1.1. Introduction.

Continuous one-parameter semigroups of bounded operators occur in many branches of mathematics, both pure and applied. The calculus of functions of one real variable can be formulated in terms of the translation semigroup, solutions of the equations connected with classical phenomena such as heat propagation are described by semigroups, and one-parameter groups and semigroups also describe the dynamics of quantum mechanical systems. Although semigroups occur in many other areas the development and scope of the general theory covered in this chapter is well illustrated by the foregoing examples. Hence we begin with a brief discussion of each of them.

The semigroup of right translations on $C_0(\mathbb{R})$, the continuous functions over the real line which vanish at infinity, is defined by

$$f \in C_0(\mathbb{R}) \longmapsto S_t f \in C_0(\mathbb{R}) ,$$

where

$$\left(S_t f\right)(x) = f(x-t) .$$

Thus one has the semigroup property

$$S_s S_t = S_{s+t} , \qquad s, t \geq 0$$

and

$$S_0 = I$$

where I is the identity operator. Moreover S is strongly continuous, i.e.,

$$\lim_{t\to 0+} \|S_t f - f\|_\infty = 0 \ , \qquad f \in C_0(\mathbb{R}) \ ,$$

where $\|\cdot\|_\infty$ indicates the supremum norm. Infinitesimally the action of this semigroup is left differentiation and globally S corresponds in some sense to the exponential of the differentiation operator, e.g., if f is analytic

$$\left(S_t f\right)(x) = \sum_{n\geq 0} \frac{(-t)^n}{n!} \frac{d^n f(x)}{dx^n} \ .$$

Alternatively, the passage from the infinitesimal action $-d \ dx$ to the semigroup S can be described as integration,

$$\left(S_t f\right)(x) = f(x-t) = \int_t^\infty ds \left(-\frac{d}{ds}\right) f(x-s) \ .$$

Thus this example illustrates how differentiation, integration, and approximation theory, underlie the general theory of one-parameter semigroups.

An alternative way of describing the translates $S_t f$ of a function $f \in C_0(\mathbb{R})$ are as solutions of the first-order partial differential equation

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} = 0 \ ,$$

in $C_0\left(\mathbb{R}^2\right)$, and this is the natural way of viewing the second example, the heat semigroup.

The heat equation

$$\frac{\partial f}{\partial t}(x, t) = \frac{\partial^2 f}{\partial x^2}(x, t)$$

describes the infinitesimal change with time t of the spatial heat distribution of an idealized one-dimensional rod. If $f \in C_0(\mathbb{R})$ describes the initial heat distribution, $f(x) = f(x, 0)$, of the infinitely long rod then at time t it is described by the solution $T_t f \in C_0(\mathbb{R})$ of the above equation,

$$f(x, t) = \left(T_t f\right)(x) = (4\pi t)^{-\frac{1}{2}} \int_{-\infty}^{\infty} dy\, e^{-(x-y)^2/4t} f(y) .$$

Again $T = \{T_t\}_{t\geq 0}$ is a strongly continuous semigroup of bounded operators acting on $C_0(\mathbb{R})$, with $T_0 = I$ and once more this semigroup corresponds to exponentiation of the operator $-\partial^2/\partial x^2$ describing the infinitesimal heat flow, e.g., if f is analytic

$$\left(T_t f\right)(x) = \sum_{n\geq 0} \frac{t^n}{n!} \frac{\partial^{2n}}{\partial x^{2n}} f(x) .$$

Thus solution of the heat equation can be viewed as construction of the semigroup from its infinitesimal action.

The translation semigroup and heat semigroup may be integrated on other function spaces such as $L^p(\mathbb{R})$, $p \in [1, \infty]$, but there are also interesting evolution equations in more general spaces than function spaces. For example the theory of quantum mechanics can be phrased in terms of observables A, B, C, ... which are bounded operators on a Hilbert space $\mathcal{H}$ and the change $(A, t) \mapsto A_t$ of these observables with time is given by the Heisenberg equation of motion

$$\frac{\partial A_t}{\partial t} = i\left(HA_t - A_t H\right) ,$$

where H is a self-adjoint operator, the Hamiltonian, and $A_0 = A$. Formally the solution of this equation is

$$A_t = U_t \, A U_{-t}$$

where U_t describes the solution of the Schrödinger equation

$$\frac{\partial \psi_t}{\partial t} = i \, H\psi_t$$

on the Hilbert space $\mathcal{H}$, i.e., $\psi_t = U_t \psi_0$. Thus the evolution of the quantum mechanical observables is described by a semigroup $A_t = S_t A$ acting on the space of all bounded operators $\mathcal{L}(\mathcal{H})$ on $\mathcal{H}$. The infinitesimal action of the semigroup is given by

$$A \mapsto \delta(A) = i(HA - AH)$$

and solution of the Heisenberg equations of motion again corresponds to 'exponentiation' of this action.

The general problem of semigroup theory is to study differential equations of the form

$$\frac{\partial a_t}{\partial t} + Ha_t = 0$$

under a variety of circumstances, to establish criteria for existence of solutions, to develop constructive methods of solution, and to analyze stability properties of the solutions. Each of these aspects will be discussed in this chapter. Formally the solution is always $a_t = \exp\{-tH\}a$ and the key problem is to define the exponential of the infinitesimal operator H. But there are also several important subsidiary factors to consider.

The translation semigroup and the quantum-mechanical semigroup, which were briefly sketched above, both extend to one-parameter groups which are isometric, e.g., $\|S_t f\|_\infty = \|f\|_\infty$ for all $f \in C_0(\mathbb{R})$. The heat semigroup cannot be extended in this manner but it is nevertheless contractive, i.e., $\|T_t f\|_\infty \leq \|f\|_\infty$. In the context of dynamics these conditions of isometry and contraction are connected with conservation laws, e.g., the contractive property of the heat equation reflects the fact that no heat is created in the isolated system, but it can dissipate. Continuity properties are also important. The translation semigroup is strongly continuous on any of the spaces $C_0(\mathbb{R})$ or $L^p(\mathbb{R})$ with $p \in [1, \infty\rangle$ but this is certainly not the case on $L^\infty(\mathbb{R})$. Nevertheless one has the residual continuity property

$$\lim_{t\to 0+} (S_t f, g) = \lim_{t\to 0+} \int_{-\infty}^{\infty} dx\,(S_t f)(x)g(x)$$

$$= \int_{-\infty}^{\infty} dx\, f(x)g(x) = (f, g)$$

for all $f \in L^{\infty}(\mathbb{R})$ and $g \in L^{1}(\mathbb{R})$. Since L^{∞} is the dual of L^{1} this corresponds to weak*-continuity, i.e., weak continuity with respect to the predual. Similarly the heat semigroup is only weak*-continuous on $L^{\infty}(\mathbb{R})$ and the quantum-mechanical semigroup is weak*-continuous on $\mathcal{L}(H)$. Finally each of these semigroups is positive in a natural sense; the translation semigroup and the heat semigroup map positive functions into positive functions, and the quantum-mechanical semigroup maps positive operators into positive operators. Again this form of positivity can often be interpreted in terms of physical conservation laws.

Motivated by these examples we concentrate in this chapter on strongly continuous contraction semigroups and partially describe the theory of weak*-continuous semigroups and groups of isometries. In Chapter 2 we examine positive semigroups.