4. BANACH SPACE-VALUED ANALYTIC FUNCTIONS

In this section we generalize the theory of complex-valued analytic functions of a complex variable by considering functions with values in a complex Banach space Y. The reason for considering the letter Y instead of the usual letter X is that we shall later consider $Y = BL(X)$, where X is a given complex Banach space.

Throughout this section D will denote a nonempty open connected set in $\mathbb{C}$.

A function $f : D \to Y$ is said to be <u>analytic on</u> D if for every $z_0 \in D$,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists in Y; it then will be denoted by $f'(z_0)$ and called the <u>derivative of</u> f <u>at</u> z_0.

If f is analytic on D and if $y^* \in Y^*$, then it follows from the conjugate linearity and the continuity of y^* that the map $z \mapsto \langle f(z), y^* \rangle$ is a complex-valued analytic function for z in D, and

$$\langle f(\cdot), y^* \rangle'(z) = \langle f'(z), y^* \rangle . \tag{4.1}$$

Dunford's theorem states the amazing fact that if $z \mapsto \langle f(z), y^* \rangle$ is analytic for z in D for every $y^* \in Y^*$, then, in fact, f is analytic on D ([L], 9.5). This result will allow us to transfer many interesting formulae from the theory of $\mathbb{C}$-valued functions to the case of Y-valued functions.

Before we discuss the integration of Y-valued functions, we deduce some useful results for Y-valued analytic functions.

PROPOSITION 4.1 (a) (Liouville's theorem) If f is analytic on $\mathbb{C}$ and is bounded there, then f is a constant function.

(b) (Maximum norm theorem) Let f be analytic on D and let $z_0 \in D$ be such that $\|f(z)\| \leq \|f(z_0)\|$ for all $z \in D$. Then

$$\|f(z)\| = \|f(z_0)\| , \quad z \in D ;$$

if Y is strictly convex, then, in fact, we have

$$f(z) = f(z_0) , \quad z \in D .$$

(c) Let a sequence (f_n) of analytic functions on D converge to a function f, uniformly on every compact subset of D. Then f is analytic on D.

Proof (a) Let $z_0 \in D$, and consider

$$g(z) = f(z) - f(z_0) , \quad z \in D .$$

Let $y^* \in Y^*$. Since the function $z \mapsto \langle g(z), y^* \rangle$ is bounded and analytic for z in $\mathbb{C}$, and has value 0 at $z = z_0$, we see by the usual Liouville theorem that $\langle g(z), y^* \rangle = 0$ for all $z \in D$. Since this is true for every $y^* \in Y^*$, we obtain $g(z) = 0$, i.e., $f(z) = f(z_0)$ for all $z \in D$.

(b) By Corollary 1.2, there is $y_0^* \in Y^*$ such that

$$\langle f(z_0), y_0^* \rangle = \|f(z_0)\| \text{ and } \|y_0^*\| = 1 .$$

By the fundamental inequality (1.3), we see that for all $z \in D$,

$$(4.2) \qquad |\langle f(z), y_0^* \rangle| \leq \|f(z)\| \leq \|f(z_0)\| = \langle f(z_0), y_0^* \rangle .$$

Since the function $z \mapsto \langle f(z), y_0^* \rangle$ is analytic for z in D, we conclude by the usual maximum modulus theorem that

$$\langle f(z), y_0^* \rangle = \langle f(z_0), y_0^* \rangle , \quad z \in D . \tag{4.3}$$

Hence equality relations hold in (4.2), i.e.,

$$\|f(z)\| = \|f(z_0)\| \quad \text{for all} \quad z \in D .$$

Now, let Y be strictly convex. If $f(z_0) = 0$, then clearly $f(z) = 0 = f(z_0)$ for all $z \in D$. Assume then $f(z_0) \neq 0$, and let $z \in D$. Now, the functional y_0^* attains its norm at $y_0 = f(z_0)/\|f(z_0)\|$, and by (4.3) at $y = f(z)/\|f(z_0)\|$, where $\|y_0\| = 1 = \|y\|$. Since y_0^* attains its norm also at $(y_0+y)/2$, we must have, in fact, $\|(y_0+y)/2\| = 1$. Then the strict convexity of Y implies that $y = y_0$, i.e., $f(z) = f(z_0)$.

(c) For $y^* \in Y^*$, the sequence $(\langle f_n(z), y^* \rangle)$ converges to $\langle f(z), y^* \rangle$, uniformly on every compact subset of D. Hence the function $z \mapsto \langle f(z), y^* \rangle$ is analytic for z in D. By Dunford's theorem, we see that f is analytic on D. //

Remark 4.2 The condition of strict convexity in the part (b) of the above proposition is merely sufficient to obtain the strong version $f(z) = f(z_0)$ of the maximum norm theorem. While it is not necessary, it cannot be dropped either. The space $Y = \mathbb{C}^2$ is not strictly convex under either of the norms

$$\|[z_1, z_2]^t\|_1 = |z_1| + |z_2| ,$$
$$\|[z_1, z_2]^t\|_\infty = \max\{|z_1|, |z_2|\} .$$

That the strong version does not hold for Y with the $\| \|_\infty$ norm follows by considering $D = \{z : |z| < 1\}$, $z_0 = 0$ and $f(z) = [z,1]^t$. On the other hand, the strong version does hold for Y with the $\| \|_1$ norm. We omit the proof, but refer the reader to a necessary and sufficient condition for the strong version to hold given in [TW].

We now consider the integration of a Banach space-valued function on a rectifiable curve Γ in $\mathbb{C}$, i.e., when Γ is a continuous complex-valued function on $[a,b]$ for which the total variation

$$V(\Gamma) = \sup\left\{ \sum_{i=1}^{n} |\Gamma(t_i) - \Gamma(t_{i-1})| : n \text{ a positive integer}, \right.$$
$$\left. a = t_0 < t_1 < \ldots < t_n = b \right\}$$

is finite. Note that a piecewise continuously differentiable function Γ has finite total variation; in fact,

$$V(\Gamma) = \int_a^b |\dot{\Gamma}(t)| dt , \tag{4.4}$$

where dot denotes differentiation with respect to t.

Let $f : \Gamma \to Y$ be continuous. For a partition $\mathbb{P} : a = t_0 < t_1 < \ldots < t_n = b$ and points $\$ = \{s_1, \ldots, s_n\}$ with $s_i \in [t_{i-1}, t_i]$, $i = 1, \ldots, n$, we let

$$\mu(\mathbb{P}) = \max\{|t_i - t_{i-1}| : i = 1, \ldots, n\} ,$$
$$\Sigma(\mathbb{P}, \$) = \sum_{i=1}^{n} f(\Gamma(s_i))[\Gamma(t_i) - \Gamma(t_{i-1})] .$$

$\mu(\mathbb{P})$ is the *mesh* of the partition $\mathbb{P}$ and $\Sigma(\mathbb{P}, \$)$ is a Riemann-Stieltjes sum.

When no ambiguity is likely to arise we denote the image $\Gamma([a,b])$ of $[a,b]$ under Γ by Γ itself.

THEOREM 4.3 Let Γ be a rectifiable curve in $\mathbb{C}$ and $f : \Gamma \to Y$ be continuous. Then there is a unique $y_0 \in Y$ with the following property: for every $\epsilon > 0$, there is $\delta > 0$ such that whenever $\mu(\mathbb{P}) < \delta$, we have

$$\|\Sigma(\mathbb{P},\$) - y_0\| < \epsilon .$$

We denote this y_0 by $\int_\Gamma f(z)dz$ and call it the <u>integral</u> <u>of</u> f <u>over</u> Γ. If $(\mathbb{P}_n)$ is a sequence of partitions with $\mu(\mathbb{P}_n) \to 0$, and $\$_n$ denotes a set of corresponding points, then

$$\int_\Gamma f(z)dz = \lim_{n\to\infty} \Sigma(\mathbb{P}_n,\$_n) . \tag{4.5}$$

Proof For a partition $\mathbb{P}$ of $[a,b]$, let

$$M(\mathbb{P}) = \max\{\|f(\Gamma(t) - f(\Gamma(s))\| : |t-s| \le \mu(\mathbb{P})\} .$$

If $\tilde{\mathbb{P}}$ is a refinement of $\mathbb{P}$ and $\tilde{\$}$ is a set of corresponding points, then one can check that

$$\|\Sigma(\mathbb{P},\$) - \Sigma(\tilde{\mathbb{P}},\tilde{\$})\| \le M(\mathbb{P})\ V(\Gamma) .$$

Let $\mathbb{Q}_n$ denote the partition of $[a,b]$ into 2^n equal parts at $t_0,\dots,t_{2^n}$ and let $\mathbb{T} = \{t_0,\dots,t_{2^n-1}\}$. Now, if $m > n$, then $\mathbb{Q}_m$ is a refinement of $\mathbb{Q}_n$. Hence

$$\|\Sigma(\mathbb{Q}_m,\mathbb{T}_m) - \Sigma(\mathbb{Q}_n,\mathbb{T}_n)\| \le M(\mathbb{Q}_n)\ V(\Gamma) .$$

But since $f\circ\Gamma$ is uniformly continuous on $[a,b]$, we see that $M(\mathbb{Q}_n) \to 0$ as $n \to \infty$. Thus, $(\Sigma(\mathbb{Q}_n,\mathbb{T}_n))$ is a Cauchy sequence in Y, and since Y is complete, it converges to some y_0 in Y.

If $\mathbb{P}$ is any partition of $[a,b]$, let $\tilde{\mathbb{Q}}_n$ denote the partition obtained by considering the nodes of $\mathbb{P}$ as well as $\mathbb{Q}_n$, and let $\tilde{\mathbb{T}}_n$ be any set of corresponding points between the nodes of $\tilde{\mathbb{Q}}_n$. Then

$$\begin{aligned}\|\Sigma(\mathbb{P},\$) - y_0\| &\le \|\Sigma(\mathbb{P},\$) - \Sigma(\tilde{\mathbb{Q}}_n,\tilde{\mathbb{T}}_n)\| \\ &\quad + \|\Sigma(\tilde{\mathbb{Q}}_n,\tilde{\mathbb{T}}_n) - \Sigma(\mathbb{Q}_n,\mathbb{T}_n)\| \\ &\quad + \|\Sigma(\mathbb{Q}_n,\mathbb{T}_n) - y_0\| \\ &\le [M(\mathbb{P})+M(\mathbb{Q}_n)]\ V(\Gamma) + \|\Sigma(\mathbb{Q}_n,\mathbb{T}_n) - y_0\| .\end{aligned}$$

From this relation we obtain the desired result. The uniqueness of y_0 and the validity of (4.5) are immediate. //

COROLLARY 4.4 With the notations of Theorem 4.3, we have

(4.6) $$\left\langle\int_\Gamma f(z)dz, y^*\right\rangle = \int_\Gamma \langle f(z), y^*\rangle dz$$

for every $y^* \in Y^*$. Also, $\int_\Gamma f(z)dz$ is the unique element of Y which satisfies (4.6).

Proof The statement follows from (4.5) by using the continuity and the conjugate linearity of $y^* \in Y^*$. //

Many properties of the integral $\int_\Gamma f(z)dz$ can be proved by using either (4.5) or (4.6). We enumerate some of them for later use. Let Γ be a rectifiable curve in $\mathbb{C}$.

(i) If f is continuous on Γ, and $M(f) = \max\{\|f(z)\| : z \in \Gamma\}$, then

(4.7) $$\left\|\int_\Gamma f(z)dz\right\| \le M(f)\ V(\Gamma) .$$

(ii) If a sequence (f_n) of continuous functions converges to f uniformly on Γ, then

(4.8) $$\lim_{n\to\infty}\int_\Gamma f_n(z)dz = \int_\Gamma f(z)dz .$$

(iii) Let $-\Gamma$ denote the <u>reversed curve</u> of Γ defined by

$$-\Gamma(t) = \Gamma(a+b-t) , \quad a \leq t \leq b .$$

If f is continuous on Γ, then

$$\int_{-\Gamma} f(z)dz = -\int_{\Gamma} f(z)dz . \tag{4.9}$$

(iv) If Γ_1 and Γ_2 are rectifiable curves in $\mathbb{C}$ and $F : \Gamma_1 \times \Gamma_2 \to Y$ is continuous, then

$$\int_{\Gamma_1} \left[\int_{\Gamma_2} F(z_1,z_2)dz_2 \right] dz_1 = \int_{\Gamma_2} \left[\int_{\Gamma_1} F(z_1,z_2)dz_1 \right] dz_2 . \tag{4.10}$$

If Γ is a *simple* closed curve, then by $\operatorname{Int}\Gamma$ and $\operatorname{Ext}\Gamma$ we shall denote the interior region and the exterior region of the curve traced out by Γ. Such a curve Γ is said to be <u>positively oriented</u>, if as the parameter t increases from a to b, $\operatorname{Int}\Gamma$ lies on the left of the curve being traced out in $\mathbb{C}$. Unless otherwise is stated explicitly, we shall denote by Γ a *simple closed positively oriented rectifiable curve*.

We now state the main results in the integration theory.

THEOREM 4.5 Let D be a simply connected domain and $f : D \to Y$ be analytic.

(a) (Cauchy's theorem) For every Γ in D,

$$\int_{\Gamma} f(z)dz = 0 .$$

(b) (Cauchy's integral formulae) Let $f^{(0)} = f$, and for $n \geq 1$, let $f^{(n)}$ denote the n-th derivative of f. For Γ in D and $z \in \operatorname{Int}\Gamma$,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(w)}{(w-z)^{n+1}} \, dw .$$

(c) (Cauchy's inequalities) If $z \in D$ and the disk $\{w \in \mathbb{C} : |w-z| \leq r\}$ lies in D, then

$$\|f^{(n)}(z)\| \leq \frac{n!}{r^n} \max\{\|f(w)\| : |w-z| = r\} .$$

Proof The results in (a) and (b) follow from the corresponding results for complex-valued functions by using Corollary 4.4. The part (c) follows from the part (b) by taking $\Gamma(t) = re^{it}$, $t \in [0,2\pi]$. //

We now study the convergence of series in a Banach space and obtain the important result that analytic functions are precisely the functions which have convergent power series expansions.

PROPOSITION 4.6 For s in a set S and $k = 0,1,2,\ldots,$ let $c_k(s) \in Y$. If

$$\gamma = \overline{\lim_{k\to\infty}} \left[\sup_{s\in S} \|c_k(s)\|^{1/k}\right] < 1 ,$$

then $\sum_{k=0}^{\infty} c_k(s)$ converges absolutely in Y, the convergence being uniform for $s \in S$.

Proof Find β such that $\gamma < \beta < 1$. Then there is k_0 such that for all $k \geq k_0$ and $s \in S$,

$$\|c_k(s)\| \leq \beta^k .$$

Since $\sum_{k=0}^{\infty} \beta^k$ converges, we see that $\sum_{k=0}^{\infty} \|c_k(s)\|$ converges uniformly for $s \in S$. Since Y is Banach, $\sum_{k=0}^{\infty} c_k(s)$ also converges uniformly for $s \in S$. ([L], 8.2) //

COROLLARY 4.7 Let $c_k \in Y$ for $k = 0,1,2,\ldots$, and $\gamma = \overline{\lim_{k\to\infty}} \|c_k\|^{1/k}$. Then $\sum_{k=0}^{\infty} c_k$ converges in Y if $\gamma < 1$, and it diverges if $\gamma > 1$.

Proof If $\gamma < 1$, then the result follows from the above proposition. If $\gamma > 1$, then there is a subsequence (c_{k_j}) such that $\|c_{k_j}\|^{1/k_j} \geq 1$ so that $\|c_{k_j}\| \geq 1$ for all j. Hence (c_k) does not tend to zero as $k \to \infty$, showing that $\sum_{k=0}^{\infty} c_k$ diverges. //

Before we consider the power series expansions of analytic functions, let us note that if $a_0,\ldots,a_n \in Y$, then the polynomial $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ is an analytic Y-valued function for z in $\mathbb{C}$.

THEOREM 4.8 (Taylor) Let $z_0 \in \mathbb{C}$ and $a_0, a_1, \ldots$ be in Y. Let

$$\alpha = 1 / \overline{\lim_{k\to\infty}} \|a_k\|^{1/k} .$$

The series

$$\sum_{k=0}^{\infty} a_k (z-z_0)^k$$

converges absolutely in Y for all z in the disk

$$D = \{z : |z-z_0| < \alpha\} ,$$

the convergence is uniform on every closed subset of D, and the sum f is an analytic function on D with

$$a_k = \frac{f^{(k)}(z_0)}{k!} .$$

Conversely, if f is analytic on a disk $\tilde{D} = \{z = |z-z_0| < \tilde{\alpha}\}$, then for $z \in \tilde{D}$, we have

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z-z_0)^k .$$

Proof Let $r < \alpha$. The first part follows by letting $S = \{z : |z-z_0| \leq r\}$ and $c_k(z) = a_k(z-z_0)^k$ for $z \in S$ in Proposition 4.6, since

$$\overline{\lim_{k\to\infty}} \left[\sup_{z\in S} \|c_k(z)\|^{1/k}\right] = r \overline{\lim_{k\to\infty}} \|a_k\|^{1/k} < 1 .$$

Also, the sum $f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$ is analytic for z in D because it is the uniform limit of analytic functions $\sum_{k=0}^{n} a_k(z-z_0)^k$ on every compact subset of D (Proposition 4.1(c)). The remaining parts of the theorem follow from the usual Taylor's theorem by noting that

$$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$$

if and only if for every $y^* \in Y^*$,

$$\langle f(z),y^*\rangle = \sum_{k=0}^{\infty} \langle a_k,y^*\rangle(z-z_0)^k$$

and by using (4.1) repeatedly. //

The above result says that the power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ converges if $|z-z_0| < \alpha$, while it follows from Corollary 4.7 that it diverges if $|z-z_0| > \alpha$. For this reason, $\alpha = 1/\overline{\lim_{k\to\infty}} \|a_k\|^{1/k}$ is called the <u>radius of convergence</u> of the power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$.

THEOREM 4.9 (Laurent) Let $z_0 \in \mathbb{C}$ and for $a_0, a_1, \ldots, b_1, b_2, \ldots$ in Y,

$$\alpha = 1 \,/\, \overline{\lim_{k\to\infty}} \|a_k\|^{1/k} \;, \quad \beta = \overline{\lim_{k\to\infty}} \|b_k\|^{1/k} \;.$$

The series

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k \quad \text{and} \quad \sum_{k=1}^{\infty} b_k(z-z_0)^{-k}$$

both converge absolutely in Y for all z in the annulus

$$D = \{z : \beta < |z-z_0| < \alpha\} \;;$$

the convergence is uniform on every closed subset of D. Let

$$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k + \sum_{k=1}^{\infty} b_k(z-z_0)^{-k} \;. \tag{4.11}$$

Then f is analytic on D and

$$a_k = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{(w-z_0)^{k+1}} \, dw \;, \quad k = 0,1,2,\ldots, \tag{4.12}$$

$$b_k = \frac{1}{2\pi i} \int_{\Gamma} f(w)(w-z_0)^{k-1} dw \;, \quad k = 1,2,\ldots, \tag{4.13}$$

where $\Gamma(t) = re^{it}$, $0 \le t \le 2\pi$, $\beta < r < \alpha$.

Conversely, if f is analytic on some annulus $\tilde{D} = \{z : \tilde{\beta} < |z-z_0| < \tilde{\alpha}\}$, then for every $z \in \tilde{D}$, we have the expansion (4.11) of f, where the a_k's and the b_k's are given by (4.12) and (4.13), with $\tilde{\beta} < r < \tilde{\alpha}$.

The proof of this theorem is similar to that of Taylor's theorem and hence we omit it. Note that to obtain the formulae (4.12) and (4.13) we should use (4.6). Taylor's theorem is, in fact, a special case of this theorem since if f is analytic in $\{z : |z-z_0| < \alpha\}$, then by Cauchy's theorem and integral formulae,

$$b_k = \frac{1}{2\pi i}\int_\Gamma f(w)(w-z_0)^{k-1}dw = 0 \ , \quad k = 1,2,\ldots,$$

and

$$a_k = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{(w-z_0)^{k+1}}\,dw = \frac{f^{(k)}(z_0)}{k!} \ , \quad k = 0,1,2,\ldots$$

If f is analytic on $\{z \in \mathbb{C} : 0 < |z-z_0| < \alpha\}$, then z_0 is called an <u>isolated singularity</u> of f . It is <u>removable</u> if all $b_k = 0$, $k = 1,2,\ldots;$ it is a <u>pole</u> if there is a positive integer ℓ such that $b_\ell \neq 0$, but $b_k = 0$ for all $k \geq \ell + 1$, ℓ being the <u>order</u> of the pole; otherwise it is called an <u>essential singularity</u>. In this case, $b_1 = \frac{1}{2\pi i}\int_\Gamma f(w)dw$, where $\Gamma(t) = re^{it}$, $0 < r < \alpha$, is called the <u>residue of</u> f <u>at</u> z_0 . Another special case occurs when f is analytic on $\{z \in \mathbb{C} : \beta < |z|\}$, and also <u>at infinity</u>, i.e., if $\lim_{w\to 0} f(\frac{1}{w})$ exists, so that the function g defined by

$$g(w) = \begin{cases} f(1/w) \ , & \text{if } 0 < |w| < 1/\beta \ , \\ \lim_{w\to 0} f(1/w) \ , & \text{if } w = 0 \ . \end{cases}$$

is analytic at 0 . In this case,

(4.14) $$f(z) = a_0 + \sum_{k=0}^{\infty} b_k z^{-k} \ ,$$

where $a_0 = g(0)$ and

$$b_k = \frac{-1}{2\pi i}\int_{\Gamma^{-1}} [g(w)/w^{k+1}]dw$$

where $\Gamma^{-1}(t) = e^{-it}/r$, $0 \leq t \leq 2\pi$, $r > \beta$. (See Problem 4.8.) The series (4.14) converges if $|z| > \overline{\lim_{k\to\infty}} \|b_k\|^{1/k}$.

Examples of Y-valued analytic functions

(i) Let $Y = \mathbb{C}^m$. If $f : D \to \mathbb{C}^m$, then

$$f(z) = [f_1(z),\dots,f_m(z)]^t , \quad z \in D ,$$

where $f_j : D \to \mathbb{C}$, $j = 1,\dots,m$. It is easily seen that f is analytic on D if and only if each f_j is analytic on D , and then

$$f'(z) = \left[f_1'(z),\dots,f_m'(z)\right]^t .$$

Also, f is continuous on Γ if and only if each f_j is, and then

$$\int_\Gamma f(z)dz = \left[\int_\Gamma f_1(z)dz,\dots, \int_\Gamma f_m(z)dz\right]^t ,$$

as can be seen by using (4.6).

(ii) Let $Y = BL(X)$, where X is a complex Banach space. Let $f : D \to BL(X)$ be analytic. For a fixed $x \in X$, consider $f_x : D \to X$ given by

$$f_x(z) = f(z)x , \quad z \in D .$$

Then since convergence in the norm of $BL(X)$ implies pointwise convergence, we see that each f_x is analytic and

$$f'(z)x = (f_x)'(z) . \tag{4.15}$$

It is interesting to note that the converse is true ([TL], p.267, Theorem 1.2).

Let $f : \Gamma \to BL(X)$ be continuous and fix $x \in X$. Then $f_x : \Gamma \to X$ is continuous, and it follows by (4.5) that

$$\left[\int_\Gamma f(z)dz\right]x = \int_\Gamma f_x(z)dz . \tag{4.16}$$

This can also be proved by using (4.6) as follows. Let $x^* \in X^*$, and consider $F \in (BL(X))^*$ given by

$$\langle T,F\rangle = \langle Tx,x^*\rangle \ , \quad T \in BL(X) \ .$$

Then

$$\begin{aligned}\langle \int_\Gamma f_x(z)dz,x^*\rangle &= \int_\Gamma \langle f_x(z),x^*\rangle dz = \int_\Gamma \langle f(z),F\rangle dz \\ &= \langle \int_\Gamma f(z)dz,F\rangle = \langle \left[\int_\Gamma f(z)dz\right]x,x^*\rangle \ .\end{aligned}$$

Since this is true for all $x^* \in X^*$, we obtain (4.16). As an example, let $X = C([0,1])$ and $f : \mathbb{C} \to BL(X)$ be given by

$$(f(z)x)(u) = e^{zu} \int_0^u e^{-zs}x(s)ds \ , \ x \in X \ , \ u \in [0,1] \ .$$

Then for $x \in X$ and $u \in [0,1]$, and Γ any rectifiable curve in $\mathbb{C}$,

$$\begin{aligned}\left[\int_\Gamma f(z)dz\right]x(u) &= \left[\int_\Gamma f(z)x\ dz\right](u) \\ &= \int_\Gamma (f(z)x)(u)dz \\ &= \int_\Gamma \left[e^{zu} \int_0^u e^{-zs}x(s)ds\right]dz \ .\end{aligned}$$

Now, for fixed x and u, $z \mapsto e^{zu} \int_0^u e^{-zs}x(s)ds$ is a complex-valued analytic function. Hence if Γ is a simple closed curve, then by Cauchy's theorem

$$\left[\int_\Gamma f(z)dz\right]x(u) = 0 \ , \quad x \in X \ , \quad u \in [0,1] \ ,$$

i.e., $\int_\Gamma f(z)dz = 0$. This is true for every simple closed Γ in $\mathbb{C}$ and f is clearly continuous, Hence by Morera's theorem (cf. Problem 4.5), f is analytic on $\mathbb{C}$.

The Banach space $Y = BL(X)$ has a multiplicative structure, and for $T \in BL(X)$, we have $T^* \in BL(X^*)$. We now study the behaviour of the integral with respect to these operations.

Let $f : \Gamma \to BL(X)$ be continuous. For $T \in BL(X)$ and $x \in X$

$$\left[\int_\Gamma f(z)dz\right]Tx = \int_\Gamma f_{Tx}(z)dz = \int_\Gamma (f(z)T)x\, dz$$
$$= \left[\int_\Gamma f(z)T\, dz\right]x \ ,,$$

by (4.16). Hence

$$(4.17)\qquad \begin{cases} \left[\int_\Gamma f(z)dz\right]T = \int_\Gamma f(z)T\, dz \ , \text{ and similarly} \\ \\ T\left[\int_\Gamma f(z)dz\right] = \int_\Gamma Tf(z)dz \ . \end{cases}$$

To consider $\left[\int_\Gamma f(z)dz\right]^*$, we define the <u>conjugate curve</u> $\bar{\Gamma}$ of Γ as follows:

$$\bar{\Gamma}(t) = \overline{\Gamma(a+b-t)} \ , \quad t \in [a,b] \ .$$

Note that since Γ is positively oriented, so is $\bar{\Gamma}$.

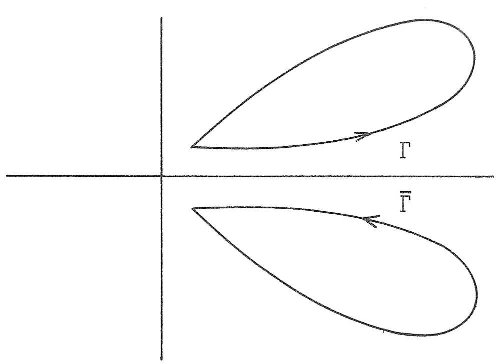

Figure 4.1

We shall make use of the following relation for a continuous function $h : \Gamma \to \mathbb{C}$.

$$(4.18)\qquad \overline{\int_\Gamma h(z)dz} = -\int_{\bar{\Gamma}} \overline{h(\bar{w})}dw \ .$$

For $x \in X$ and $x^* \in X^*$, we have

$$\langle \left[\int_\Gamma f(z)dz \right] x, x^* \rangle = \langle \int_\Gamma f_x(z)dz, x^* \rangle , \quad \text{by (4.16)}$$

$$= \int_\Gamma \langle f_x(z), x^* \rangle dz , \quad \text{by (4.6)}$$

$$= \int_\Gamma \langle x, [f(z)]^* x^* \rangle dz$$

$$= \overline{-\int_{\bar{\Gamma}} \overline{\langle x, [f(\bar{w})]^* x^* \rangle} dw} , \quad \text{by (4.18)}$$

$$= \overline{-\int_{\bar{\Gamma}} \langle f[(\bar{w})]^* x^*, x \rangle dw}$$

$$= \overline{-\langle \int_{\bar{\Gamma}} [f(\bar{w})]^* x^* dw, x \rangle} , \quad \text{by (4.6)}$$

$$= \langle x, \left[-\int_{\bar{\Gamma}} [f(\bar{w})]^* dw \right] x^* \rangle , \quad \text{by (4.16)} .$$

Since this is true for all $x \in X$ and $x^* \in X^*$, we have

(4.19) $$\left[\int_\Gamma f(z)dz \right]^* = - \int_{\bar{\Gamma}} [f(\bar{w})]^* dw .$$

Note that (4.18) is a special case of (4.19) when $X = \mathbb{C} = BL(X)$.

Problems

4.1 Let the functions $z \mapsto c(z) \in \mathbb{C}$, $z \mapsto x(z) \in X$ and $z \mapsto f(z) \in BL(X)$ be analytic for z in D . Then the functions $z \mapsto c(z)x(z) \in X$ and $z \mapsto f(z)x(z) \in X$ are analytic for z in D .

4.2 (Identity theorem) Let $f : D \to Y$ be analytic. If $\{z_k\}$ is a set in D having a limit point in D and if $f(z_k) = 0$ for all k , then $f \equiv 0$ on D . (Note that D is open and connected.)

4.3 (Vitali) Let $f_n : D \to Y$ be analytic and $\|f_n(z)\| \leq \alpha$ for all n and $z \in D$. Suppose there is a set $\{z_k\}$ in D having a limit point in D such that for each $k = 1,2,\ldots,$ $\lim_{n\to\infty} f_n(z_k)$ exists. Then $\lim_{n\to\infty} f_n(z)$

exists for each $z \in D$, the convergence is uniform for z in every compact subset of D, and the limit function is analytic on D.

4.4 Let $\alpha : [c,d] \to [a,b]$ be strictly increasing, continuous and satisfy $\alpha(c) = a$, $\alpha(d) = b$. If $\Gamma : [a,b] \to \mathbb{C}$ is rectifiable and $f : \Gamma \to Y$ is continuous, then $\int_{\Gamma\circ\alpha} f(z)dz = \int_\Gamma f(z)dz$, showing that $\int_\Gamma f(z)dz$ does not depend on the parametrization of Γ.

4.5 (Morera's theorem) Let D be a simply connected domain and $f : D \to Y$ be continuous. If $\int_\Gamma f(z)dz = 0$ for every simple closed rectifiable curve Γ in D, then f is analytic on D.

4.6 Let $f : \Gamma \to Y$ be continuous. Then the function

$$g(z) = \int_\Gamma \frac{f(w)}{w-z}\,dw\ , \quad z \notin \Gamma\ ,$$

is analytic for $z \in \mathbb{C} \setminus \Gamma$, and $g'(z) = \int_\Gamma \frac{f(w)}{(w-z)^2}\,dw$.

4.7 Let $f : \Gamma \to Y$ be continuous. Then for every $y^* \in Y^*$,

$$\langle y^*, \int_\Gamma f(z)dz\rangle = -\int_{\bar{\Gamma}} \langle y^*, f(\bar{w})\rangle dw\ .$$

4.8 Let Γ be a continuously differentiable curve in $\mathbb{C}$ and let $f : \Gamma \to Y$ be continuous. Let h be a complex-valued analytic function on a neighbourhood of Γ, which is one to one. If $\tilde{\Gamma} = h\circ\Gamma$, then

$$\int_{\tilde{\Gamma}} f(h^{-1}(w))dw = \int_\Gamma f(z)h'(z)dz\ .$$

In particular, if $z_0 \notin \Gamma$, and $h(z) = 1/(z-z_0)$, then we have

$$\int_\Gamma f(z)dz = -\int_{\tilde{\Gamma}} f(z_0+\tfrac{1}{w})\frac{dw}{w^2}\ .$$