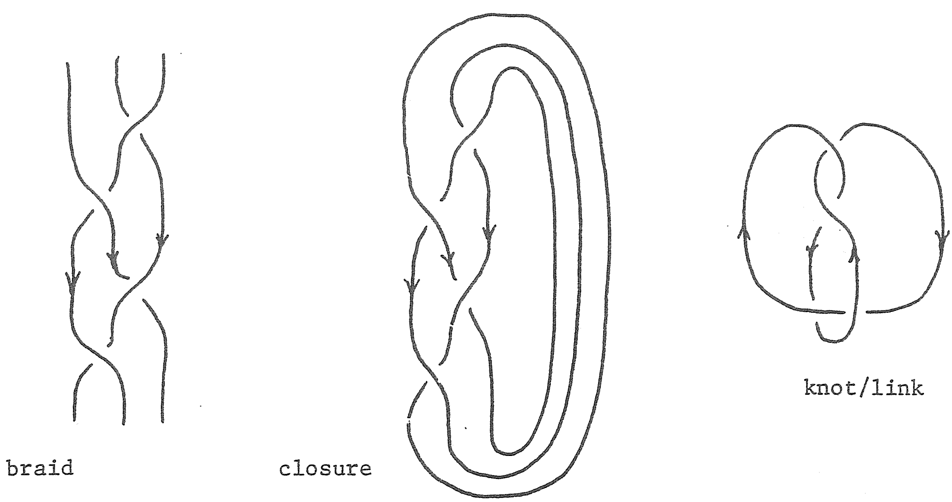

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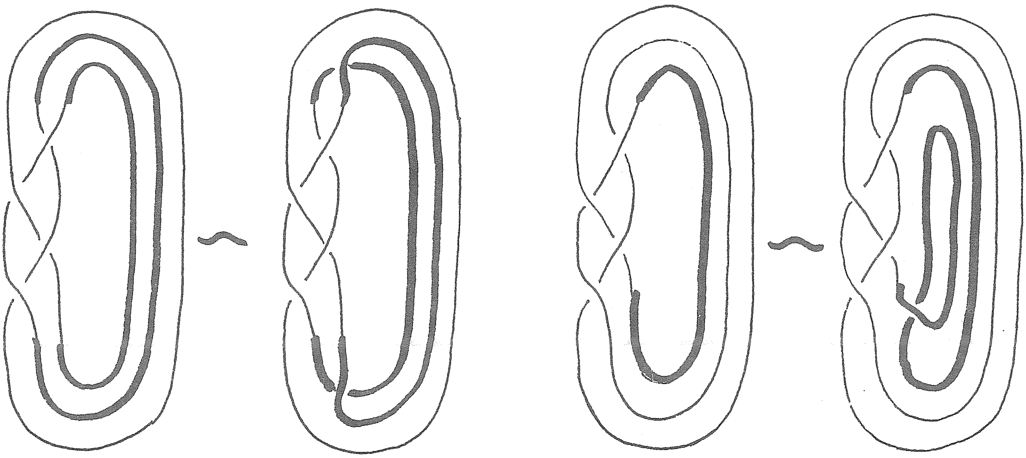
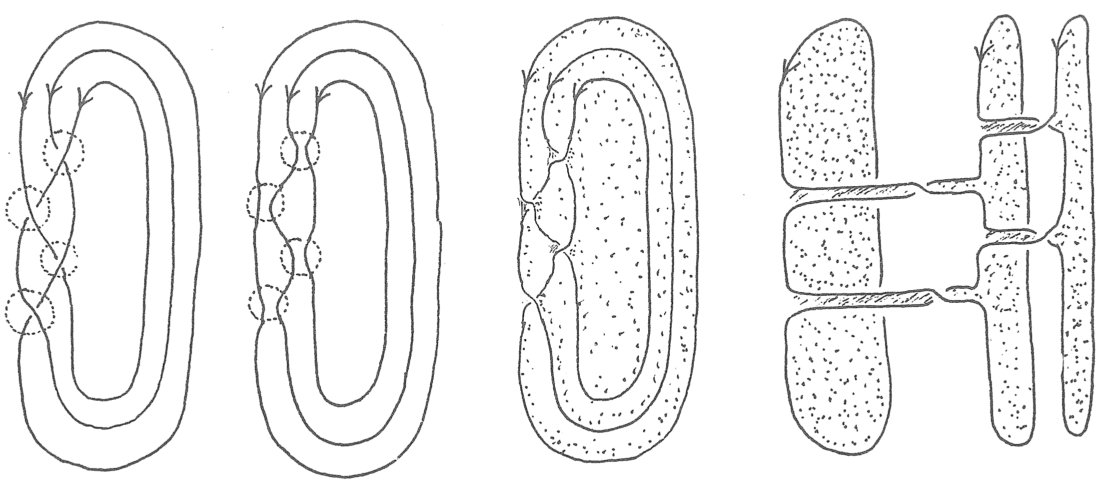


Markov move I

Markov move II

Figure 8

Figure 9



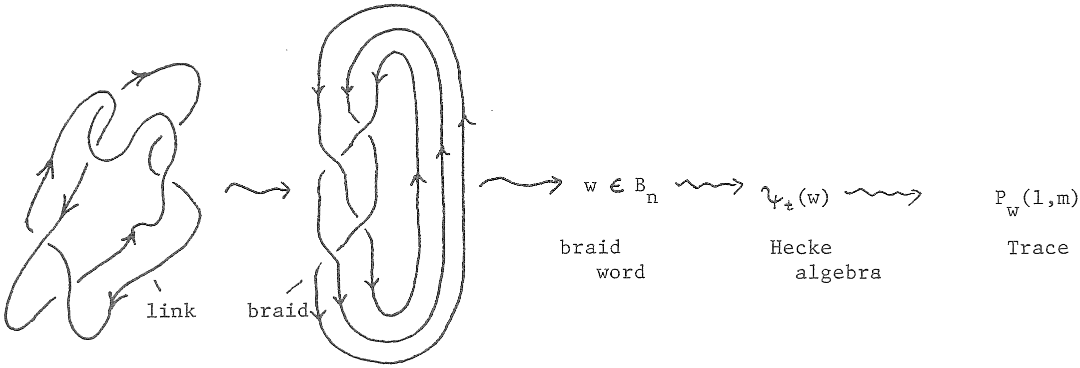
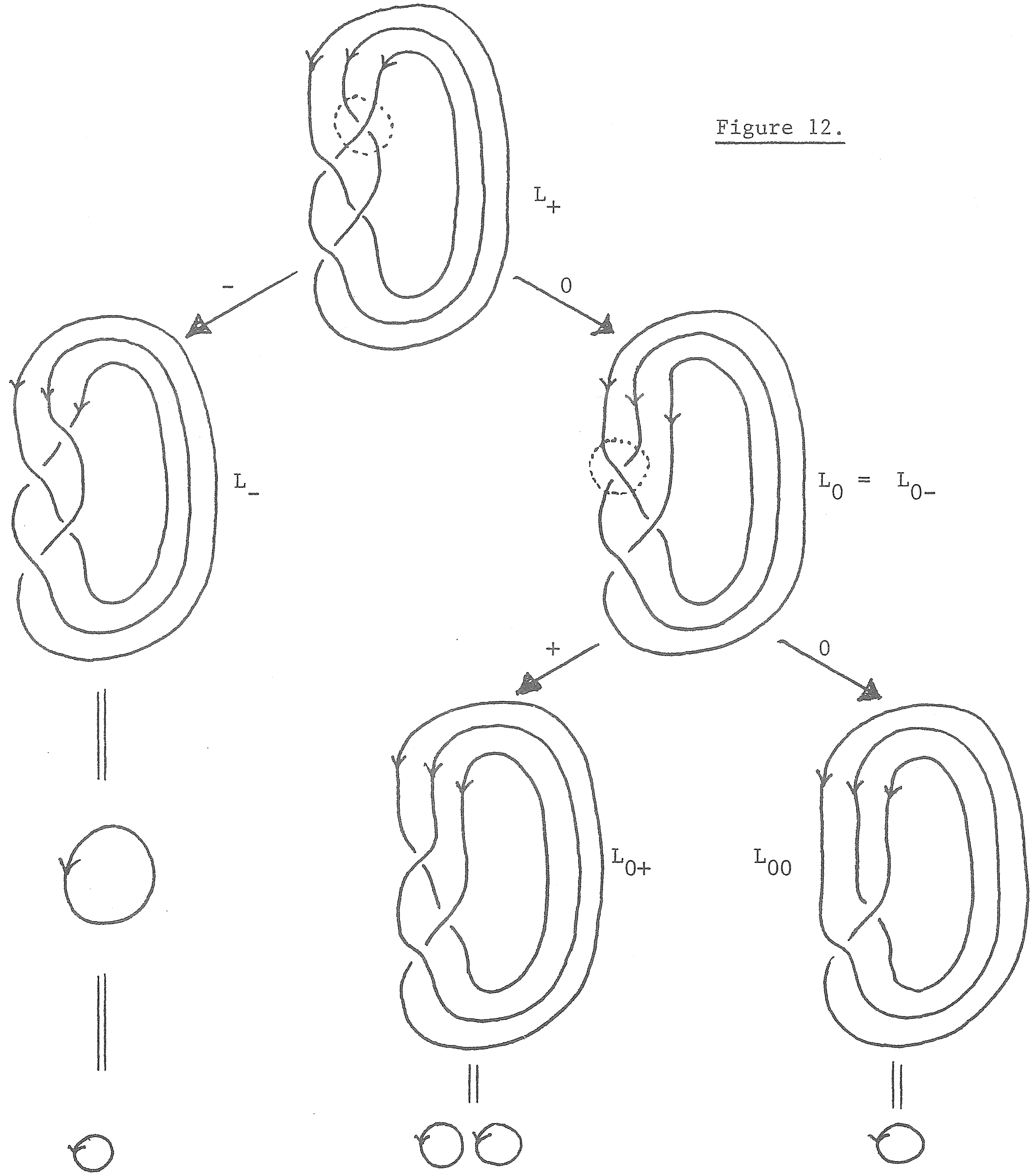


Figure 10.

Figure 11.

Figure 12.



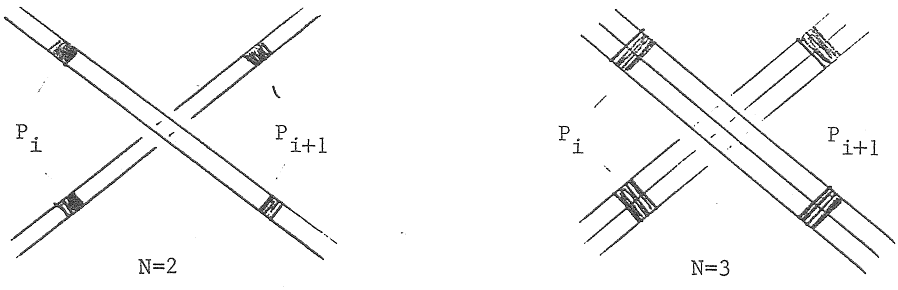
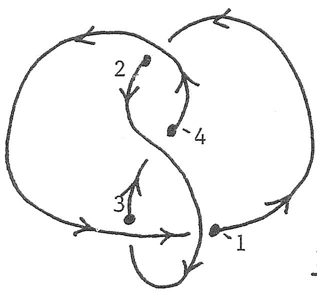


Figure 13. Fusion



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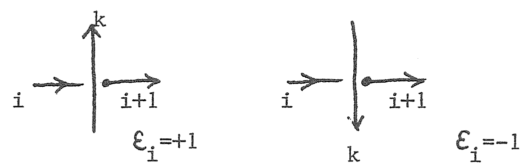

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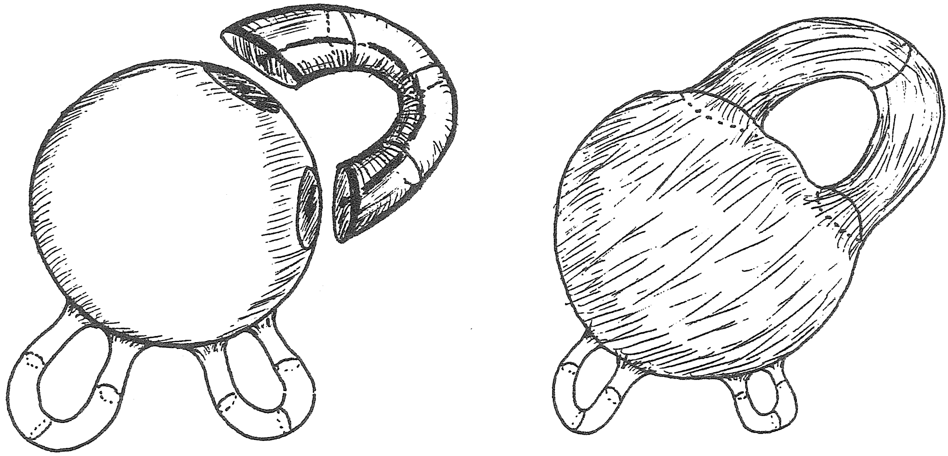

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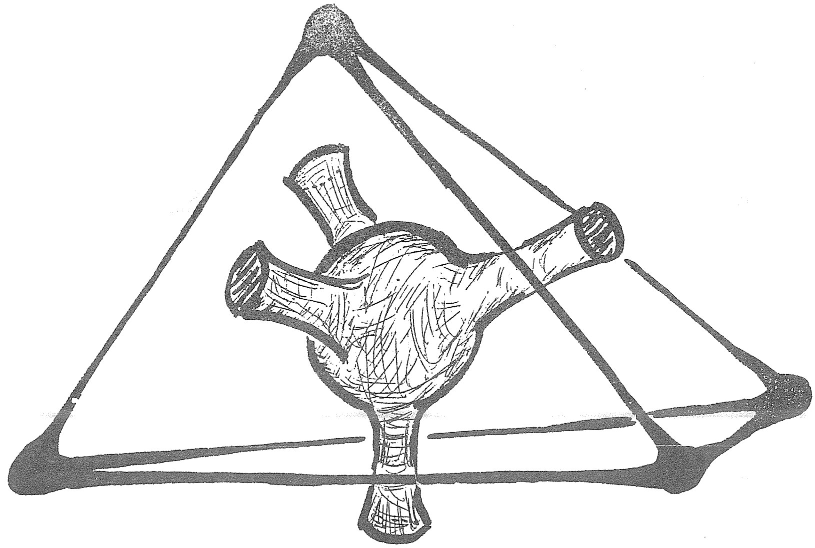

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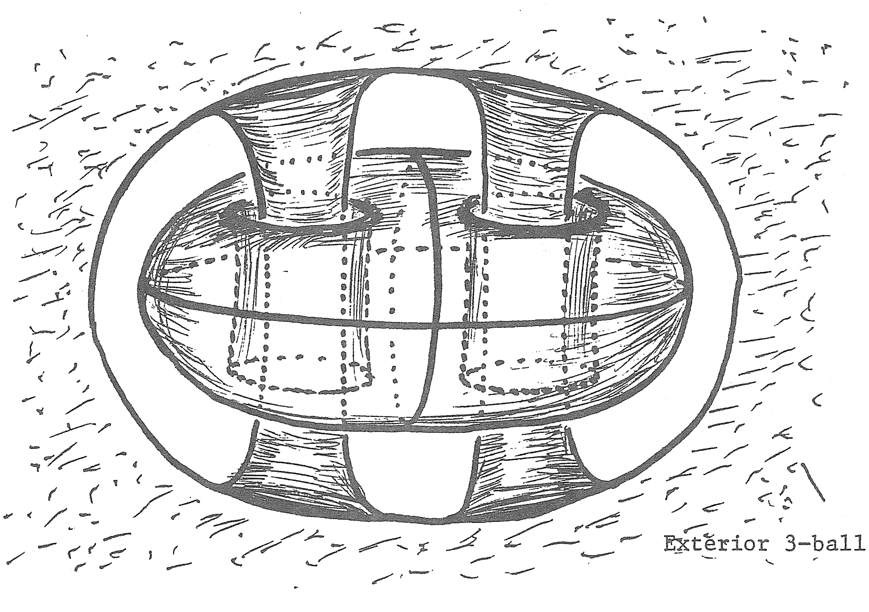

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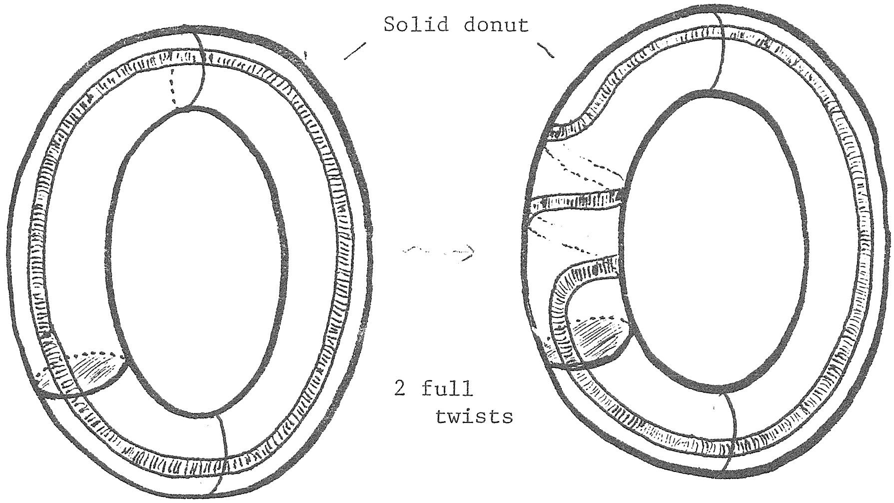

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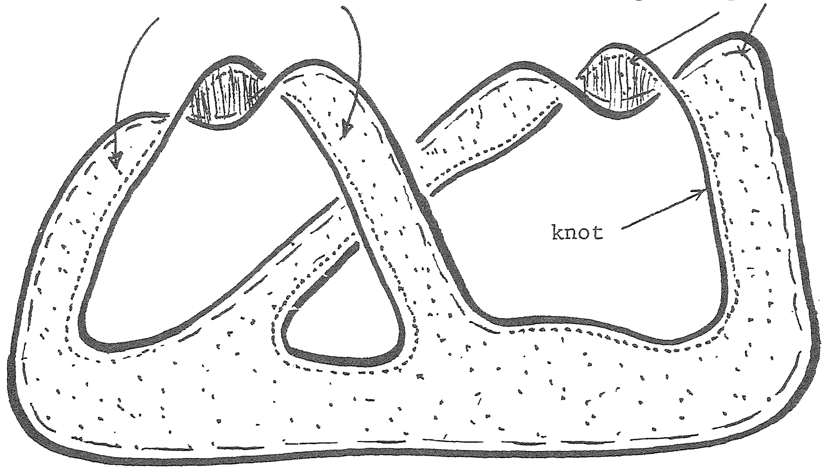
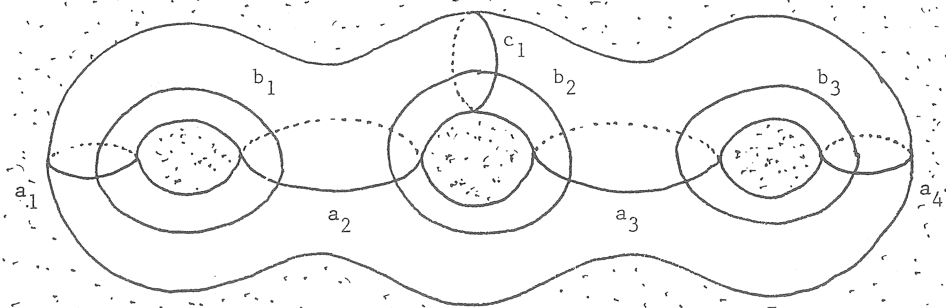


Figure 20.

Genus 3 surface: generators for diffeomorphisms by Dehn twists



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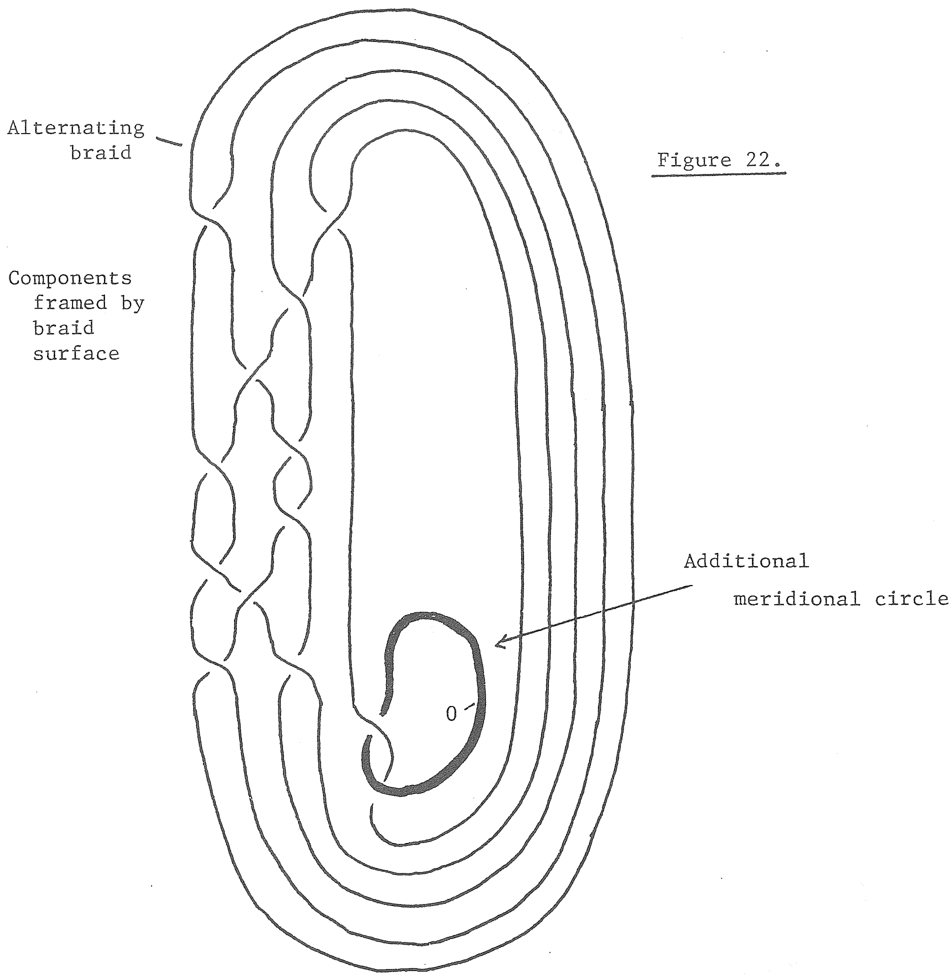
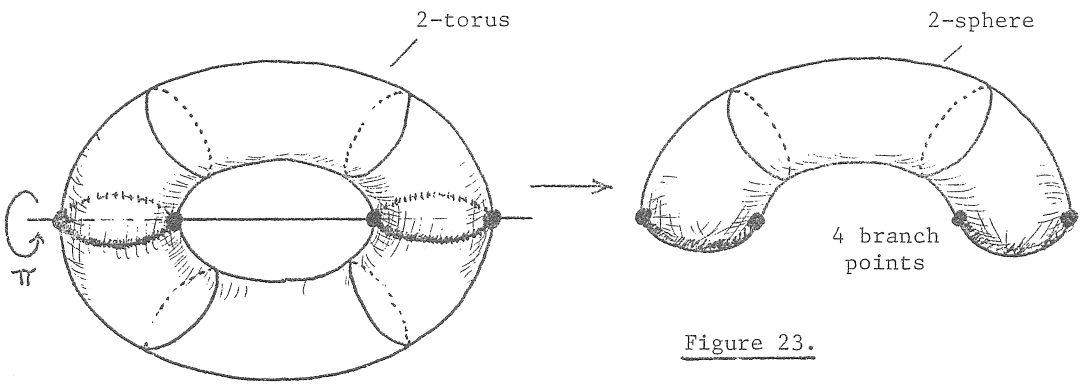


Figure 22.

Hyperelliptic involution:



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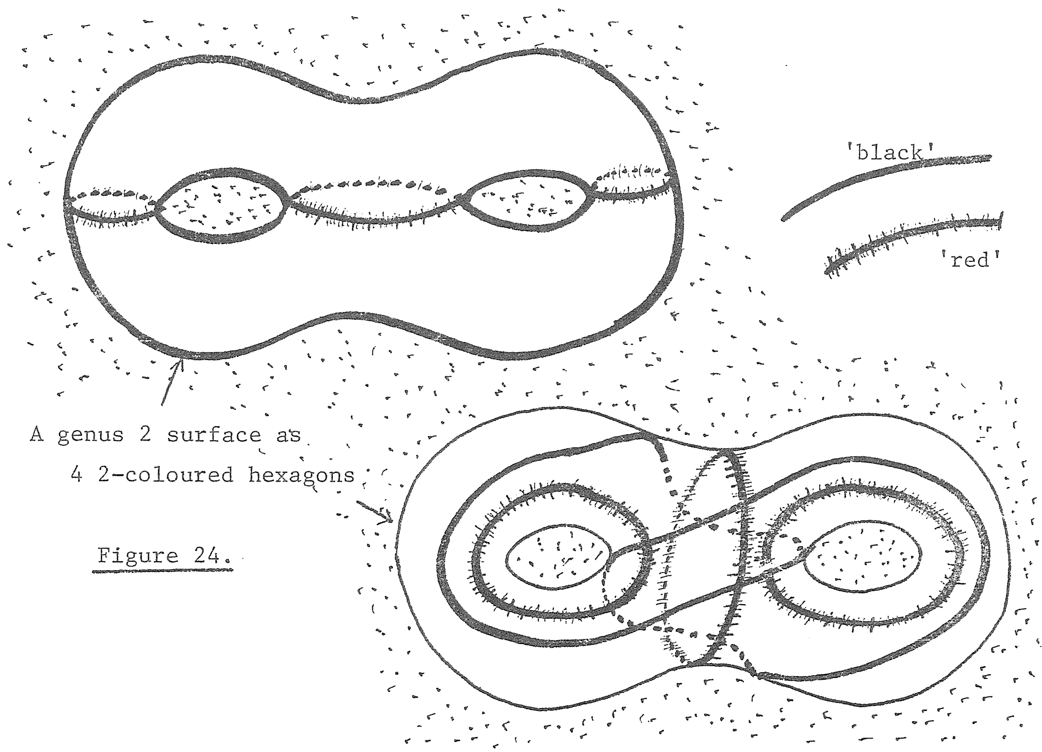

A genus 2 surface as
4 2-coloured hexagons

Figure 24.