ELLIPTIC EQUATIONS IN NON-DIVERGENCE FORM

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This article is concerned with certain local estimates of Harnack and Hölder type that have been established recently for linear and non-linear elliptic partial differential equations in the works of Krylov and Safonov, [10], [11], Trudinger [19], [20], and Evans [7], [8]. Crucial to the derivation of these results is a maximum principle that was discovered about twenty years ago by Aleksandrov [2] and Bakelman [5]. Since the Aleksandrov-Bakelman maximum principle has received only scant attention in expository monographs such as [9] we will also supply its proof below.

Let Ω be a bounded domain in Euclidean n space, $\mathbb{R}^n$ and let $A = [a^{ij}]$ be a measurable, real $n \times n$ symmetric matrix valued function on Ω. We assume that A is positive in Ω so that the partial differential operator L, given by

(1) $$Lu = a^{ij}D_{ij}u$$

for $u \in C^2(\Omega)$ is *elliptic* in Ω. (As is customary we adopt the summation convention that repeated indices indicate summation from 1 to n). Following the notation in [19], we let λ, Λ, $\mathcal{D}$ denote respectively the minimum eigenvalue, maximum eigenvalue and determinant of A. Setting $\mathcal{D}^* = \mathcal{D}^{1/n}$, we thus have

$$(2) \qquad 0 < \lambda \leq \mathcal{D}^* \leq \Lambda .$$

If u is a continuous function on Ω, we define the *upper contact set* of u, denoted Γ^+ or Γ_u^+, to be the subset of Ω where the graph of u lies below a support hyperplane in $\mathbb{R}^{n+1}$, that is

$$(3) \qquad \Gamma^+ = \left\{ y \in \Omega \mid u(x) \leq u(y) + p \cdot (x-y) \quad \text{for all} \quad x \in \Omega , \quad \text{for some} \quad p = p(y) \in \mathbb{R}^n \right\} .$$

Clearly if u is a concave function, then $\Gamma^+ = \Omega$. When $u \in C^1(\Omega)$ we must have $p = Du(y)$ in (3) as any support hyperplane must then be a tangent hyperplane to the graph of u. Furthermore when $u \in C^2(\Omega)$, the Hessian matrix, $D^2u = [D_{ij}u]$ is non-positive on Γ^+. In general the contact set Γ^+ is closed in Ω.

We can now state the following version of the Aleksandrov-Bakelman maximum principle.

<u>THEOREM 1</u>. *For any function* $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *with* $u \leq 0$ *on* $\partial\Omega$ *we have*

$$(4) \qquad \sup_{\Omega} u \leq Cd \| Lu/\mathcal{D}^* \|_{L^n(\Gamma^+)} ,$$

where C *depends only on* n *and* d *is the diameter of* Ω. *(In fact we can take* $C = \dfrac{1}{n(\omega_n)^{1/n}}$ *).*

The salient features which distinguish Theorem 1 from the classical Hopf maximum principle are the presence of the integral norm, the restriction to the set Γ^+ and the dependence on the coefficient determinant. Note that we have $Lu \le 0$ on Γ^+ since $D^2u \le 0$ there.

The proof of Theorem 1 utilizes the concept of normal mapping. If u is an arbitrary continuous function on Ω, the *normal mapping*, $\chi(y) = \chi_u(y)$ of a point $y \in \Omega$ is the set of "slopes" of support hyperplanes at y lying above the graph of u, that is

(5) $$\chi(y) = \left\{p \in \mathbb{R}^n \mid u(x) \le u(y) + p\cdot(x-y) \quad \text{for all} \quad x \in \Omega\right\} .$$

Clearly $\chi(y) \neq \phi$ if and only if $y \in \Gamma^+$. If $u \in C^1(\Omega)$, then $\chi(y) = Du(y)$ for $y \in \Gamma^+$. If, for example, Ω is a ball, $B_R(0)$, and k the conical function

$$k(x) = h\left(1 - \frac{|x|}{R}\right) , \quad (h = \text{constant}) ,$$

then

(6) $$\chi_k(y) = \begin{cases} -\dfrac{hy}{R|y|} & \text{for} \quad y \neq 0 \\ B_{h/R}(0) & \text{for} \quad y = 0 . \end{cases}$$

When $u \in C^2(\Omega)$, the n dimensional Lebesgue measure of $\chi(\Omega) = \bigcup_{y\in\Omega} \chi(y)$ is given by

(7) $$\begin{aligned} |\chi(\Omega)| &= |\chi(\Gamma^+)| \\ &= |Du(\Gamma^+)| \\ &= \int_{\Gamma^+} |\det D^2u| \end{aligned}$$

since $D^2u \le 0$ on Γ^+.

Let us now show that u can be estimated in terms of $|\chi(\Omega)|$. Suppose that u takes a positive maximum at a point $y \in \Omega$ and let k be the function whose graph is the cone with vertex $(y,u(y))$ and base $\partial\Omega$. Then $\chi_k(\Omega) \subset \chi_u(\Omega)$ since for each supporting hyperplane to the graph of k, there exists a parallel tangent hyperplane to the graph of u. Now let $\tilde{k}$ be the function whose graph is the cone with vertex $(y,u(y))$ and base $B_d(y)$. Clearly $\chi_{\tilde{k}}(\Omega) \subset \chi_k(\Omega)$ and consequently

$$|\chi_{\tilde{k}}(\Omega)| \leq |\chi_u(\Omega)| .$$

But then, using (6) and (7), we have

$$\omega_n \left(\frac{u(y)}{d}\right)^n \leq \int_{\Gamma^+} |\det D^2 u|$$

and hence

$$u(y) \leq \frac{d}{\omega_n^{1/n}} \left(\int_{\Gamma^+} |\det D^2 u|\right)^{1/n} . \tag{8}$$

To conclude the proof of Theorem 1, we invoke the matrix inequality,

$$\det A \det B \leq \left(\frac{\operatorname{trace} AB}{n}\right)^n , \quad A , B \geq 0 .$$

Consequently on Γ^+ we have

$$\begin{aligned} |\det D^2 u| &= \det(-D^2 u) \\ &\leq \frac{1}{\mathcal{D}} \left(\frac{-Lu}{n}\right)^n \end{aligned}$$

and hence by (8),

$$u(y) \leq \frac{d}{n\omega_n^{1/n}} \left\{\int_{\Gamma^+} \frac{|Lu|^n}{\mathcal{D}}\right\}^{1/n} \tag{9}$$

as required.

COROLLARY 1. *For any function* $u \in C^0(\bar{\Omega}) \cap W^{2,n}_{loc}(\Omega)$ *with* $u \leq 0$ *on* $\partial\Omega$, *we have*

$$\sup_{\Omega} u \leq Cd\| (Lu)^-/\mathcal{D}^* \|_{L^n(\Omega)} \tag{10}$$

where C *is the constant in* (9).

Corollary 1 follows from Theorem 1 by approximation. Since the argument was only carried out for uniformly elliptic L in earlier works such as [5], [17], we supply a complete proof here. Suppose first that L is uniformly elliptic, that is $\Lambda/\lambda \leq \gamma_0$ for some constant γ_0. Let $\{u_m\}$ be a sequence of functions in $C^2(\Omega)$ converging in the sense of $W^{2,n}_{loc}(\Omega)$ to u. For arbitrary $\varepsilon > 0$, we can then assume $\{u_m\}$ converges to u in $W^{2,n}(\Omega_\varepsilon)$ and $u_m \leq \varepsilon$ on $\partial\Omega_\varepsilon$ for some domain $\Omega_\varepsilon \subset \Omega$. Consequently by Theorem 1,

$$\sup_{\Omega_\varepsilon} u_m \leq \varepsilon + \frac{C}{d} \|\frac{1}{\lambda} a^{ij} D_{ij}(u_m - u)\|_{L^n(\Omega_\varepsilon)} + \frac{C}{d} \|\frac{(Lu)^-}{\mathcal{D}^*}\|_{L^n(\Omega_\varepsilon)}$$

and hence, letting $m \to \infty$ and using the fact that u_m converges uniformly to u on Ω_ε, we have

$$\sup_{\Omega_\varepsilon} u \leq \varepsilon + \|\frac{(Lu)^-}{\mathcal{D}^*}\|_{L^n(\Omega_\varepsilon)} , \tag{11}$$

from which (10) follows by letting $\varepsilon \to 0$.

To remove the condition of uniform ellipticity we consider for $\eta > 0$, the operators

$$L_\eta = \eta\Lambda\Delta + L$$

which will be uniformly elliptic. We obtain by virtue of (11),

$$\sup_{\Omega_\varepsilon} u \leq \varepsilon + \frac{C}{d} \left\{ \|\frac{\eta\Lambda\Delta u}{\mathcal{D}^*_\eta}\|_{L^n(\Omega_\varepsilon)} + \|\frac{(Lu)^-}{\mathcal{D}^*}\|_{L^n(\Omega_\varepsilon)} \right\}$$

so that letting $\eta \to 0$ and using the dominated convergence theorem we get inequality (11) again. Corollary 2 now follows by letting $\varepsilon \to 0$.

The proof of Theorem 1 can be extended to cover more general linear and quasilinear operators by appropriately weighting the area of the normal mapping; (see [3] or [6] for details). In particular for linear operators of the general form.

$$Lu = a^{ij}D_{ij}u + b^{i}D_{i}u + cu \tag{12}$$

with $c \leq 0$, we obtain an estimate of the form (10) but where the constant C also depends on $\|b/\mathcal{D}^*\|_{L^n(\Omega)}$.

It is interesting to note that Theorem 1 arose as a byproduct of the work of Aleksandrov [1] and Bakelman [4] on the Monge-Ampère equation,

$$\det D^2u = g \ , \tag{13}$$

where $g \in L^1(\Omega)$ is positive. If u is a convex solution of (13) and $u \geq 0$ on $\partial\Omega$, then the estimate (8) (with the appropriate sign change) implies that

$$\inf_{\Omega} u \geq -\frac{d}{\omega_n^{1/n}} \left(\int_{\Omega} g\right)^{1/n} . \tag{14}$$

We will later in this article derive an estimate for derivatives of solutions of (13) as a consequence of the local estimates which we will derive from Theorem 1.

Let us now turn to local estimates. For uniformly elliptic operators of the form (1) the following two important estimates were established by Krylov and Safonov in [10], [11]. In their formulation we will denote by B_R a ball of radius R in $\mathbb{R}^n$ and $B_{\sigma R}$ the concentric ball of radius σR , $\sigma < 1$.

<u>THEOREM 3.</u> *(<u>Hölder estimate</u>) For any function* $u \in W^{2,n}(B_R)$, *we have*

(15) $$\operatorname*{osc}_{B_{\sigma R}} u \le C\sigma^{\alpha}\left(\operatorname*{osc}_{B_R} u + \frac{R}{\lambda}\|Lu\|_{L^n(B_R)}\right)$$

where C *and* α *are positive constants depending only on* n *and* λ_0 .

THEOREM 4. *(Harnack inequality)* *For any non-negative function* $u \in W^{2,n}(B_R)$, *we have*

(16) $$\sup_{B_{\sigma R}} u \le C\left(\inf_{B_R} u + \frac{R}{\lambda}\|Lu\|_{L^n(B_R)}\right)$$

where C *depends only on* n , γ_0 *and* σ .

Alternative proofs of Theorems 3 and 4 are provided in the paper [19] in addition to the following two complementary results.

THEOREM 5. *(Local maximum principle)* *For any function* $u \in W^{2,n}(B_R)$ *and* $p > 0$, *we have*

(17) $$\sup_{B_{\sigma R}} u \le C\left\{\left[R^{-n}\int_{B_R} (u^+)^p\right]^{1/p} + R\|(Lu)^-/\mathcal{D}^*\|_{L^n(B_R)}\right\}$$

where C *depends on* n , γ_0 , σ *and* p .

THEOREM 6. *(Weak Harnack inequality)* *For any non-negative function* $u \in W^{2,n}(B_R)$ *and some* $p > 0$, *we have*

(18) $$\left(R^{-n}\int_{B_{\sigma R}} u^p\right)^{1/p} \le C\left\{\inf_{B_{\sigma R}} u + R\|(Lu)^+/\mathcal{D}^*\|_{L^n(B_R)}\right\}$$

where C *depends on* n , γ_0 *and* σ , p *depends on* n *and* γ_0 *only.*

The Harnack inequality, Theorem 4, is a simple consequence of the combination of Theorems 3 and 4. Also the Hölder estimate, Theorem 3, can be readily deduced from Theorem 6 by a standard method; (see [9], p.190). While the weak Harnack inequality, Theorem 6, is basically a variant of a key measure theoretic result of Krylov and Safonov, the local maximum principle, Theorem 5, is substantially new. Its proof differs from those of the other results in that it depends crucially on the restriction in the Aleksandrov-Bakelman maximum principle, Theorem 1, to the upper contact set Γ^+. For the other proofs, a weaker version of the estimate (4) with Γ^+ replaced by Ω is sufficient. Furthermore the proof of Theorem 6, as given in [19], extends to embrace a large class of quasilinear, *non-uniformly* elliptic operators. A byproduct of this extension, namely Harnack inequalities for non-uniformly elliptic divergence structure equations, is taken up by the author in [20]. For operators of the form (1), the condition of uniform ellipticity can be weakened to a condition,

$$\Lambda/\mathcal{D}^* \in L^q(B_R)$$

for some $q > n$; the constant C in (17) will then depend on q and $R^{-n/q}\,\|/\mathcal{D}^*\|_{L^q(B_R)}$ instead of γ_0.

As detailed proofs of these results are carried out in [19], we will confine attention here to illustrating the use of Theorem 1. By a coordinate stretching, we can assume that $R = 1$. Fix $\beta \leq 1$ and set

$$\eta(x) = (1-|x|^2)^\beta \quad , \quad v = \eta u$$

so that

$$Lv = \eta Lu + 2a^{ij}D_i u D_j \eta + uL\eta \; .$$

Theorem 5 now follows by application of Theorem 1 and appropriate choice of β. The first derivatives of u are estimated on the contact set Γ^+ by

$$(19) \qquad \begin{aligned} |Du| &\leq \frac{1}{\eta}(|Dv|+u|D\eta|) \\ &\leq \frac{1}{\eta}\left(\frac{v}{1-|x|} + u|D\eta|\right) \\ &\leq 2(1+\beta)\eta^{-1/\beta}u \; . \end{aligned}$$

Note that the definition of Γ^+ is crucial for this estimation.

To prove Theorem 6, we normalize L so that $\mathcal{D}^* = 1$ and set

$$g_0 = \|(Lu)^+\|_{L^n(B_1)}$$

which we also assume initially to be positive. Then for $w = -\log(u+g_0)$, $v = \eta w$ we have

$$\begin{aligned} Lv &= \eta Lw + 2a^{ij}D_i\eta D_j w + wL\eta \\ &\geq -\eta(Lu)^+/g_0 + \eta a^{ij}D_i w D_j w + 2a^{ij}D_i\eta D_j w + wL\eta \\ &\geq -\eta(Lu)^+/g_0 - \frac{1}{\eta}a^{ij}D_i\eta D_j\eta + wL\eta \ , \quad \text{by Schwarz's inequalities.} \end{aligned}$$

Now for fixed $\alpha \in (0,1)$ and sufficiently large β , we find that $L\eta \geq 0$ for $|x| \geq \alpha$. Application of the Aleksandrov-Bakelman maximum principle, Theorem 1, then yields a bound for v in the ball B_1 provided the measure of the subset of B_α where w is positive is sufficiently small. From this point we can proceed with the aid of an ingenious measure-theoretic induction argument of Krylov and Safonov to estimate w from above and hence equivalently u from below.

Theorems 3 to 6 all extend readily to more general linear and quasilinear elliptic operators. For linear operators of the form (12), Theorems 3, 4 and 6 continue to hold provided the coefficients b and c satisfy

$$b/\mathcal{D}^* \in L^{2n}(B_R) \quad , \quad c/\mathcal{D}^* \in L^n(B_R)$$

while Theorem 5 holds provided

$$b/\mathcal{D}^* \in L^q(B_R) \quad , \quad c/\mathcal{D}^* \in L^n(B_R) \quad \text{for some } q > n \ .$$

The constants C in the resultant estimates will then depend also upon the appropriate norms of the coefficients b and c . These results are all proved in [19], (see also [18]), together with appropriate extensions to quasilinear operators. The Hölder estimate for quasilinear elliptic equations is also treated in [13] and [14]. An interesting consequence of these extensions is the interior gradient estimate for quasilinear equations satisfying the "natural conditions" of Ladyzhenskaya and Ural'tseva, (see [9], [13]. [19]).

Liouville theorems also follow from the Harnack inequalities. For example, Theorem 4 implies that a $W^{2,n}_{loc}(\mathbb{R}^n)$ function satisfying $Lu = 0$ in $\mathbb{R}^n$ and bounded on one side must be constant. Furthermore Theorems 3, 5 and 6 admit extensions to balls B_R which intersect the boundary $\partial\Omega$ of a domain Ω where L and u are defined. In particular if, $u \leq 0$ on $B_R \cap \partial\Omega$, then Theorem 5, and its extensions to more general operators, holds with $B_{\sigma R}$ and B_R replaced by $B_{\sigma R} \cap \Omega$ and $B_R \cap \Omega$ respectively. If at a point $y \in \partial\Omega$, the generalized cone condition,

$$\liminf_{R\to 0} \frac{|B_R(y)-\Omega|}{|B_R(y)|} = \theta > 0 , \tag{20}$$

is satisfied, then we obtain a Hölder estimate for u at y in terms of the trace of u on $\partial\Omega$.

To conclude this article we indicate how the weak Harnack inequality can be used to give a simplified derivation of recent important Hölder estimates of Evans [7] for the second derivatives of fully nonlinear elliptic equations. For an application of Theorem 5 to such equations the reader is referred to [21].

Let us consider equations of the form

$$F(D^2u) = g \tag{21}$$

in a domain $\Omega \subset \mathbb{R}^n$, where F is a *concave* function in $C^2(\mathbb{R}^n)$.

Let $u \in C^4(\Omega)$ be a solution of (21) and suppose that F is *uniformly elliptic* for u, that is, there exist positive constants λ, Λ such that

$$\lambda|\xi|^2 \le F_{ij}(D^2u)\xi_i\xi_j \le \Lambda|\xi|^2 \tag{22}$$

for all $\xi \in \mathbb{R}^n$, where $F_{ij} = \frac{\partial F}{\partial r_{ij}}$. Let γ be a unit vector in $\mathbb{R}^n$ and differentiate equation (21) twice in the direction γ. We obtain thus

$$F_{ij}D_{ij}D_\gamma u = D_\gamma \phi \tag{23}$$

$$F_{ij}D_{ij}D_{\gamma\gamma}u + F_{ij,k\ell}D_{ij\gamma}uD_{k\ell\gamma}u = D_{\gamma\gamma}\phi \tag{24}$$

where $F_{ij,k\ell} = \frac{\partial^2 F}{\partial r_{ij} r_{k\ell}}$. By the concavity of F we see that the function $w = D_{\gamma\gamma}u$ satisfies the differential inequality

$$F_{ij}D_{ij}w \ge D_{\gamma\gamma}\phi \ . \tag{25}$$

Using the concavity of F again, we obtain for any $x,y \in \Omega$,

$$\begin{aligned} F_{ij}(D^2u(y))(D_{ij}u(x)-D_{ij}u(y)) &\ge F(D^2u(x)) - F(D^2u(y)) \\ &= g(x) - g(y) \ . \end{aligned} \tag{26}$$

We now adopt the key trick of Evans [7] and invoke a result from matrix theory [15] which guarantees the existence of a natural number $N > n$ and unit vectors $\gamma_1,\ldots\gamma_N$, depending only on n, λ and Λ, such that

$$\begin{aligned} &F_{ij}(D^2u(y))(D_{ij}u(x)-D_{ij}u(y)) \\ &\quad = \sum_{j=1}^{N} \beta_j(y)(D_{\gamma_j\gamma_j}u(x)-D_{\gamma_j\gamma_j}u(y)) \\ &\quad = \sum_{j=1}^{N} \beta_j(w_j(x)-w_j(y)) \end{aligned} \tag{27}$$

where $w_j = D_{\gamma_j\gamma_j}u$ and $\beta_1,\ldots\beta_N$ satisfy

$$(28) \qquad 0 < \lambda^* \le \beta_j \le \Lambda^*$$

for positive constants λ^*, Λ^* also depending only on n, λ and Λ. Therefore by (26) we obtain

$$(29) \qquad \sum_{j=1}^{N} \beta_j (w_j(y) - w_j(x)) \le g(y) - g(x) .$$

Now we are in a position to exploit the weak Harnack inequality. Let B_R, B_{2R} be concentric balls in Ω of radii R, $2R$ respectively and set

$$M_j^{(1)}, M_j^{(2)} = \sup_{B_R, B_{2R}} w_j ,$$

$$m_j^{(1)}, m_j^{(2)} = \inf_{B_R, B_{2R}} w_j .$$

Applying Theorem 6 to the functions $M_j^{(2)} - w_j$ we obtain

$$(30) \quad \left\{ R^{-n} \int_{B_R} (M_j^{(2)} - w_j)^p \right\}^{1/p} \le C \left\{ M_j^{(2)} - M_j^{(1)} + R \| (D_{\gamma_j \gamma_j} g)^- \|_{L^n(B_{2R})} \right\}$$

where C and p are positive constants depending only on n, λ and Λ. Let us now sum the estimates (30) over $j \ne k$ for some fixed index k. We obtain thus, (taking $p \le 1$),

$$(31) \quad \left\{ R^{-n} \int_{B_R} \left[\sum_{j \ne k} (M_j^{(2)} - w_j) \right]^p \right\}^{1/p} \le N^{1/p} \sum_{j \ne k} \left\{ R^{-n} \int_{B_R} (M_j^{(2)} - w_j)^p \right\}^{1/p}$$

$$\le C \left\{ \omega(2R) - \omega(R) + R g_2 \right\}$$

where $\omega(R)$, $\omega(2R) = \sum_{j=1}^{N} M_j^{(1)} - m_j^{(1)}, M_j^{(2)} - m_j^{(2)}$,

and $\quad g_2 = \|D^2 g\|_{L^n(\Omega)}$.

Now by (29) we have, for $x \in B_{2R}$, $y \in B_R$,

$$\beta_k(w_k(y)-w_k(x)) \le g(y) - g(x) + \sum_{j\neq k} \beta_j(w_j(x)-w_j(y))$$

so that

$$w_k(y) - m_k^{(2)} \le \frac{1}{\lambda^*}\left\{\underset{B_{2R}}{\operatorname{osc}}\, g + \Lambda^* \sum_{j\neq k} (M_j^{(2)}-w_j(y))\right\} .$$

Consequently by (31),

$$(32) \qquad \left\{R^{-n}\int_{B_R} (w_k-m_k^{(2)})^p\right\}^{1/p} \le C\left\{\omega(2R)-\omega(R) + \underset{B_{2R}}{\operatorname{osc}}\, g + g_2R\right\}$$

where the constant C again depends only on n, λ and Λ. By adding (30) and (31) and summing we then obtain

$$\omega(2R) \le C(\omega(2R)-\omega(R) + \underset{B_{2R}}{\operatorname{osc}}\, g + g_2R)$$

whence

$$(33) \qquad \omega(R) \le \delta\omega(2R) + C(\underset{B_{2R}}{\operatorname{osc}}\, g + g_2R) \quad , \quad \delta = 1 - \frac{1}{C} .$$

Hölder estimates for the functions w_j, $j = 1, N$, now follow in the standard way; (see [9], p.191). By suitable choice of additional unit vectors, namely $\gamma_{ij} = \frac{1}{\sqrt{2}}(e_i \pm e_j)$, $i,j = 1,\ldots n$, or by the Schauder estimates we get the full second derivative Hölder estimates.

<u>THEOREM 7.</u> *Let* u *be a* $C^4(\Omega)$ *solution of equation* (21). *Then for any ball* $B_R \subset \Omega$ *and* $\sigma < 1$.

$$(34) \qquad \underset{B_{\sigma R}}{\operatorname{osc}}\, D^2u \le C\sigma^\alpha\left\{\underset{B_R}{\operatorname{osc}}\, D^2u + R\left(\sup_{B_R}|Dg| + \|D^2g\|_{L^n(B_R)}\right)\right\}$$

where C *and* α *are positive constants depending only on* n, λ *and* Λ.

We remark that Theorem 7 continues to remain valid when we only assume F is concave, uniformly elliptic for u and $u, g \in W^{2,n}(\Omega)$, [21]. Furthermore by interpolation, (see [9], p.124), we obtain an estimate for the $C^{2,\alpha}(\Omega')$ norm of u, for any $\Omega' \subset\subset \Omega$, in terms of $\sup_{\Omega} |u|$, n, λ, Λ, Ω', Ω and $\|g\|_{W^{2,n}(\Omega)}$. These estimates also embrace the Bellman-Pucci equation

$$\inf_{k=1,..m} L_k u = g \tag{35}$$

where

$$L_k u = a_k^{ij} D_{ij} u$$

and the coefficient matrices $[a_k^{ij}]$, $k = 1, ..m$ are constant matrices satisfying

$$\lambda|\xi|^2 \le a_k^{ij}\xi_i\xi_j \le \Lambda|\xi|^2 \tag{36}$$

for all $\xi \in \mathbb{R}^n$. This was the prime example of Evans [7]. In the later work [8], Evans considers more general nonlinear equations thereby including the general Bellman-Pucci equations with variable coefficients. The approach described above extends analogously to cover these situations, the main additional technique being a device used by Ladyzhenskaya and Ural'tseva, (see [9] p.270 or [12] p.340), in connection with gradient Hölder estimates for quasilinear equations.

Finally we note that the proof of Theorem 7 yields an alternative derivation of interior $C^{2,\alpha}$ estimates for the Monge-Ampère equation (13) to that of Pogorelev, (see [16]). Note that the function

$$r \to |\det r|^{1/n}$$

is concave for $r = [r_{ij}] > 0$. It also extends to the complex case as considered by Yau [22].

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