1.9. Perturbation Theory

The next aspect of stability that we describe is stability of a semigroup under perturbations of its generator. Let H be the generator of a C_0-semigroup of contractions on the Banach space $\mathcal{B}$ and P a linear operator on $\mathcal{B}$. Our aim is to describe conditions on P which ensure that $H + P$ also generates a C_0-semigroup of contractions. In applications the perturbation P is often an unbounded operator and the notion of relatively bounded operator is useful.

Let H and P be linear operators on a Banach space. Then P is defined to be *relatively bounded* with respect to H, or H-*relatively bounded*, if the following two conditions are satisfied:

1. $D(P) \supseteq D(H)$

2. $\|Pa\| \leq \alpha\|a\| + \beta\|Ha\|$

for all $a \in D(H)$ and some $\alpha, \beta > 0$.

The greatest lower bound of the β for which this last relation is valid is called the *relative bound* of P with respect to H, or the H-*bound*.

The key result concerning relative bounded perturbations of generators of contraction semigroups is the following:

THEOREM 1.9.1. *Let* $S_t = \exp\{-tH\}$ *be a* C_0*-semigroup of contractions on the Banach space* $\mathcal{B}$ *and assume* P *is* H-*relatively bounded with*

H-bound $\beta_0 < 1$.

If P , *or* H + P , *is norm-dissipative then* H + P *generates a* C_0*-semigroup of contractions.*

Proof. First note that it follows from Theorems 1.3.1 and 1.4.1 that $D(H)$ is norm dense and $\mathrm{Re}(f_a, Ha) \geq 0$ for all tangent functionals f_a at $a \in D(H)$. Second since $D(H) \subseteq D(P)$ the latter set is norm dense. Hence if P is norm-dissipative $\mathrm{Re}(f_a, Pa) \geq 0$ for all tangent functionals at $a \in D(H)$ by Theorem 1.4.1. Therefore $\mathrm{Re}(f_a, (H+\lambda P)a) \geq 0$ for all $\lambda \geq 0$ and $H + \lambda P$ is norm-dissipative. Alternatively if $H + P$ is norm-dissipative then $\mathrm{Re}(f_a(H+P)a) \geq 0$ and

$$\begin{aligned}\mathrm{Re}(f_a, (H+\lambda P)a) &= (1-\lambda)\mathrm{Re}(f_a, Ha) + \lambda\mathrm{Re}(f_a, (H+P)a) \\ &\geq 0\end{aligned}$$

for $0 \leq \lambda \leq 1$. Thus in both cases $H + \lambda P$ is norm-dissipative for $0 \leq \lambda \leq 1$.

Next we exploit the relative bound.

Let us assume that

$$\|Pa\| \leq \alpha\|a\| + \beta\|Ha\|$$

for all $a \in D(H)$ where $\alpha > 0$ and $\beta < 1$. Therefore

$$\begin{aligned}\|\lambda P(I+\lambda H)^{-1}a\| &\leq \alpha\|\lambda(I+\lambda H)^{-1}a\| + \beta\|(I-(I+\lambda H)^{-1})a\| \\ &\leq (\alpha\lambda+2\beta)\|a\|\end{aligned}$$

where we have used $\|(I+\lambda H)^{-1}\| \le 1$. Thus if $0 \le \lambda_1 \le (2\beta)^{-1}$ one may choose $\lambda_0 > 0$ such that $\lambda_1(\alpha\lambda+2\beta) < 1$ for $0 < \lambda < \lambda_0$ and then the operator $P_\lambda = \lambda_1\lambda P(I+\lambda H)^{-1}$ is bounded with $\|P_\lambda\| < 1$. Hence $I + P_\lambda$ has a bounded inverse. But

$$\left(I+\lambda\left(H+\lambda_1 P\right)\right) = \left(I+P_\lambda\right)(I+\lambda H)$$

and since $R(I+\lambda H) = \mathcal{B}$ one has

$$\begin{aligned} R\left(I+\lambda\left(H+\lambda_1 P\right)\right) &= R\left(I+P_\lambda\right) \\ &= D\left(\left(I+P_\lambda\right)^{-1}\right) = \mathcal{B} . \end{aligned}$$

Therefore $H + \lambda_1 P$ is the generator of a C_0-semigroup of contractions by Theorem 1.3.5.

To continue the proof we remark that

$$\|Pa\| \le \|a\| + \beta\left\|\left(H+\lambda_1 P\right)a\right\| + \beta\lambda_1\|Pa\|$$

and since $\lambda_1 \le (2\beta)^{-1}$ one has

$$\|Pa\| \le 2\alpha\|a\| + 2\beta\left\|\left(H+\lambda_1 P\right)a\right\| .$$

We may now choose $0 \le \lambda_2 \le (4\beta)^{-1}$ and repeat the above argument to deduce that $H + \left(\lambda_1+\lambda_2\right)P$ is the generator of a C_0-semigroup of contractions. Iteration of this argument n times proves that $H + \lambda P$ is a generator for all $0 \le \lambda < \left(1-2^{-n}\right)/\beta$. Choosing n sufficiently large, but finite, one obtains the desired result. $\square$

Next we examine a more restricted class of perturbations. If $S_t = \exp\{-tH\}$ is a C_0-semigroup and P is a linear operator, on the Banach space $\mathcal{B}$, then P is called a *Phillips perturbation* of S if the following three conditions are satisfied:

1. P is closed.

2. For each $t > 0$ one has $S_t\mathcal{B} \subseteq D(P)$ and PS_t has bounded closure.

3. $\int_0^1 dt\, \|PS_t\| < +\infty$.

Note that if S is a group then each Phillips perturbation P of S is automatically bounded because $P = \big(PS_t\big)S_{-t}$ for each $t > 0$. More generally, for semigroups, P is relatively bounded. To see this consider the case that S is a contraction semigroup. Consequently

$$(\lambda I+H)^{-1}a = \int_0^\infty dt\, e^{-\lambda t}S_t a$$

for each $a \in \mathcal{B}$ and $\lambda > 0$. But one also has

$$\Big\|\int_0^\infty dt\, e^{-\lambda t}PS_t a\Big\| \le \|a\|\Big(\int_0^\delta dt\, \|PS_t\| + \|PS_\delta\|/\lambda\Big)$$

for any $0 < \delta < 1$. Since P is closed a simple Riemann approximation argument establishes that $(\lambda I+H)^{-1}a \in D(P)$, i.e., $D(H) \subseteq D(P)$, and

$$P(\lambda I+H)^{-1}a = \int_0^\infty dt\, e^{-\lambda t}PS_t a \ .$$

Therefore setting $b = (\lambda I+H)^{-1}a$ and using the foregoing estimate one finds

$$\|Pb\| = \left\|\int_0^\infty dt\ e^{-\lambda t}PS_t a\right\|$$

$$\leq \|(\lambda I+H)b\|\left(\int_0^\delta dt\ \|PS_t\|+\|PS_\delta\|/\lambda\right)$$

$$\leq (\lambda\|b\|+\|Hb\|)\left(\int_0^\delta dt\ \|PS_t\|+\|PS_\delta\|/\lambda\right) .$$

Thus P is H-relatively bounded. Moreover choosing δ to be small and λ to be large one sees that P has H-bound zero. The same conclusion is indeed valid for a general C_0-semigroup but one must use the bound $\|S_t\| \leq M\exp\{\omega t\}$ and take $\lambda > \omega$.

Theorem 1.9.1 can now be strengthened for the class of Phillips perturbations.

THEOREM 1.9.2. *Let* $S_t = \exp\{-tH\}$ *be a* C_0*-semigroup of contractions on the Banach space* $\mathcal{B}$ *and* P *a Phillips perturbation of* S .

If P *, or* $H + P$ *, is norm-dissipative then* $H + P$ *generates a* C_0*-semigroup of contractions* S^P . *Moreover*

$$S_t^P a = S_t a + (-1)^n \sum_{n\geq 1} \int_{0\leq t_n\leq t_{n-1}\leq\ldots\leq t_1\leq t} dt_1 \ldots dt_n$$

$$S_{t-t_1}PS_{t_1-t_2}P \ldots PS_{t_n}a$$

for all $a \in \mathcal{B}$ *, where the integrals exist in the norm topology and define a series of bounded operators which converges in norm* *uniformly for* t *in any finite interval of the form* $\langle \varepsilon, 1/\varepsilon \rangle$ *where* $0 < \varepsilon < 1$.

Proof. The first statement of the theorem follows from Theorem 1.8.1 and the foregoing observation that a Phillips perturbation P of H is H-relatively bounded with H-bound zero.

Now consider the perturbation series for S^P . It follows from the definition of a Phillips perturbation that each term is well defined as a bounded operator and is strongly continuous for $t > 0$. But if $S_t^{(n)}$ denotes the n-th term then

$$S_t^{(0)} = S_t \ , \qquad S_t^{(n)} = (-1) \int_0^t ds\, S_{t-s} P S_s^{(n-1)} \ .$$

Hence, by iteration,

$$\|S_t^{(n)}\| \le g * f^{n*}(t)$$

where

$$g(t) = \|S_t\|, \qquad f(t) = \|PS_t\| \ ,$$

the $*$ denotes the convolution product, and f^{n*} denotes the n-fold convolution of f with itself.

Now let us examine bounds on f .

Since S is contractive f is non-increasing and the integral

$$I_\lambda = \int_0^\infty dt\, e^{-\lambda t} f(t)$$

is finite for each $\lambda > 0$. Moreover $I_\lambda \to 0$ as $\lambda \to \infty$. But for $0 < s < t$ one has $f(t) \leq f(t-s)$ and hence

$$2\left[e^{-\lambda t}f(t)\right] \leq \left(e^{-\lambda(t-s)}f(t-s)+e^{-\lambda s}\right)^2 .$$

Therefore

$$t\left[e^{-\ t}f(t)\right]^{\frac{1}{2}} \leq \int_0^{t/2} ds \; \left(e^{-\lambda(t-s)}f(t-s)+e^{-\lambda s}\right)$$

$$\leq I_\lambda + 1/\lambda .$$

Consequently for λ sufficiently large

$$\int_0^\infty dt\; e^{-\lambda t}f(t) \leq 1/2^4$$

and

$$f(t) \leq e^{\lambda t}/2^4t^2 .$$

Moreover since S is contractive there is an $M > 0$ such that

$$\int_0^\infty dt\; e^{-\lambda t}g(t) \leq M , \qquad g(t) \leq Me^{\lambda t}/t^2$$

for this same range of large λ .

Next we examine the propagation of these bounds.

Suppose two positive integrable functions f_1 , f_2 , on $[0, \infty\rangle$ satisfy

$$\int_0^\infty dt\; e^{-\lambda t}f_i(t) \leq M_i , \qquad f_i(t) \leq M_ie^{\lambda t}/t^2 .$$

Then

$$\int_0^\infty dt\ e^{\lambda t}\left(f_1 * f_2\right)(t) = \int_0^\infty dt\ e^{-\lambda t} f_1(t) \int_0^\infty ds\ e^{-\lambda s} f_2(s)$$

$$\leq M_1 M_2 \leq 8 M_1 M_2 \ .$$

Moreover

$$\left(f_1 * f_2\right)(t) \leq e^{\lambda t} \int_0^t ds\ \left(e^{\lambda(t-s)} f_1(t-s)\right)\left(e^{-\lambda s} f_2(s)\right)$$

$$\leq e^{\lambda t} \int_0^{t/2} ds\ \frac{M_1}{(t-s)^2}\ e^{-\lambda s} f\ (s) + e^{\lambda t} \int_{t/2}^t ds\ e^{-\lambda(t-s)} f_1(t-s)\ \frac{M_2}{s^2}$$

$$\leq 8 M_1 M_2 e^{\lambda t}/t^2 \ .$$

Thus the bounds propagate.

Combining the foregoing estimates one concludes that

$$\int_0^\infty dt\ e^{-\lambda t} g * f^{n*}(t) \leq M/2^n$$

and

$$g * f^{n*}(t) \leq M e^{\lambda t}/2^n t^2 \ .$$

Consequently the perturbation series for S^P is majorized in norm by the series

$$\sum_{n \geq 0} M e^{\lambda t}/2^n t^2 = 2 M e^{\lambda t}/t^2$$

and this immediately implies the convergence statements for the perturbation series.

It remains to prove that S^P is a C_0-semigroup with generator $H + P$.

First strong continuity at the origin follows from the integrability of $t \mapsto \|PS_t\|$ at the origin and the straightforward estimate

$$\|(S_t^P - S_t)a\| \leq \sum_{n\geq 1} \int_{0\leq t_n\leq \dots \leq t_1 \leq t} dt_1 \dots dt_n \, \|PS_{t_1-t_2}\| \, \|PS_{t_2-t_3}\| \dots \|PS_n\| \|a\|$$

$$\leq \sum_{n\geq 1} \left\{ \int_0^t ds \, \|PS_s\| \right\}^n \|a\| .$$

Second note that S^P satisfies the integral equation

$$S_t^P = S_t - \int_0^t ds \, S_{t-s} P S_s^P$$

and hence

$$S_{t_1}^P S_{t_2}^P = S_{t_1} S_{t_2}^P - \int_0^{t_1} ds \, S_{t_1-s} P S_s^P S_{t_2}^P$$

$$= S_{t_1+t_2} - \int_0^{t_2} ds \, S_{t_1+t_2-s} P S_s^P - \int_0^{t_1} ds \, S_{t_1-s} P S_s^P S_{t_2}^P$$

$$= S_{t_1+t_2}^P + \int_0^{t_1} ds \, S_{t_1-s} P \left\{ S_{s+t_2}^P - S_s^P S_{t_2}^P \right\} .$$

Thus the family of operator-valued functions

$$\lambda \in \mathbb{C} \mapsto F_{t_1}(\lambda) = S_{t_1}^{\lambda P} S_{t_2}^{\lambda P} - S_{t_1+t_2}^{\lambda P}$$

is entire analytic, in the norm topology, and satisfies the homogeneous integral equations

$$F_t(\lambda) = \lambda \int_0^t ds\ S_{t-s} P F_s(\lambda) .$$

It then follows from Taylor's series that $F_t(\lambda) = 0$, i.e., the semigroup property

$$S^P_{t_1} S^P_{t_2} = S^P_{t_1+t_2}$$

is valid.

Finally let K denote the generator of S^P . For λ sufficiently large one has

$$(\lambda I+K)^{-1} = \int_0^\infty dt\ e^{-\lambda t} S^P_t .$$

But using the integral equation for S^P one finds

$$\begin{aligned}(\lambda I+K)^{-1} &= \int_0^\infty dt\ e^{-\lambda t} S_t - \int_0^\infty dt \int_0^t ds\ e^{-\lambda t} S_{t-s} P S^P_s \\ &= (\lambda I+H)^{-1} - \int_0^\infty dt\ e^{-\lambda t} S_t\ P \int_0^\infty ds\ e^{-\lambda s} S^P_s \\ &= (\lambda I+H)^{-1} - (\lambda I+H)^{-1} P (\lambda I+K)^{-1} .\end{aligned}$$

This establishes that

$$(\lambda I+H+P)(\lambda I+K)^{-1} = I .$$

But

$$(\lambda I+H+P) = \left(I+P(\lambda I+H)^{-1}\right)(\lambda I+H)$$

and

$$\|P(\lambda I+H)^{-1}\| \le \int_0^\infty dt\ e^{-\lambda t}\|PS_t\| < 1$$

for λ sufficiently large. Therefore $(\lambda I+H+P)$ is invertible with bounded inverse. Consequently

$$(\lambda I+H+P)^{-1} = (\lambda I+K)^{-1}$$

and

$$K = H + P . \qquad \square$$

Remark 1.9.3. One can obtain an analogue of Theorem 1.9.2 without assuming that S is contractive or P norm-dissipative. *If* $S_t = \exp\{-tH\}$ *is a* C_0*-semigroup and* P *a Phillips perturbation of* S *then* $H + P$ *generates a* C_0*-semigroup* S^P *which can be defined by the perturbation series of Theorem 1.9.2.* The proof of this generalization is very similar to the above proof but the estimates necessary for the convergence of the series are slightly more onerous because of the growth of $\|S_t\|$.

Example 1.9.4. Let $\mathcal{B} = L^p(\mathbb{R}^\nu)$ and let S be the semigroup generated by the Laplacian, i.e.,

$$(S_tf)(x) = (\mu_t * f)(x)$$

where

$$\mu_t(x) = (4\pi t)^{-\nu/2}\exp\{\frac{-x^2}{4t}\} .$$

Next let V be a multiplication operator

$$(Vf)(x) = V(x)f(x)$$

where $V \in L^q(\mathbb{R}^\nu)$ and $q > \nu/2$, $q \geq p$. Then by successively applying Hölder's and Young's inequalities

$$\begin{aligned} \|VS_t f\|_p &\leq \|V\|_q\|\mu_t * f\|_r \\ &\leq \|V\|_q\|\mu_t\|_s\|f\|_p \end{aligned}$$

where $p^{-1} = q^{-1} + r^{-1}$, $r^{-1} + 1 = s^{-1} + p^{-1}$, and $1 \leq p, q, r, s \leq \infty$. But

$$\|\mu_t\|_s \leq ct^{(\nu/2)(s^{-1}-1)} = ct^{-\nu/2q} .$$

Thus $\|VS_t\|_p$ is integrable at the origin and V is a Phillips perturbation of S .

Exercises.

1.9.1. Let P be relatively bounded with respect to H with H-bound less than one. Prove that $H + P$ is closable if, and only if, H is closable and in this case the closures have the same domain.

1.9.2. If P is relatively bounded with respect to H with H-bound $\beta < 1$ prove that P is relatively bounded with respect $H + P$ with H+P-bound $\beta(1-\beta)^{-1}$.

1.9.3. Let H be the generator of a C_0-contraction semigroup on a Banach space $\mathcal{B}$ and suppose P is relatively bounded with respect to H. Prove that if $\lambda > 0$ then

$$\|P(\lambda I+H)^{-1}\| \le \alpha\lambda^{-1} + 2\beta .$$

Moreover if $\mathcal{B}$ is a Hilbert space

$$\|P(\lambda I+H)^{-1}\| \le \alpha\lambda^{-1} + \beta .$$

Hint: In the Hilbert space case use norm-dissipativity to prove that

$$\|(\lambda I+H)a\|^2 \ge \lambda^2\|a\|^2 + \|Ha\|^2 .$$

1.9.4. If $a \in L^2(\mathbb{R}^3)$ has partial derivatives in $L^2(\mathbb{R}^3)$ prove that

$$\int d^3x \; |a(x)|^2 / |x|^2 \le 4 \int d^3x \; |\nabla a(x)|^2 .$$

Hint: Calculate $\nabla |x|^{\frac{1}{2}} a(x)$.

1.9.5. Let H denote the Laplacian on $L^p(\mathbb{R}^\nu)$ and χ_Λ the operator of multiplication by the characteristic function of the open bounded set $\Lambda \subset \mathbb{R}^\nu$. Define $S^{(n)}$ to be the C_0-semigroup generated by the perturbed Laplacian $H_n = H + n(I-\chi_\Lambda)$. Prove that $S^{(n)}$ converges strongly on $L^p(\Lambda)$ to the semigroup generated by the Laplacian with Dirichlet boundary conditions, as $n \to \infty$.