6. SPECTRAL DECOMPOSITION

In this section we develope a powerful method of decomposing an operator $T \in BL(X)$ in such a way that the spectrum $\sigma(T)$ of T becomes the *disjoint* union of the spectra of the restrictions of T. It also allows us to determine the coefficients in the Laurent expansion of the resolvent operator $R(z)$. We start with a simple result.

PROPOSITION 6.1 Let $T \in BL(X)$ be decomposed by (Y,Z). Then

(6.1) $$\rho(T) = \rho(T_Y) \cap \rho(T_Z) ,$$

or, equivalently

(6.2) $$\sigma(T) = \sigma(T_Y) \cup \sigma(T_Z) .$$

In fact, for z in $\rho(T)$, we have

(6.3) $$R(T,z)|_Y = R(T_Y,z) \quad \text{and} \quad R(T,z)|_Z = R(T_Z,z) ,$$

while for $z \in \rho(T_Y) \cap \rho(T_Z)$, we have

(6.4) $$R(T_Y,z)P + R(T_Z,z)(I-P) = R(T,z) ,$$

where P is the projection on Y along Z.

Proof The formula (6.3) can be verified easily and since P commutes with T (Proposition 2.1) the formula (6.4) also follows. Hence the relations (6.1) and (6.2) hold. //

We remark that when $T = T_Y \oplus T_Z$, $\sigma(T)$ need not be the *disjoint* union of $\sigma(T_Y)$ and $\sigma(T_Z)$, since the parts T_Y and T_Z of T can, in general, have common spectral values. The simplest example is given by the identity operator I on $X = \mathbb{C}^2$, $Y = \{[z_1,0]^t : z_1 \in \mathbb{C}\}$ and $Z = \{[0,z_2]^t : z_1 \in \mathbb{C}\}$, so that $\sigma(T) = \sigma(T_Y) = \sigma(T_Z) = \{1\}$.

To describe a special way of decomposing an operator T for which the union in (6.2) is disjoint, we introduce the following notations.

Unless otherwise stated, Γ denotes a simple closed positively oriented rectifiable curve in $\mathbb{C}$. Let $T \in BL(X)$. If $\Gamma \subset \rho(T)$, define

$$P_\Gamma(T) = -\frac{1}{2\pi i}\int_\Gamma R(z)dz , \tag{6.5}$$

and for $z_0 \notin \Gamma$, let

$$S_\Gamma(T,z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{R(z)}{z - z_0}\, dz . \tag{6.6}$$

When there is no ambiguity, we shall denote $P_\Gamma(T)$ simply by P_Γ or by P , and $S_\Gamma(T,z_0)$ by $S_\Gamma(z_0)$ or $S(z_0)$.

Cauchy's theorem (Theorem 4.5(a)) can be used to show that if Γ is continuously deformed in $\rho(T)$ to another curve $\tilde{\Gamma}$, then $P_\Gamma = P_{\tilde{\Gamma}}$ and if this process can be carried out in $\rho(T) \setminus \{z_0\}$, then $S_\Gamma(z_0) = S_{\tilde{\Gamma}}(z_0)$.

PROPOSITION 6.2 Let $T \in BL(X)$, $\Gamma \subset \rho(T)$ and $z_0 \notin \Gamma$. Denote $P_\Gamma(T)$ by P , and $S_\Gamma(T,z_0)$ by S .

(a) The operators T , P and S commute with each other.

(b) $P^2 = P$, i.e., P is a projection, and

$$TP = -\frac{1}{2\pi i}\int_\Gamma zR(z)dz . \tag{6.7}$$

(c) If $z_0 \in \text{Int}\,\Gamma$, then

$$SP = 0 \quad\text{and}\quad (T-z_0I)S = I - P , \tag{6.8}$$

while if $z_0 \in \text{Ext}\,\Gamma$, then

(6.9) $$SP = S \quad \text{and} \quad (T-z_0I)S = -P \ .$$

Proof Since P and S are the limits in $BL(X)$ of the respective Riemann-Stieltjes sums (4.5), and since for z and w in $\rho(T)$, $R(z)$ commutes with $R(w)$, we see that T , P and S commute with each other. This proves (a).

To calculate P^2 , let us consider a curve $\tilde{\Gamma}$ in $\rho(T)$ which can be continuously deformed in $\rho(T)$ to Γ , and which encloses Γ in its interior. This is possible since $\Gamma \subset \rho(T)$ and $\rho(T)$ is open. Then $P_\Gamma = P_{\tilde{\Gamma}}$.

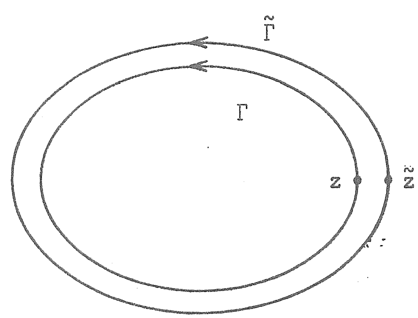

Figure 6.1

By using (4.17) and (5.5), we have

$$P^2 = P_\Gamma P_{\tilde{\Gamma}} = \frac{1}{(2\pi i)^2}\int_\Gamma R(z)dz \int_{\tilde{\Gamma}} R(\tilde{z})d\tilde{z}$$

$$= \frac{1}{(2\pi i)^2}\int_\Gamma \left[\int_{\tilde{\Gamma}} R(z)R(\tilde{z})d\tilde{z}\right]dz$$

$$= \frac{1}{(2\pi i)^2}\int_\Gamma \left[\int_{\tilde{\Gamma}} \frac{R(z)-R(\tilde{z})}{z-\tilde{z}}\, d\tilde{z}\right]dz$$

$$= \frac{1}{(2\pi i)^2}\int_\Gamma R(z)\left[\int_{\tilde{\Gamma}} \frac{d\tilde{z}}{z-\tilde{z}}\right]dz \ ,$$

since $\int_\Gamma \left[\int_{\tilde{\Gamma}} \frac{R(\tilde{z})d\tilde{z}}{z-\tilde{z}}\right]dz = \int_{\tilde{\Gamma}} \left[R(\tilde{z}) \int_\Gamma \frac{dz}{z-\tilde{z}}\right]d\tilde{z}$ by (4.10), and

$\int_\Gamma \frac{dz}{z - \tilde{z}} = 0$ for every $\tilde{z} \in \tilde{\Gamma}$ by Cauchy's theorem. But as $\int_{\tilde{\Gamma}} \frac{d\tilde{z}}{z - \tilde{z}} = -2\pi i$ for every $z \in \Gamma$, we see that

$$P^2 = -\frac{1}{2\pi i} \int_\Gamma R(z)dz = P .$$

This proves (b). Now, by (4.17) and (5.4),

$$\begin{aligned} TP &= -\frac{1}{2\pi i} \int_\Gamma TR(z)dz \\ &= -\frac{1}{2\pi i} \int_\Gamma [I+zR(z)]dz \\ &= -\frac{1}{2\pi i} \int_\Gamma zR(z)dz , \end{aligned}$$

which proves (6.7).

As for the part (c), let $z_0 \notin \Gamma$, and $\tilde{\Gamma}$ be a curve which encloses Γ in its interior and can be continuously deformed to Γ in $\rho(T) \setminus \{z_0\}$. Thus, $z_0 \in \text{Int}\,\Gamma$ if and only if $z_0 \in \text{Int}\,\tilde{\Gamma}$. Also, $S_{\tilde{\Gamma}}(z_0) = S$. Again as before,

$$\begin{aligned} PS &= \frac{-1}{(2\pi i)^2} \int_\Gamma R(z)dz \int_{\tilde{\Gamma}} \frac{R(\tilde{z})}{\tilde{z} - z_0} d\tilde{z} \\ &= \frac{-1}{(2\pi i)^2} \int_\Gamma \left[\int_{\tilde{\Gamma}} \frac{R(z)-R(\tilde{z})}{(z-\tilde{z})(\tilde{z}-z_0)} d\tilde{z} \right] dz \\ &= \frac{-1}{(2\pi i)^2} \int_\Gamma R(z) \left[\int_{\tilde{\Gamma}} \frac{d\tilde{z}}{(z-\tilde{z})(\tilde{z}-z_0)} \right] dz , \end{aligned}$$

since $\int_\Gamma \left[\int_{\tilde{\Gamma}} \frac{R(\tilde{z})d\tilde{z}}{(z-\tilde{z})(\tilde{z}-z_0)} \right] dz = \int_{\tilde{\Gamma}} \left[\frac{R(\tilde{z})}{\tilde{z}-z_0} \int_\Gamma \frac{dz}{z-\tilde{z}} \right] d\tilde{z}$ by (4.10), and $\int_\Gamma \frac{dz}{z - \tilde{z}} = 0$ for every $\tilde{z} \in \tilde{\Gamma}$ by Cauchy's theorem. But

$$\int_{\tilde{\Gamma}} \frac{d\tilde{z}}{(z-\tilde{z})(\tilde{z}-z_0)} = \frac{1}{z-z_0}\left[\int_{\tilde{\Gamma}} \frac{d\tilde{z}}{z-\tilde{z}} + \int_{\tilde{\Gamma}} \frac{d\tilde{z}}{\tilde{z}-z_0}\right]$$

$$= \begin{cases} 0 & , \text{ if } z_0 \in \operatorname{Int} \Gamma \\ \dfrac{-2\pi i}{z-z_0} & , \text{ if } z_0 \in \operatorname{Ext} \Gamma \end{cases} .$$

Hence

$$PS = SP = \begin{cases} 0, & \text{if } z_0 \in \operatorname{Int} \Gamma \\ S, & \text{if } z_0 \in \operatorname{Ext} \Gamma \end{cases} .$$

Finally, since $TR(z) = I + zR(z)$, we have

$$(T-z_0I)S = \frac{1}{2\pi i}\int_{\Gamma} \frac{(T-z_0I)R(z)}{z-z_0}\, dz$$

$$= \frac{1}{2\pi i}\int_{\Gamma} \frac{I + (z-z_0)R(z)}{z-z_0}\, dz$$

$$= \begin{cases} I - P, & \text{if } z_0 \in \operatorname{Int} \Gamma \\ -P, & \text{if } z_0 \in \operatorname{Ext} \Gamma \end{cases} .$$

Thus, (6.8) and (6.9) are proved. //

The commutation relations of part (a) of the above proposition can be used to characterize $P_\Gamma(T)$ and $S_\Gamma(z_0)$, $z_0 \in \operatorname{Int} \Gamma$. See Problem 6.5.

Now we come to the major result of this section.

THEOREM 6.3 (Spectral decomposition theorem) Let $T \in BL(X)$ and $\Gamma \subset \rho(T)$. Then T is decomposed by $Y = R(P_\Gamma)$ and $Z = Z(P_\Gamma)$, and $\sigma(T)$ is the *disjoint* union of $\sigma(T_Y)$ and $\sigma(T_Z)$. In fact

$$(6.10)\qquad \begin{aligned}\sigma(T_Y) &= \sigma(T) \cap \text{Int}\,\Gamma \\ \sigma(T_Z) &= \sigma(T) \cap \text{Ext}\,\Gamma\end{aligned} .$$

Also, for $z_0 \in \text{Int}\,\Gamma$,

$$(6.11)\qquad R(T_Z, z_0) = S_\Gamma(z_0)|_Z ,$$

and for $z_0 \in \text{Ext}\,\Gamma$,

$$(6.12)\qquad R(T_Y, z_0) = -S_\Gamma(z_0)|_Y .$$

Proof By Proposition 6.2, $P_\Gamma \equiv P$ is a projection and it commutes with T . Hence T is decomposed by $Y = R(P)$ and $Z = Z(P)$ (Proposition 2.1). Also, by Proposition 6.1,

$$(6.13)\qquad \sigma(T) = \sigma(T_Y) \cup \sigma(T_Z) .$$

For $z_0 \notin \Gamma$, the operator $S_\Gamma(z_0) \equiv S(z_0)$ commutes with P , and hence maps Y into Y , and Z into Z .

Let $z_0 \in \text{Int}\,\Gamma$. By the part (c) of Proposition 6.2, we have

$$S(z_0)(T-z_0I) = I - P = (T-z_0I)S(z_0) .$$

Considering restrictions to the closed subspace Z , we obtain

$$S(z_0)|_Z(T_Z-z_0I_Z) = I_Z = (T_Z-z_0I_Z)S(z_0)|_Z .$$

This shows that $z_0 \in \rho(T_Z)$ and $S(z_0)_{|Z}$ is the inverse of $T_Z - z_0I_Z$. This proves (6.11) and we have

$$(6.14)\qquad \text{Int}\,\Gamma \subset \rho(T_Z) .$$

Next, let $z_0 \in \text{Ext}\,\Gamma$. Then, by (6.9) we have

$$-S(z_0)(T-z_0I) = P = (T-z_0I)(-S(z_0)) .$$

Considering now restrictions to the closed subspace Y, we see that $z_0 \in \rho(T_Y)$ and $-S(z_0)|_Y$ is the inverse of $T_Y - z_0 I_Y$. This proves (6.12) and we have

$$\text{Ext}\, \Gamma \subset \rho(T_Y) \ . \tag{6.15}$$

The relations (6.13), (6.14) and (6.15) imply (6.10) since $\Gamma \subset \rho(T)$. It shows, in particular, that $\sigma(T)$ is the *disjoint* union of $\sigma(T_Y)$ and $\sigma(T_Z)$. //

The above theorem tells us that if we wish to study only a part of the spectrum $\sigma(T)$ of T, which is separated by a closed curve Γ from the rest of $\sigma(T)$, then we need to study only a part of the operator T, namely T_Y, where Y is the range of P_Γ.

We now investigate the range of P_Γ. Let $z_0 \in \text{Int}\, \Gamma$, and $x \in X$ with $(T-z_0 I)^n x = 0$ for some nonnegative integer n. Then

$$0 = (I-P_\Gamma)(T-z_0)^n x = (T-z_0 I)^n (I-P_\Gamma)x \ .$$

But by (6.11), $(T_Z - z_0 I_Z)$ and hence $(T_Z - z_0 I_Z)^n$ are invertible, where $Z = (I-P_\Gamma)(X)$. In particular, $(T-z_0 I)^n|_Z$ is one to one. Hence

$$(I-P_\Gamma)x = 0 \ , \quad \text{or} \quad x = P_\Gamma x \ ,$$

i.e., $x \in R(P_\Gamma)$. Thus, if for some $z_0 \in \text{Int}\, \Gamma$ and some nonnegative integer n, $(T-z_0 I)^n x = 0$, then x is in the range of P_Γ. Of course, such an element x is nonzero only if $z_0 \in \sigma(T) \cap \text{Int}\, \Gamma$. The case $n = 1$ is of particular importance. If $x \neq 0$ and $Tx = z_0 x$, then x is called <u>an eigenvector of</u> T corresponding to the <u>eigenvalue</u> z_0. More generally, a nonzero element x with $(T-z_0 I)^n x = 0$ for some $n \geq 1$ is called <u>a generalized eigenvector of</u> T corresponding to z_0 and it is said to be of <u>grade</u> n if $(T-z_0 I)^{n-1} \neq 0$; in this case, z_0 is an eigenvalue of T with a eigenvalue of T with a corresponding eigenvector $(T-z_0I)^{n-1}x$. When z_0 is an eigenvalue of T, the space $Z(T-z_0I)$ is called the corresponding <u>eigenspace</u> and the space $\{x \in X : (T-z_0I)^n x = 0$ for some $n = 1,2,\ldots\}$ is called the corresponding <u>generalized eigenspace.</u> As a trivial example, let $X = \mathbb{C}^2$, T be represented by the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and Γ be a closed curve enclosing the point 1. Then $P_\Gamma(X) = X$, which is spanned by the eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the generalized eigenvector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ of T corresponding to the only eigenvalue 1 of T. Thus, the range of P_Γ contains all generalized eigenspaces corresponding to the eigenvalues of T in $\operatorname{Int} \Gamma$.

For $z_0 \in \operatorname{Int} \Gamma$, we have by (6.8) and (6.11),

$$S_\Gamma(z_0) = S_Y \oplus S_Z ,$$

where $S_Y = 0$, and $S_Z = (T_Z - z_0 I_Z)^{-1}$.

These considerations allow us to give appropriate names to the operators which we have introduced: $P_\Gamma(T)$ is called the <u>spectral projection associated with</u> T <u>and</u> Γ, and the closed subspace $Y = R(P_\Gamma)$ is called the <u>associated spectral subspace</u>. For $z_0 \in \operatorname{Int} \Gamma$, the operator $S_\Gamma(z_0)$ is called the <u>reduced resolvent of</u> $(T-z_0I)$ on the closed subspace $Z = Z(P_\Gamma)$.

We introduce another operator which vanishes on $Z(P_\Gamma)$ and which tells us how T differs from a scalar multiple of the identity operator on $R(P_\Gamma)$.

For $z_0 \in \mathbb{C}$, let

$$D_\Gamma(z_0) = (T-z_0I)P_\Gamma . \tag{6.16}$$

Then it follows that $D_\Gamma(z_0)$ commutes with P_Γ, so that

$$D_\Gamma(z_0) = D_Y \oplus D_Z ,$$

where $D_Y = T_Y - z_0 I_Y$ and $D_Z = 0$.

Also, it can be seen that

$$D_\Gamma^2(z_0) = (T-z_0 I)D_\Gamma(z_0) . \tag{6.17}$$

We now characterize the spectra of $S_\Gamma(z_0)$ and $D_\Gamma(z_0)$.

PROPOSITION 6.4 Let $\Gamma \subset \rho(T)$.

(a) $P_\Gamma = 0$ if and only if $\sigma(T) \subset \mathrm{Ext}\,\Gamma$, and then

$$S_\Gamma(z_0) = R(z_0) \quad \text{for} \quad z_0 \in \mathrm{Int}\,\Gamma ,$$

$$D_\Gamma(z_0) = 0 \quad \text{for} \quad z_0 \in \mathbb{C} .$$

(b) $P_\Gamma = I$ if and only if $\sigma(T) \subset \mathrm{Int}\,\Gamma$, and then

$$S_\Gamma(z_0) = 0 \quad \text{for} \quad z_0 \in \mathrm{Int}\,\Gamma ,$$

$$D_\Gamma(z_0) = T - z_0 I \quad \text{for} \quad z_0 \in \mathbb{C} .$$

(c) Let $0 \neq P_\Gamma \neq I$. Then for $z_0 \in \mathrm{Int}\,\Gamma$, we have

$$\sigma(S_\Gamma(z_0)) = \{0\} \cup \{\frac{1}{\lambda - z_0} : \lambda \in \sigma(T) \cap \mathrm{Ext}\,\Gamma\} . \tag{6.18}$$

Also, for $z_0 \in \mathbb{C}$, we have

$$\sigma(D_\Gamma(z_0)) = \{\lambda - z_0 : \lambda \in \sigma(T) \cap \mathrm{Int}\,\Gamma\} \cup \{0\} . \tag{6.19}$$

Proof Let $Y = R(P_\Gamma)$ and $Z = Z(P_\Gamma)$. Then we know by (6.2) that

$$\sigma(T) = \sigma(T_Y) \cup \sigma(T_Z) .$$

Now, $P_\Gamma = 0$ if and only if $Y = \{0\}$, i.e., $\sigma(T_Y) = \emptyset$, by Theorem 5.2. This is the case if and only if $\sigma(T) = \sigma(T_Z) = \{\lambda \in \sigma(T) : \lambda \in \text{Ext}\,\Gamma\}$, by Theorem 6.3. In this case, we have for $z_0 \in \text{Int}\,\Gamma$, $(T-z_0I)S_\Gamma(z_0) = I - P = I$ by (6.8), so that $S_\Gamma(z_0) = R(z_0)$. Also, $D_\Gamma(z_0) = (T-z_0I)P_\Gamma = (T-z_0I)0 = 0$. This proves (a). The proof of (b) is exactly similar.

Let, now, $0 \neq P_\Gamma \neq I$. For $z_0 \in \text{Int}\,\Gamma$, we have

$$S_\Gamma(z_0) = S_Y \oplus S_Z ,$$

where $S_Y = 0$ and $S_Z = (T_Z - z_0 I_Z)^{-1}$ by (6.8) and (6.11). Since $Y \neq \{0\}$, we see that $\sigma(S_Y) = \{0\}$, and

$$\sigma(S_Z) = \{\frac{1}{\lambda - z_0} : \lambda \in \sigma(T_Z)\}$$

$$= \{\frac{1}{\lambda - z_0} : \lambda \in \sigma(T) \cap \text{Ext}\,\Gamma\} ,$$

by (6.10). Since $\sigma(S_\Gamma(z_0)) = \sigma(S_Y) \cup \sigma(S_Z)$, we obtain (6.18). The proof of (6.19) is very similar. //

We are now in a position to find the coefficients in the Laurent exansion of $R(z)$ in an annulus about z_0.

THEOREM 6.5 (Laurent expansion of $R(z)$) Let $\Gamma \subset \rho(T)$, $z_0 \in \text{Int}\,\Gamma$, and write $P = P_\Gamma$, $S = S_\Gamma(z_0)$, $D = D_\Gamma(z_0)$. Then

(6.20) $$\max\{|\lambda - z_0| : \lambda \in \sigma(T) \cap \text{Int}\,\Gamma\} = r_\sigma(D) \equiv r_1 ,$$

(6.21) $$\min\{|\lambda - z_0| : \lambda \in \sigma(T) \cap \text{Ext}\,\Gamma\} = \frac{1}{r_\sigma(S)} \equiv r_2 .$$

Let $r_1 < r_2$. The resolvent operator $R(z)$ is analytic on the annulus $\{z \in \mathbb{C} : r_1 < |z-z_0| < r_2\}$, and we have

$$(6.22)\qquad R(z) = \sum_{k=0}^{\infty} S^{k+1}(z-z_0)^k - \frac{P}{z-z_0} - \sum_{k=1}^{\infty} \frac{D^k}{(z-z_0)^{k+1}} .$$

Proof The expressions for $r_\sigma(D)$ and $1/r_\sigma(S)$ follow immediately from Proposition 6.4 upon making use of the convention that if a set of nonnegative numbers reduces to the empty set, then its maximum is 0, while its minimum is infinity.

Now, assume that $z \in \mathbb{C}$ is such that the two infinite series on the right hand side of (6.22) converge in $BL(X)$. Let $f(z)$ denote the right hand side of (6.22) and for $n = 1,2,\ldots,$ let

$$f_n(z) = \sum_{k=0}^{n} S^{k+1}(z-z_0)^k - \frac{P}{z-z_0} - \sum_{k=1}^{n} \frac{D^k}{(z-z_0)^{k+1}} .$$

Since $z_0 \in \text{Int}\,\Gamma$, we see by (6.8) that $(T-z_0I)S = I - P$. Also, by (6.16) and (6.17), we have $(T-z_0I)P = D$, $(T-z_0I)D = D^2$. Hence

$$\begin{aligned} f_n(z)(T-zI) &= (T-zI)f_n(z) \\ &= (T-z_0I)f_n(z) - (z-z_0)f_n(z) \\ &= \sum_{k=0}^{n} (I-P)S^k(z-z_0)^k - \frac{D}{z-z_0} - \sum_{k=1}^{n} \frac{D^{k+1}}{(z-z_0)^{k+1}} \\ &\qquad - \sum_{k=0}^{n} S^{k+1}(z-z_0)^{k+1} + P + \sum_{k=1}^{n} \frac{D^k}{(z-z_0)^k} \\ &= (I-P) - \frac{D^{n+1}}{(z-z_0)^{n+1}} - S^{n+1}(z-z_0)^{n+1} + P \\ &= I - \left[\frac{D}{z-z_0}\right]^{n+1} - [(z-z_0)S]^{n+1} , \end{aligned}$$

which tends to I as $n \to \infty$, since the latter two terms are the $(n+1)$-st terms of convergent series. This shows that $z \in \rho(T)$ and $R(z) = f(z)$.

Now, we know from Theorem 4.9 that the series $\sum_{k=0}^{\infty} S^{k+1}(z-z_0)^k$ and $\sum_{k=1}^{\infty} \frac{D^k}{(z-z_0)^{k+1}}$ both converge if

$$r_\sigma(D) = \overline{\lim_{k\to\infty}} \|D^k\|^{1/k} < |z-z_0| < 1 \Big/ \overline{\lim_{k\to\infty}} \|S^k\|^{1/k} = \frac{1}{r_\sigma(S)} .$$

Thus, for $r_1 < |z-z_0| < r_2$, the expansion (6.22) of $R(z)$ is valid and $R(z)$ is analytic there. //

We remark that the coefficients in the Laurent expansion of $R(z)$ around z_0 are given by

$$\begin{aligned} a_k &= S^{k+1}(z_0) , \quad k = 0,1,2,\ldots, \\ b_1 &= -P , \\ b_k &= -D^{k-1}(z_0) , \quad k = 2,3,\ldots \end{aligned} \tag{6.23}$$

Thus, they are determined by three operators: $S(z_0)$, P and $D(z_0)$. In fact, since $D(z_0) = (T-z_0I)P$ and $P = I - (T-z_0I)S(z_0)$, we see that the single operator $S(z_0)$ determines all these coefficients. Another noteworthy feature of these coefficients is as follows: If some $a_j = S^{j+1}(z_0) = 0$, then for all $k > j$, we have $a_k = S^{k+1}(z_0) = 0$; and if some $b_j = -D^{j-1}(z_0) = 0$, then for all $k > j$, we have $b_k = -D^{k-1}(z_0) = 0$. This fact would be used quite fruitfully in the sequel.

We also note that if $0 \in \text{Int}\,\Gamma$ and $\sigma(T) \subset \text{Int}\,\Gamma$, then $P_\Gamma = I$, $D_\Gamma(0) = T$ and $S_\Gamma(0) = 0$. Hence the Laurent expansion (6.22) reduces to

$$R(z) = -\sum_{k=0}^{\infty} T^k z^{-(k+1)} \quad \text{for} \quad |z| > r_\sigma(T) .$$

This coincides with the first Neumann expansion (5.8), which we have obtained earlier.

Problems Let Γ be a simple closed positively oriented rectifiable curve in $\rho(T)$.

6.1 Let $\tilde{\Gamma}$ be another closed rectifiable curve in $\rho(T)$. If $\text{Int}\,\Gamma \cap \text{Int}\,\tilde{\Gamma} = \emptyset$, then $P_{\tilde{\Gamma}}P_{\Gamma} = 0$. If $\text{Int}\,\Gamma \subset \text{Int}\,\tilde{\Gamma}$, then $P_{\tilde{\Gamma}}P_{\Gamma} = P_{\Gamma}$, and $P = P_{\tilde{\Gamma}} - P_{\Gamma}$ is a projection. If $Y = R(P)$ and $Z = Z(P)$, then

$$\sigma(T_Y) = \sigma(T) \cap (\text{Int}\,\tilde{\Gamma} \cap \text{Ext}\,\Gamma) ,$$
$$\sigma(T_Z) = \sigma(T) \cap (\text{Ext}\,\tilde{\Gamma} \cup \text{Int}\,\Gamma) .$$

6.2 Let $z_0 \in \text{Int}\,\Gamma$. Then for $n = 1,2,\ldots$,

$$Z(P_\Gamma) \subset R((T-z_0I)^n) .$$

6.3 Let $z_0 \in \text{Ext}\,\Gamma$. If $P_\Gamma = 0$, then $S_\Gamma(z_0) = 0$; if $P_\Gamma = I$, then $S_\Gamma(z_0) = -R(z_0)$; if $0 \neq P_\Gamma \neq I$, then

$$\sigma(S_\Gamma(z_0)) = \{0\} \cup \left\{\frac{-1}{\lambda - z_0} : \lambda \in \sigma(T) \cap \text{Int}\,\Gamma\right\} .$$

6.4 For $z_0 \in \mathbb{C}$, let $D = D_\Gamma(z_0)$ and for $z_0 \notin \Gamma$, let $S = S_\Gamma(z_0)$. Then $DP = PD = D$. If $z_0 \in \text{Int}\,\Gamma$ then $DS = SD = 0$, while if $z_0 \in \text{Ext}\,\Gamma$, then $DS = SD = -P$. For $k = 1,2,\ldots,$

$$-\frac{1}{2\pi i}\int_\Gamma (z-z_0)^k R(z) = D^k ,$$

$$\frac{1}{2\pi i}\int_\Gamma \frac{R(z)}{(z-z_0)^k}\,dz = \begin{cases} S^k , & \text{if } z_0 \in \text{Int}\,\Gamma , \\ (-1)^{k-1}S^k , & \text{if } z_0 \in \text{Ext}\,\Gamma . \end{cases}$$

Theorem 6.5 can be proved by noting that if $r_1 < r_2$, then Γ can be continuously deformed in $\rho(T) \setminus \{z_0\}$ to $\tilde{\Gamma}$, where $\tilde{\Gamma}(t) = z_0 + re^{it}$, $t \in [0,2\pi]$, $r_1 < r < r_2$, and showing that $a_k = S^{k+1}$ for $k = 0,1,\ldots,$ $b_1 = -P$ and $b_k = -D^{k-1}$, $k = 2,3,\ldots$. (See (4.12) and (4.13).)

6.5 Let Q be a (bounded) projection such that $Q(X) = P_\Gamma(X)$. (i) For $x, y \in X$, we have $x = y$ if and only if $Qx = Qy$ and $S_\Gamma(z_0)x = S_\Gamma(z_0)y$ for some $z_0 \in \mathrm{Int}\,\Gamma$. (ii) $Q = P_\Gamma$ if and only if Q comutes with T . (iii) Let $z_0 \in \mathrm{Int}\,\Gamma$ and $A \in BL(X)$. Then $A = S_\Gamma(z_0)$ if and only if $AP_\Gamma = 0$, $A(T-z_0I) = I - P_\Gamma$ and A commutes with T .

6.6 Let $X = Y \oplus Z$ with $T(Y) \subset Y$. Let Q be the projection on Y along Z and $\tilde{T}_Z = (I-Q)T|_Z$. Then $\sigma(T) \subset \sigma(T_Y) \cup \sigma(\tilde{T}_Z)$. (Hint: If $z \in \rho(T_Y) \cap \rho(\tilde{T}_Z)$, then $R(T_Y,z)Q + [I-R(T_Y,z)QT]R(\tilde{T}_Z,z)(I-Q)$ is the inverse of $T - zI$.)

6.7 Let Y be a closed subspace of X such that $T(X) \subset Y$. Then for $0 \neq z \in \rho(T)$, $R(z)(Y) \subset Y$, and if Γ does not enclose 0 , then $P_\Gamma(X) \subset T(X) \subset Y$. Moreover,

$$P_\Gamma = \frac{-T}{2\pi i} \int_\Gamma \frac{R(z)}{z}\, dz \; .$$