NON—REFLEXIVE REALIZATIONS OF NON—DISTRIBUTIVE SUBSPACE LATTICES

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Abstract. Let $\mathcal{F}$ be a subspace lattice on a (complex) Hilbert space H . A subspace lattice $\mathcal{G}$ on a Hilbert space K is a *realization* of $\mathcal{F}$ on K if $\mathcal{G}$ is lattice—isomorphic to $\mathcal{F}$. In 1975 it was proved that if $\mathcal{F}$ is completely distributive every realization $\mathcal{G}$ of it is reflexive (that is, $\mathcal{G}$ is the set of invariant subspaces of a family of operators). A partial converse has recently been found: If every realization of $\mathcal{F}$ is reflexive and $\mathcal{F}$ has a finite—dimensional realization, then $\mathcal{F}$ is completely distributive. This is proved by showing that every non—distributive subspace lattice on a finite—dimensional space has a non—reflexive realization on the same space.

1. Introduction

This expository article concerns a problem that arose about 14 years ago. A partial solution to it has recently been found. This solution precipitated after discussions with algebraists (in particular, R.S. Freese) at the

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University of Hawaii.

After some preliminary remarks concerning notation and terminology I will

describe how the problem arose. After this, I will discuss, and outline the

steps of, the recently found proof. This discussion will include the

consideration of a special case in an attempt to clarify why the proof works.

Finally, some open problems will be mentioned. For more details, the reader

is referred to [9].

Throughout, H denotes a complex Hilbert space. The algebra of

(bounded, linear) operators on H is denoted by $\mathcal{B}(H)$ and the lattice of

(closed, linear) subspaces of H is denoted by $\mathcal{C}(H)$. For any subset $\mathcal{E}$ of

H , $\langle\mathcal{E}\rangle$ denotes the linear span of $\mathcal{E}$. The closed linear span of a family

$\{M_\gamma : \gamma \in \Gamma\}$ of subspaces of H is denoted by $\vee_\Gamma M_\gamma$. The inner—product on

H is denoted by $(\cdot|\cdot)$. If $T \in \mathcal{B}(H)$, $G(T)$ denotes the graph of T ; so

$G(T) = \{(x, Tx) : x \in H\}$. Two (abstract) lattices L_1 and L_2 are

isomorphic if there exists a bijection $\varphi : L_1 \to L_2$ satisfying $a \le b$ if and

only if $\varphi(a) \le \varphi(b)$. A lattice L is *distributive* if

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

and its dual statement hold identically in L .

For every subset $\mathcal{F} \subseteq \mathcal{C}(H)$ define Alg $\mathcal{F}$ by

$$\text{Alg } \mathcal{F} = \{T \in \mathcal{B}(H) : TM \subseteq M \text{ , for every } M \in \mathcal{F}\} .$$

Dually, for every subset $\mathcal{O}l \subseteq \mathcal{B}(H)$ define Lat $\mathcal{O}l$ by

$$\text{Lat } \mathcal{O}l = \{M \in \mathcal{C}(H) : TM \subseteq M \text{ , for every } T \in \mathcal{O}l \} .$$

Then Alg $\mathcal{F}$ is an algebra of operators on H and Lat $\mathcal{O}l$, the set of

invariant subspaces of $\mathcal{O}l$, is a lattice of subspaces of H . For any subset

$\mathcal{F}$, $\mathcal{F} \subseteq$ Lat Alg $\mathcal{F}$. Call $\mathcal{F}$ *reflexive* if $F = $ Lat Alg $\mathcal{F}$. (Alg is a

mapping from the power set of $\mathcal{C}(H)$ into the power set of $\mathcal{B}(H)$, and Lat is a mapping in the reverse direction. If we partially order these power sets by inclusion, the pair Alg , Lat is a Galois connection; see [13, p.70] for a general definition. The Galois closed subsets of $\mathcal{C}(H)$ are precisely the reflexive lattices. The Galois closed subsets of $\mathcal{B}(H)$ are called *reflexive algebras*. These have been the subject of intensive study; see [11] for further references.) The notion of reflexivity presented here, with its topological overtones, is due to Halmos in 1968 [2] . A purely algebraic version of the notion was considered by Thrall in 1952 [14] .

A subset $\mathcal{F} \subseteq \mathcal{C}(H)$ is reflexive if and only if $\mathcal{F} = \text{Lat } \mathcal{U}$ for some family $\mathcal{U}$ of operators on H . This characterization is more descriptive than the definition; reflexive lattices are invariant subspace lattices. Obvious necessary conditions for the reflexivity of $\mathcal{F}$ are (1) (0) , $H \in \mathcal{F}$, (2) for every family $\{M_\gamma : \gamma \in \Gamma\} \subseteq \mathcal{F}$, $\vee_\Gamma M_\gamma \in \mathcal{F}$ and $\cap_\Gamma M_\gamma \in \mathcal{F}$. A subset satisfying conditions (1) and (2) is called a *subspace lattice* on H .

2.   Reflexive lattices

A characterization of reflexivity has yet to be found, even in finite-dimensions. There are many partial results some of which will now be mentioned. Of course, $\mathcal{C}(H)$ is reflexive: $\mathcal{C}(H) = \text{Lat } \mathbb{C}I$. A *double triangle lattice* is a five-element lattice, with a greatest and a least element, in which every pair of non-trivial elements are complementary, that is, a lattice with Hasse diagram . If dim $H < \infty$, every non-distributive subspace lattice on it contains a double triangle sublattice by the Birkhoff-Dedekind criteria [13, p.90]. Consider the double triangle subspace lattice $\mathcal{D}$ on $H \oplus H$ with non-trivial elements consisting of the two 'axes' $H \oplus (0)$ and $(0) \oplus H$ and the 'diagonal' $G(I) = \{(x,x) : x \in H\}$. It is

easy to see that $\text{Alg } \mathcal{D} = \{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in \mathcal{B}(H) \}$. Clearly, $G(\alpha I) \in \text{Lat Alg } \mathcal{D}$

for every $\alpha \in \mathbb{C}$. In fact, $\text{Lat Alg } \mathcal{D} = \{ (0),\ (0) \oplus H,\ H \oplus H \} \cup \{ G(\alpha I) :$

$\alpha \in \mathbb{C} \}$. The Hasse diagrams of $\mathcal{D}$ and $\text{Lat Alg } \mathcal{D}$ are represented by Figures
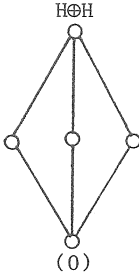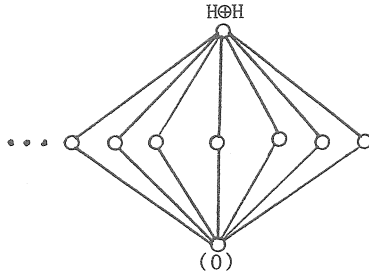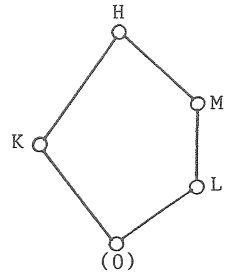
1 and 2 respectively.

Fig.1   Fig.2   Fig.3

Of course $\mathcal{D}$ is not reflexive; $\text{Lat Alg } \mathcal{D}$ is.

The above serves to indicate why the following is true: Every finite,

reflexive subspace lattice $\mathcal{F}$ on a finite—dimensional space is distributive.

(This was first proved by Jans in 1957 [6], and, independently, by Johnson

[7].) For, if $\mathcal{F}$ were non—distributive, it would contain a double triangle

sublattice $\mathcal{D}_o$. Then $\text{Lat Alg } \mathcal{D}_o \subseteq \text{Lat Alg } \mathcal{F} = \mathcal{F}$. But $\text{Lat } \mathcal{D}_o$ is

infinite; so then is $\mathcal{F}$. Contradiction. This result is false in

infinite—dimensions: there exists a reflexive pentagon $\mathcal{P}$ [3]. The Hasse

diagram of $\mathcal{P} = \{ (0),\ K,\ L,\ M,\ H \}$ is Figure 3. Note that

$M \cap (K \vee L) = M \neq L = (M \cap K) \vee (M \cap L)$. The converse, that finite

distributive subspace lattices are reflexive, is true [4,5]. (This was first

proved, for finite—dimensional spaces, in 1952 by Thrall [14], and

independently by Johnson [7].)

In 1965 Ringrose [12] proved that every totally ordered subspace lattice

$\mathcal{F}$ is reflexive. He did this by showing that $\mathcal{F} = \text{Lat } \mathcal{R}$ where $\mathcal{R}$ denotes

the set of rank one operators of $\text{Alg } \mathcal{F}$. Along the way he defined, for every

element $M \in \mathcal{F}$, the element $M_- \in \mathcal{F}$ by $M_- = \vee\{N \in \mathcal{F} : N \subset M\}$ where '⊂'
denotes strict inclusion and where, by convention $\vee\phi = (0)$, so that
$(0)_- = (0)$. For every pair $e,f \in H$ of non-zero vectors let $e \otimes f$ denote
the rank one operator on H given by $e \otimes f(x) = (x|e)f$. Ringrose showed that
$e \otimes f \in \text{Alg } \mathcal{F}$ if and only if $f \in M$ and $e \in M_-^\perp$ for some $M \in \mathcal{F}$. In other
interesting special cases of reflexivity obtained by Harrison [5] and Halmos
[3], including the reflexive pentagon, it again turned out that $\mathcal{F} = \text{Lat } \mathcal{R}$.
This suggested the question: Which subspace lattices $\mathcal{F}$ on H satisfy
$\mathcal{F} = \text{Lat } \mathcal{R}$ where $\mathcal{R}$ is the set of rank one operators of $\text{Alg } \mathcal{F}$?

Let $\mathcal{F}$ be a subspace lattice on H . Extend Ringrose's definition of
M_- . For every $M \in \mathcal{F}$, put $M_- = \vee\{N \in \mathcal{F} : M \nsubseteq N\}$ and, for every $N \in \mathcal{F}$
put $N_* = \cap\{M_- : M \in \mathcal{F} \text{ and } M \nsubseteq N\}$, with the convention that $\cap\phi = H$ so
that $H_* = H$. Then $N_* \in \mathcal{F}$ and $N \subseteq N_*$. Once again it turns out that
$e \otimes f \in \text{Alg } \mathcal{F}$ if and only if $f \in M$ and $e \in M_-^\perp$ for some $M \in \mathcal{F}$, and we
have the following results.

PROPOSITION (1975, [8]). $\text{Lat } \mathcal{R} = \{K \in \mathcal{C}(H) : N \subseteq K \subseteq N_* \text{ for some } N \in \mathcal{F} \}$.
COROLLARY. *If* $\dim(N_* \ominus N) \leq 1$ *for every* $N \in \mathcal{F}$, *then* $\mathcal{F} = \text{Lat } \mathcal{R}$.
COROLLARY. *If* $N = N_*$ *for every* $N \in \mathcal{F}$, *then* $\mathcal{F} = \text{Lat } \mathcal{R}$.

The first of these corollaries explains the reflexivity of Halmos' pentagon
$\mathcal{P}$. In $\mathcal{P}$, see Figure 3, $\dim(M \ominus L) = 1$. In the second corollary, the
condition 'for every $N \in \mathcal{F}$, $N = N_*$' is purely lattice-theoretic. It was
shown to be equivalent to the notion of 'complete distributivity', a very
strong form of distributivity. The types of subspace lattices, mentioned
earlier, considered by Ringrose, Harrison and Halmos (apart from the pentagon)
are completely distributive.

THEOREM (1975, [8]). *Every completely distributive subspace lattice* $\mathcal{F}$ *on* H *is reflexive. In fact* $\mathcal{F} = $ Lat $\mathcal{R}$ *where* $\mathcal{R}$ *is the set of rank one operators of* Alg $\mathcal{F}$.

At the time, I called completely distributive subspace lattices *strongly reflexive* because, if $\mathcal{F}$ is strongly reflexive, every subspace lattice $\mathcal{G}$ on a (complex) Hilbert space K , which is lattice—isomorphic to $\mathcal{F}$, is also reflexive. In general, for a given subspace lattice $\mathcal{L}$ on H call a subspace lattice $\mathcal{M}$ on a Hilbert space K , a *realization* or a *copy* of $\mathcal{L}$ on K if $\mathcal{M}$ is lattice—isomorphic to $\mathcal{L}$. It is not hard to find an example of a reflexive subspace lattice with a non—reflexive realization. For example, let $\mathcal{D}$ be the double triangle described earlier. Then Lat Alg $\mathcal{D}\backslash\{G(-I)\}$ is a non—reflexive copy of Lat Alg $\mathcal{D}$. Also, Halmos' pentagon $\mathcal{P}$ has a non—reflexive realization [10]. Such considerations lead to the following question concerning the appropriateness of the terminology 'strongly reflexive'.

QUESTION. If a subspace lattice $\mathcal{F}$ has the property that *every* realization of it is reflexive, must $\mathcal{F}$ be completely distributive?

The answer is not known. Recently, however, I've found that if we additionally require that $\mathcal{F}$ have a realization on a finite—dimensional space, then the answer is affirmative. In particular, $\mathcal{C}$(H) for $1 < \dim H < \infty$, has a non—reflexive realization.

3. Non—reflexive realizations

If the underlying space is finite—dimensional, a subspace lattice is completely distributive if and only if it is distributive. Thus, in order to

show that the question at the end of preceding section has an affirmative
answer, given that $\mathcal{F}$ has a finite–dimensional realization, it is enough to
prove the following.

THEOREM (1988, [9]). *Every non-distributive subspace lattice on a*
finite-dimensional complex Hilbert space has a non-reflexive realization on
the same space.

Let me discuss, and outline the steps of, the proof. A key observation is
that there are proper subfields of $\mathbb{C}$ which are field–isomorphic to $\mathbb{C}$; in
fact there are 2^{c} such subfields [1, p.233]. Throughout the remainder of
this section $\dim H < \infty$ and $\mathbb{F}$ denotes a proper subfield of $\mathbb{C}$ isomorphic
to $\mathbb{C}$ by the mapping $\sigma : \mathbb{C} \to \mathbb{F}$. So σ is a bijection satisfying
$\sigma(\alpha + \beta) = \sigma(\alpha) + \sigma(\beta)$ and $\sigma(\alpha\beta) = \sigma(\alpha)\sigma(\beta)$ for every $\alpha,\beta \in \mathbb{C}$.

First, we use σ to define a lattice–isomorphism φ of $\mathcal{C}(H)$ onto a
certain subspace lattice $\mathcal{L}$ on H . Let $\{e_1,e_2,\ldots,e_n\}$ be a basis for H .
Define $\tilde{\sigma} : H \to H$ by

$$\tilde{\sigma}\left(\sum_1^n \xi_j e_j \right) = \sum_1^n \sigma(\xi_j)e_j \quad .$$

For every subspace M of H , define $\tilde{\sigma}(M) = \{\tilde{\sigma}(x) : x \in M\}$. Then $\tilde{\sigma}(M)$ is
closed under addition, but it need not be a subspace. For example,
$\tilde{\sigma}(<e_1>) = \{\beta e_1 : \beta \in \mathbb{F}\}$. Define $\varphi : \mathcal{C}(H) \to \mathcal{C}(H)$ by $\varphi(M) = < \tilde{\sigma}(M) >$.
Then $\mathcal{L} = \{\varphi(M) : M \in \mathcal{C}(H)\}$ is a subspace lattice on H and $\varphi : \mathcal{C}(H) \to \mathcal{L}$
is an isomorphism. In particular, $\varphi(\cap_\Gamma M_\gamma) = \cap_\Gamma \varphi(M_\gamma)$ and $\varphi(\vee_\Gamma M_\gamma) = \vee_\Gamma \varphi(M_\gamma)$
for every family $\{M_\gamma : \gamma \in \Gamma\}$ of subspaces of H . Also, for any subspace
lattice $\mathcal{F}$ on H , $\varphi(\mathcal{F}) = \{\varphi(M) : M \in \mathcal{F}\}$ is a subspace lattice on H ,

isomorphic to $\mathcal{F}$ by the mapping $M \mapsto \varphi(M)$.

The mapping φ preserves dimension. In fact, if $\{x_1, x_2, \ldots, x_m\}$ is a basis for M , then $\{\tilde{\sigma}(x_1), \tilde{\sigma}(x_2), \ldots, \tilde{\sigma}(x_m)\}$ is a basis for $\varphi(M)$. Proof of the latter rests on the fact that, if A is an $n \times n$ invertible matrix all of whose entries belong to $\mathbb{F}$, then all of the entries of A^{-1} also belong to $\mathbb{F}$. Suppose $\dim H > 1$. We show that φ does not map onto $\mathcal{C}(H)$. Let $\beta \in \mathbb{C} \backslash \mathbb{F}$ and put $N = \langle e_1 + \beta e_2 \rangle$. If $\varphi(M) = N$ for some $M \in \mathcal{C}(H)$, then $N = \langle \tilde{\sigma}(e) \rangle$ for some vector $e \in H$. Thus there exists $\gamma \in \mathbb{C}$ such that $e_1 + \beta e_2 = \gamma \tilde{\sigma}(e) = \gamma \left(\sum_1^n \sigma(\xi_j) e_j \right)$ say. Comparing coefficients, $1 = \gamma \sigma(\xi_1)$ and $\beta = \gamma \sigma(\xi_2)$. Thus $\beta = \sigma(\xi_2/\xi_1) \in \mathbb{F}$. Contradiction.

We will show presently that, if $\mathcal{F}$ is non-distributive, then $\varphi(\mathcal{F})$ is non-reflexive. However, in an attempt to clarify things let us next consider the case of the non-distributive (and reflexive) subspace lattice Lat Alg $\mathcal{D}$ described earlier, with $\mathcal{D}$ the double triangle with non-trivial elements $H \oplus (0), (0) \oplus H$ and $G(I)$. We have Lat Alg $\mathcal{D} = \{(0), (0) \oplus H, H \oplus H\} \cup \{G(\alpha I) : \alpha \in \mathbb{C}\}$. As a basis for $H \oplus H$ take $\{e_1, e_2, \ldots, e_{2n}\}$ where $e_j = (f_j, 0)$ and $e_{j+n} = (0, f_j)$ for $1 \le j \le n$, where $\{f_1, f_2, \ldots, f_n\}$ is a basis for H . With $\mathbb{F}$ and σ as before, the mapping $\tilde{\sigma} : H \oplus H \to H \oplus H$ is given by

$$\tilde{\sigma}(x, y) = \tilde{\sigma}\left(\sum_1^n \xi_j f_j, \sum_1^n \eta_j f_j \right) = \left(\sum_1^n \sigma(\xi_j) f_j, \sum_1^n \sigma(\eta_j) f_j \right)$$

and $\varphi((0) \oplus H) = \langle \tilde{\sigma}(0, y) : y \in H \rangle = (0) \oplus H$. Also, for every $\alpha \in \mathbb{C}$, $\varphi(G(\alpha I)) = \langle \tilde{\sigma}(x, \alpha x) : x \in H \rangle = \{(y, \sigma(\alpha)y) : y \in H\} = G(\sigma(\alpha)I)$. Thus $\varphi(\text{Lat Alg } \mathcal{D}) = \{(0), (0) \oplus H, H \oplus H\} \cup \{G(\beta I) : \beta \in \mathbb{F}\}$ which is clearly

not reflexive.

A lattice, such as Lat Alg $\mathcal{D}$, in which every pair of distinct

non-trivial elements are complementary, is called *medial*. The lattice

Lat Alg $\mathcal{D}$ is a maximal medial subspace lattice on $H \oplus H$: there does not

exist a subspace N of $H \oplus H$ satisfying $N \cap ((0) \oplus H) = (0)$, $N \vee ((0) \oplus H)$

$= H \oplus H$ and, for every $\alpha \in \mathbb{C}$, $N \cap G(\alpha I) = (0)$, $N \vee G(\alpha I) = H \oplus H$. (The

first pair of equations would give $N = G(B)$ for some $B \in \mathcal{B}(H)$; the

remaining conditions then give that B has no eigenvalues.)

As far as mediality is concerned, the situation with general double

triangle sublattices of $\mathcal{C}(H)$ is much the same as just remarked. If

$\mathcal{T} = \{E, K, L, M, F\} \subseteq \mathcal{C}(H)$ has Hasse diagram as in Figure 4, with possibly

$E \neq (0)$ and/or $F \neq H$, then Lat Alg $\mathcal{T} = \{(0), E, M, F, H\} \cup \{M_\alpha : \alpha \in \mathbb{C}\}$

where (i) $M_0 = K$ and $M_1 = L$, (ii) $M_\alpha \cap M = E$ and $M_\alpha \vee M = F$, for every

$\alpha \in \mathbb{C}$, and (iii) $M_\alpha \cap M_\beta = E$ and $M_\alpha \vee M_\beta = F$, if $\alpha, \beta \in \mathbb{C}$ and $\alpha \neq \beta$.

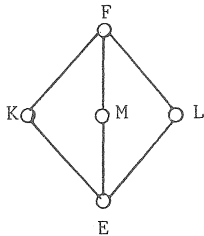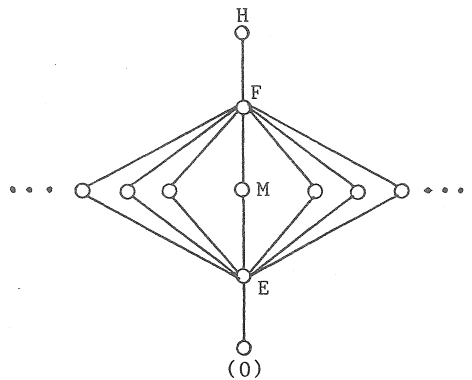The Hasse diagram of Lat Alg $\mathcal{T}$ is represented by Figure 5.

Fig.4

Fig.5

There does not exist a subspace Q of H satisfying $Q \cap M = E$, $Q \vee M = F$ and, for every $\alpha \in \mathbb{C}$, $Q \cap M_\alpha = E$, $Q \vee M_\alpha = F$.

Finally, let $\mathscr{F}$ be a non-distributive subspace lattice on H. Let $\varphi : \mathscr{C}(H) \to \mathscr{C}(H)$ be the mapping defined at the beginning of this section and put $\mathscr{G} = \varphi(\mathscr{F})$. We show that $\mathscr{G}$ is non-reflexive. Now $\mathscr{F}$ contains a double triangle sublattice $\mathscr{T} = \{E, K, L, M, F\}$ say, with Hasse diagram as in Figure 4, and $\varphi(\mathscr{T})$ is a double triangle sublattice of $\mathscr{G}$. We have Lat Alg $\mathscr{T} = \{(0), E, M, F, H\} \cup \{M_\alpha : \alpha \in \mathbb{C}\}$ and Lat Alg $\varphi(\mathscr{T}) = \{(0), \varphi(E), \varphi(M), \varphi(F), H\} \cup \{N_\alpha : \alpha \in \mathbb{C}\}$ where it can be shown that $\varphi(M_\alpha) = N_{\sigma(\alpha)}$ for every $\alpha \in \mathbb{C}$.

If $\beta \in \mathbb{C} \backslash \mathbb{F}$, then $N_\beta \in$ Lat Alg $\varphi(\mathscr{T}) \subseteq$ Lat Alg $\mathscr{G}$. But $N_\beta \notin \mathscr{G}$. For if $N_\beta \in \mathscr{G}$ then $N_\beta = \varphi(Q)$ for some $Q \in \mathscr{F}$.

Since

$\qquad$ (1) $\quad N_\beta \cap \varphi(M) = \varphi(E)$ and $N_\beta \vee \varphi(M) = \varphi(F)$,

and $\qquad$ (2) $\quad$ for every $\alpha \in \mathbb{C}$, $N_\beta \cap \varphi(M_\alpha) = \varphi(E)$ and $N_\beta \vee \varphi(M_\alpha) = \varphi(F)$,

we have

$\qquad$ (1)' $\quad Q \cap M = E$ and $Q \vee M = F$,

and $\qquad$ (2)' $\quad$ for every $\alpha \in \mathbb{C}$, $Q \cap M_\alpha = E$ and $Q \vee M_\alpha = F$.

But no such subspace Q exists. Contradiction. Hence $\mathscr{G}$ is non-reflexive.

4. Concluding remarks

(I) As we have already remarked, it is not known whether or not a subspace lattice, whose every realization is reflexive, must be completely distributive. If the underlying space H is infinite-dimensional it is not

clear how to proceed from the existence of a proper subfield $\mathbb{F}$ of $\mathbb{C}$ and a field—isomorphism $\sigma : \mathbb{C} \to \mathbb{F}$. Can useable maps $\tilde{\sigma} : H \to H$ and $\varphi : \mathscr{C}(H) \to \mathscr{C}(H)$ be defined? An additional problem is that an arbitrary non—completely—distributive subspace lattice on H need not contain a double triangle sublattice (the pentagon $\mathcal{P}$ is an example). Also, the class of subspace lattices arising as Lat Alg $\mathcal{T}$ for some double triangle sublattice $\mathcal{T}$ is not nearly as well understood as in finite—dimensions.

(II)   Concerning subspace lattices, some authors impose the additional requirement of strong closedness. A subspace lattice $\mathcal{F}$ on H is *strongly closed* if the family of orthogonal projections whose ranges belong to $\mathcal{F}$ is closed in the strong operator topology. This is a necessary condition for reflexivity [3]. With this additional requirement, reconsider the basic question raised at the end of section 2. What can be said?

(III)   If L is an abstract complete lattice, call a subspace lattice $\mathcal{F}$ on a complex Hilbert space H a *realization* of L on H if $\mathcal{F}$ is lattice—isomorphic to L. The question: Which completely distributive lattices L *have* a realization on Hilbert space? seems worth considering. It has not been investigated though there are some partial results scattered throughout the literature.

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