CHAPTER 5

HARMONIC MAPS BETWEEN SURFACES

5.1 NONEXISTENCE RESULTS

In this chapter, we want to present the existence theory for harmonic maps between closed surfaces, possibly with boundary. In the two-dimensional case, the regularity theory for minimizing maps is very easy, and the local geometry of the image does not lead to any difficulties in contrast to the situation we encountered in chapter 4 (cf. the example in section 4.1). This allows us to investigate in more detail what obstructions for the existence of harmonic maps are caused by the global topology of the image.

We first want to show some instructive nonexistence results which illustrate the difficulties we shall encounter later on when we try to prove existence results by variational methods.

Lemaire [L1] showed

PROPOSITION 5.1.1 *There is no nonconstant harmonic map from the unit disc* D *onto* S^2 *mapping* ∂D *onto a single point.*

Proof Suppose $u : D \to S^2$ is harmonic with $u(\partial D) = p \in S^2$. Since the boundary values of u are constant, u is also a critical point with respect to variations $u \circ \psi$, where $\psi : D \to D$ is a diffeomorphism, mapping ∂D onto itself, but not necessarily being the identity on ∂D.

Thus, one can use a standard argument to show that u is a conformal map (cf. [L1] or [M3], pp.369-372). Since u is constant on ∂D one can extend it by reflection as a conformal map on the whole of $\mathbb{R}^2$. But then this conformal map is constant on a curve interior to its domain of definition, namely ∂D , and thus has to be constant itself.

q.e.d.

The same argument was used independently and in a different context by H. Wente [Wt].

One can obtain examples of homotopy classes which do not contain energy minimizing maps by making use of the following special case of a result of Morrey [M2].

LEMMA 5.1.1 *For every* $\varepsilon > 0$ *there exists a map* $k : D \to S^2$ *of degree 1, mapping* ∂D *onto some point* $p \in S^2$ *and satisfying*

$$E(k) \leq \text{Area}(S^2) + \varepsilon . \tag{5.1.1}$$

Such a map k *is called* ε*-conformal.*

Proof of Lemma 5.1.1 We divide S^2 into $B(p,\delta)$ and $S^2\backslash B(p,\delta)$.

All the maps to follow will be understood to be equivariant w.r.t. the rotations of D and to those of S^2 leaving p fixed.

First of all, for sufficiently small δ , we can map $\{z \in \mathbb{C} : \frac{1}{2} \leq z \leq 1\}$ onto $B(p,\delta)$, $\{|z| = \frac{1}{2}\}$ going onto $\partial B(p,\delta)$ and $\{|z| = 1\}$ going onto p with energy smaller than ε . On the other hand, $\{z \in \mathbb{C} : |z| \leq \frac{1}{2}\}$ can be mapped conformally onto $S^2\backslash B(p,\delta)$, $\{|z| = \frac{1}{2}\}$ going again onto $\partial B(p,\delta)$, and the energy of this map, since conformal, equals the area of its image and is hence smaller than the area of S^2 . This proves the claim.

q.e.d.

It is quite instructive to look at the second map of the proof more closely. If we stereographically project S^2 onto $\mathbb{C}$, choosing the antipodal point $\bar{p}$ of p as the origin, $S^2\backslash B(p,\delta)$ is mapped onto

$\{|z| \leq N\}$ with $N \to \infty$ as $\delta \to 0$. The conformal map used above is then just given by $z \mapsto 2Nz$. Thus, the preimage of $\{|z| \leq 1\}$, which corresponds to the hemisphere centred at $\bar{p}$, under this map is $\{|z| \leq \frac{1}{2N}\}$, i.e. shrinks to a single point as $N \to \infty$. In this way, we see how a singularity is created in the limit of an energy minimizing sequence of degree 1 from D onto S^2, mapping ∂D onto p.

This heuristic reasoning will be made precise in Prop. 5.1.2 below, with the help of the following easily checked

LEMMA 5.1.2 *If* $f : \Sigma_1 \to \Sigma_2$ *is a map between surfaces, then*

$$\text{Area}(f(\Sigma_1)) \leq E(f) , \tag{5.1.2}$$

where the area is counted with appropriate multiplicity. Furthermore, equality holds in (5.1.2) if and only if f *is conformal.*

As a consequence, we have for example the following result, again due to Lemaire [L1].

PROPOSITION 5.1.2 *Let* α *be a homotopy class of maps of degree* ± 1 *from a closed surface* Σ *of positive genus onto* S^2. *Then the minimum of energy is not attained in* α.

Proof Let B be any disc in Σ and let $\varepsilon > 0$. Since B is conformally equivalent to the unit disc D, Lemmata 5.1.1 and 1.3.2 imply that we can find a map $k : B \to S^2$ of degree ± 1, mapping ∂D onto some point p, and satisfying (5.1.1). If we extend k to all of Σ by mapping $\Sigma \backslash B$ onto p, then $k : \Sigma \to S^2$ still satisfies (5.1.1) and is of degree ± 1.

If there would be an energy minimizing h in α, then h would have to satisfy consequently

$$E(h) = \text{Area}(S^2)$$

by Lemma 5.1.2, and would hence have to be conformal, by Lemma 5.1.2 again. On the other hand, a conformal map of degree ± 1 has to be a diffeomorphism which is not possible since Σ is by assumption not homeomorphic to S^2.

q.e.d.

The following example where some homotopy classes contain harmonic representatives, while others do not, is again based on the idea of Lemaire [L1].

Let D be the unit disc in the complex plane, and $k : D \to S^2$ be a conformal map mapping D onto the upper hemisphere and ∂D onto the equator. Furthermore, suppose that k is equivariant with respect to the rotations of D and S^2 (the latter ones leaving the north and south pole of S^2 fixed).

We choose the orientation on S^2 in such a way that the Jacobian of k is positive.

Let $D(0,r)$ be the plane disc with centre 0 and radius r (i.e. $D = D(0,1)$).

Let h_r be a map from $D(0,r)$ onto S^2 which maps $\partial D(0,r)$ onto the north pole, is injective in the interior of $D(0,r)$ and has a positive Jacobian there, and is ε-conformal. We introduce polar coordinates (ρ,ϕ) on D and define for $0 < r < 1$ the mapping k_r by

$$k_r(\rho,\phi) = \begin{cases} k\left(\frac{1}{1-r}\,\rho + \frac{r}{r-1},\ \phi\right) & \text{if } r \le \rho \le 1 \\ h_r(\rho,\phi) & \text{if } 0 \le \rho \le r\,. \end{cases}$$

Using Lemma 5.1.1 it is easy to see that the energy of k_r can be made arbitrarily close to 6π if we choose $r > 0$ sufficiently small.

On the other hand, 6π is just the area of the image of k_r, counted with multiplicity. Hence, if there is an energy minimizing map homotopic to

k_r , its energy has to be 6π , and it therefore has to be conformal. Since the boundary values are equivariant, this conformal map itself has to be equivariant (otherwise there would exist infinitely many homotopic conformal maps with the same boundary values which is not possible). This, however, implies that it would have to collapse a circle in D to a point which is not possible for a conformal map. Hence there is no energy minimizing map homotopic to k_r .

By letting h_r cover S^2 more than once, we obtain other classes without energy minimizing maps by a similar argument. If h_r , however, has degree -1 , then k_r is homotopic to a map of D onto the lower hemisphere and hence homotopic to an energy minimizing map. Hence, in this example, there are precisely two homotopy classes which contain energy minimizing maps, while all the others do not.

The preceding example is discussed in [BC2] by means of explicit calculations.

While Prop. 5.1.2 only excluded the existence of an energy minimizing map, one can even show

PROPOSITION 5.1.3 *If* Σ_1 *is diffeomorphic to the two-dimensional torus, and* Σ_2 *to* S^2 *, then there is no harmonic map* $h : \Sigma_1 \to \Sigma_2$ *of degree* $d(h) = \pm 1$ *, for any metrics on* Σ_1 *and* Σ_2 .

This result was obtained by Eells-Wood [EW] as a consequence of their

THEOREM 5.1.1 *Suppose that* Σ_1 *and* Σ_2 *are closed orientable surfaces,* $\chi(\Sigma)$ *denotes the Euler characteristic of a surface* Σ *, and* $d(\phi)$ *is the degree of a map* ϕ .

Suppose $h : \Sigma_1 \to \Sigma_2$ *is harmonic with respect to metrics* γ *and* g *on*

Σ_1 and Σ_2, resp. If

$$\chi(\Sigma_1) + |d(h)|\ |\chi(\Sigma_2)| > 0 ,$$

then h *is holomorphic or antiholomorphic relative to the complex structures determined by* γ *and* g .

Thm. 5.1.1, together with the existence theorem of Lemaire and Sacks-Uhlenbeck, to be proved below, also enabled Eeels and Wood to give an analytic proof of the following topological result of H. Kneser [Kn2]

THEOREM 5.1.2 *Suppose again that Σ_1 and Σ_2 are closed orientable surfaces, and furthermore $\chi(\Sigma_2) < 0$. Then for any continuous map* $\phi : \Sigma_1 \to \Sigma_2$

$$|d(\phi)|\ \chi(\Sigma_2) \geq \chi(\Sigma_1) . \tag{5.1.3}$$

Proof of Theorem 5.1.2 We introduce some metrics γ and g on Σ_1 and Σ_2, resp., and find a harmonic map h homotopic to ϕ by Thm. 5.3.1. By Thm. 5.1.1, h is (anti) holomorphic in case $|d(\phi)|\ \chi(\Sigma_2) < \chi(\Sigma_1)$. This, however, is in contradiction to the Riemann-Hurwitz formula, which says $|d(h)|\ \chi(\Sigma_2) = \chi(\Sigma_1) + r$, $r \geq 0$ for an (anti) holomorphic map h. Therefore, (5.1.3) must hold.

q.e.d.

Before proving Thm. 5.1.1, we note another consequence

COROLLARY 5.1.1 *If Σ_1 is diffeomorphic to S^2, then any harmonic map $h : \Sigma_1 \to \Sigma_2$ is (anti) holomorphic (and therefore constant, if $\chi(\Sigma_2) \leq 0$).*

This is due to Wood [W1] and Lemaire [L1].

Cor. 5.1.1 also follows from Lemma 1.3.4, since there are no nonzero holomorphic quadratic differentials on S^2 which easily follows from

Liouville's theorem.

We need some preparations for the proof of Thm. 5.1.1.

We shall make use of some computations of Schoen and Yau [SY]] in the sequel. It is convenient to use the complex notation. If $\rho^2(z)\,dzd\bar{z}$ and $\sigma^2(h)\,dhd\bar{h}$ are the metrics w.r.t. to conformal coordinate charts on Σ_1 and Σ_2, resp., then h as a harmonic map satisfies

(5.1.4) $$h_{z\bar{z}} + \frac{2\sigma_h}{\sigma} h_z h_{\bar{z}} = 0 \,, \qquad \text{cf. (1.3.4)} .$$

LEMMA 5.1.3 *At points, where* ∂h *or* $\bar{\partial} h$, *resp., is nonzero*

(5.1.5) $$\Delta \log|\partial h|^2 = K_1 - K_2(|\partial h|^2 - |\bar{\partial} h|^2)$$

(5.1.6) $$\Delta \log|\bar{\partial} h|^2 = K_1 + K_2(|\partial h|^2 - |\bar{\partial} h|^2) \,,$$

where K_i *denotes the Gauss curvature of* Σ_i, *and*

$$|\partial h|^2 = \frac{\sigma^2}{\rho^2} h_z \cdot \bar{h}_{\bar{z}} \,, \qquad |\bar{\partial} h|^2 = \frac{\sigma^2}{\rho^2} \bar{h}_z h_{\bar{z}} \,.$$

Proof For any positive smooth function f on Σ_1,

(5.1.7) $$\Delta \log f = \frac{1}{f}\Delta f - \frac{1}{f^2}\cdot\frac{1}{\rho^2} f_z f_{\bar{z}} \,.$$

Furthermore,

(5.1.8) $$\Delta \log \frac{1}{\rho^2} = K_1 \,.$$

In order to abbreviate the following calculations, we define D as the covariant derivative in the bundle $h^{-1}\,T\Sigma_2$, e.g.

$$D_{\partial/\partial z}\, h_z = h_{zz} + \frac{2\sigma_h}{\sigma} h_z h_z \,.$$

(5.1.4) then is expressed as

(5.1.9) $$D_{\partial/\partial z}\, h_{\bar{z}} = 0 \,.$$

Since

$$\sigma^2 h_z \bar{h}_{\bar{z}} = \langle h_z, \bar{h}_{\bar{z}} \rangle_{h^{-1}T\Sigma_2},$$

(5.1.10) $$\Delta\sigma^2 h_z \bar{h}_{\bar{z}} = \frac{1}{\rho^2}\frac{\partial}{\partial\bar{z}}\left\langle D_{\partial/\partial z} h_z, \bar{h}_{\bar{z}}\right\rangle, \qquad \text{using (5.1.9)}$$

$$= \frac{1}{\rho^2}\left\langle D_{\partial/\partial\bar{z}} D_{\partial/\partial z} h_z, \bar{h}_{\bar{z}}\right\rangle + \frac{1}{\rho^2}\left\langle D_{\partial/\partial z} h_z, D_{\partial/\partial\bar{z}} \bar{h}_{\bar{z}}\right\rangle$$

$$= \frac{1}{\rho^2} R\left(h_*\left(\frac{\partial}{\partial\bar{z}}\right), h_*\left(\frac{\partial}{\partial z}\right), h_z, \bar{h}_{\bar{z}}\right) + \frac{1}{\rho^2}\left\langle D_{\partial/\partial z} h_z, D_{\partial/\partial\bar{z}} \bar{h}_{\bar{z}}\right\rangle,$$

where R denotes the curvature tensor of Σ_2

$$= -K_2|\partial h|^2 \cdot J(h) + \frac{1}{\rho^2}\left\langle D_{\partial/\partial z} h_z, D_{\partial/\partial\bar{z}} \bar{h}_{\bar{z}}\right\rangle,$$

where $J(h) = |\partial h|^2 - |\bar{\partial} h|^2$ is the Jacobian of h. Moreover,

(5.1.11) $$\frac{1}{\rho^2}\frac{\partial}{\partial z}\langle h_z, h_{\bar{z}}\rangle \cdot \frac{\partial}{\partial\bar{z}}\langle h_z, h_{\bar{z}}\rangle = \frac{1}{\rho^2}\langle h_z, h_{\bar{z}}\rangle\left\langle D_{\partial/\partial z} h_z, D_{\partial/\partial\bar{z}} h_{\bar{z}}\right\rangle,$$

using again (5.1.9), and the fact that the complex dimension of Σ_2 is 1.

(5.1.5) now follows from (5.1.7), (5.1.8), (5.1.10), and (5.1.11), and (5.1.6) can either be calculated in the same way or directly deduced from (5.1.5), since $|\bar{\partial} h|^2 = |\partial\bar{h}|^2$ and complex conjugation on the image can be considered as a change of orientation.

q.e.d.

LEMMA 5.1.4 *If* $h_z(z_0) = 0$, *then*

(5.1.12) $$|\partial h|^2 = \zeta\cdot|k|^2 \qquad \text{near } z = z_0,$$

where ζ *is a nonvanishing* C^2 *function, and* k *is holomorphic. A corresponding result holds for* $h_{\bar{z}}$.

Proof By (5.1.4), $f := h_z$ satisfies

$$|f_{\bar{z}}| \le c|f|.$$

Therefore, we can apply the similarity principle of Bers and Vekua (cf. [B] or [Hz1]), to obtain the representation (5.1.12) with Hölder continuous ζ. An inspection of the proof of the similarity principle shows that in our case $\zeta \in C^2$ (cf. [Hz1], p.210). (We note that a similarity principle can be derived from Cor. 5.5.2 below which also contains the existence of solutions of Beltrami equations, cf. [BJS].)

q.e.d.

Proof of Theorem 5.1.1 Lemma 5.1.4 shows that the zeros z_i of $|\partial h|^2$ are isolated, unless $\partial h \equiv 0$, and that near each z_i.

$$|\partial h|^2 = a_i |z - z_i|^{n_i} + o(|z - z_i|^{n_i}) ,$$

for some $a_i > 0$ and some $n_i \in \mathbb{N}$.

By Lemma 5.1.3 and the residue formula, unless $\partial h \equiv 0$

(5.1.13) $$\int_{\Sigma_1} K_1 - \int_{\Sigma_2} K_2(|\partial h|^2 - |\bar{\partial} h|^2) = -\Sigma n_i .$$

Similarly, if $\bar{\partial} h \not\equiv 0$,

(5.1.14) $$\int_{\Sigma_1} K_1 + \int_{\Sigma_2} K_2(|\partial h|^2 - |\bar{\partial} h|^2) = -\Sigma m_i$$

where $m_i \in \mathbb{N}$ are now the orders of the zeros of $|\bar{\partial} h|^2$. Thus, since $|\partial h|^2 - |\bar{\partial} h|^2$ is the Jacobian of h,

$$\chi(\Sigma_1) - d(h)\,\chi(\Sigma_2) \le 0 , \qquad \text{unless } \partial h \equiv 0$$

and

$$\chi(\Sigma_1) + d(h)\,\chi(\Sigma_2) \le 0 , \qquad \text{unless } \bar{\partial} h \equiv 0 ,$$

and Thm. 5.1.1 follows.

5.2 SOME LEMMATA

In this section, we want to derive some tools for our existence proofs. First of all, we note

LEMMA 5.2.1 *Suppose* B_0 *is a geodesic ball with centre* p *and radius* s, $s \le \frac{1}{3}\min(i(p), \pi/2\kappa)$, *where* κ^2 *is an upper bound for the sectional curvature of* N *and* $i(p)$ *is the injectivity radius of* p. *If* $h : \Omega \to N$ *is energy minimizing among maps which are homotopic to some map* $g : \Omega \to B_0$, *and if* $h(\partial\Omega) \subset B_0$, *then also*

$$h(\Omega) \subset B_0 .$$

(for a suitable representative of h, *again).*

Proof By assumption, we can introduce geodesic polar coordinates (r,ϕ) on $B(p,3s)$ $(0 \le r \le 3s)$.

We define a map π in the following way:

$$\begin{aligned} \pi(r,\phi) &= (r,\phi) && \text{if } r \le s \\ \pi(r,\phi) &= (\tfrac{1}{2}(3s-r),\phi) && \text{if } s \le r \le 3s \\ \pi(q) &= p && \text{if } q \in N\backslash B(p,3s) . \end{aligned}$$

(Here, we have identified a point in $B(p,3s)$ with its representation in geodesic polar coordinates.)

Using Lemma 2.2.1, it is easily seen that π can be approximated by a map satisfying the assumptions of Lemma 4.10.1.

q.e.d.

Moreover, we have the following result, based on an idea of Lebesgue and extensively used by Courant in his study of minimal surfaces (cf. e.g. [Co]).

Suppose Ω is an open subset of some two-dimensional Riemannian manifold Σ of class C^3, while S is any Riemannian manifold.

LEMMA 5.2.2 *Let* $u \in H_2^1(\Omega,S)$, $E(u) \le D$, $x_0 \in \Sigma$, $-\lambda^2$ *a lower bound for the curvature* K *of* Σ, $\delta < \min(1, i(\Sigma)^2, 1/\lambda^2)$. *Then there exists some* $r \in (\delta,\sqrt{\delta})$ *for which* $u|\partial B(x_0,r) \cap \bar{\Omega}$ *is absolutely continuous and*

$$d(u(x_1),\ u(x_2)) \le 4\pi \cdot D^{\frac{1}{2}} \cdot (\log 1/\delta)^{-\frac{1}{2}}$$

for all x_1, $x_2 \in \partial B(x_0,r) \cap \bar{\Omega}$.

Proof We introduce polar coordinates on $B(x_0,r)$, i.e. $ds^2 = dr^2 + G^2(r,\theta)\, d\theta^2$.

Since $K = -\frac{G_{rr}}{G}$ (cf. [Bl], p.153) and $G(0,\theta) = 0$, we infer

$$G(r,\theta) \le 1/\lambda \sinh \lambda r. \tag{5.2.1}$$

Now for x_1, $x_2 \in \partial B(x_0,r)$ and almost all r, since u is a Sobolev function $u|\partial B(x_0,r)$ is absolutely continuous and

$$\begin{aligned} d(u(x_1),\ u(x_2)) &\le \int_0^{2\pi} |u_\theta(x)|\, d\theta \\ &\le 2\pi \left(\int_0^{2\pi} |u_\theta|^2\, d\theta\right)^{\frac{1}{2}} \end{aligned} \tag{5.2.2}$$

where we assumed w.l.o.g. $B(x_0,r) \subset \Omega$.

The Dirichlet integral of u on $B(x_0,r)$ is

$$E(u;B(x_0,r)) = \frac{1}{2}\int_{B(x_0,r)} \left(|u_r|^2 + \frac{1}{G^2}|u_\theta|^2\right) G\, dr\, d\theta.$$

Thus, we can find some $r \in (\delta,\sqrt{\delta})$ with

$$\int_0^{2\pi} |u_\theta(r,\theta)|^2\, d\theta \le \frac{2D}{\int_\delta^{\sqrt{\delta}} \frac{1}{G(\rho,\theta)}\, d\theta} \le \frac{2D}{\log 1/\rho} \tag{5.2.3}$$

since for $r \le \sqrt{\delta} \le 1/\lambda$, $G(r,\theta) \le 2r$ by (5.2.1).

The lemma follows from (5.2.2) and (5.2.3).

q.e.d.

Finally, we shall need the two-dimensional version of Theorem 4.10.1. This also follows from Morrey's work on the minima of two-dimensional variational problems. We shall present a proof which already illustrates some of the ideas of the arguments in later sections and is based on Lemmata 4.10.2 and 5.2.2.

LEMMA 5.2.3 *Suppose* $\partial\Omega \neq \emptyset$, $B(p,M)$ *is a disc in some surface* Σ *with radius* $M < \frac{\pi}{2\kappa}$, *where* $\kappa^2 \ge 0$ *is an upper bound of the Gauss curvature of* $B(p,M)$, *and* $g : \partial\Omega \to B(p,M)$ *is continuous and admits an extension* $\bar{g} \in H_2^1(\Omega, B(p,M))$.†

Then there exists a harmonic map $h : \Omega \to B(p,M)$ *with boundary values* g , *and* h *minimizes the energy with respect to these boundary values. Vice versa, each such energy minimizing map is harmonic. The modulus of continuity of* h *can be estimated in terms of* λ , $i(\Sigma_1)$, M , κ , *and* $E(\bar{g})$ *and the modulus of continuity of* g .

Proof (The idea is taken from the proof of Thm. 4.1 in [HW1].) As in Lemma 4.10.3, we find a weakly harmonic map which minimizes energy among all maps into $B(p,M)$ with boundary values g .

By Prop. 2.4.2, every two points in $B(p,M)$ can be joined by a unique geodesic arc in $B(p,M)$, and this arc is free of conjugate points. Suppose $q \in B(p,M)$, v_1 and v_2 are unit vectors in $T_q\Sigma$, and c_1, c_2 are the geodesic parametrized by arc length and starting at q with tangent vectors

† Here, we can again define $H_2^1(\Omega, B(p,M))$ unambiguously with the help of the global coordinates on $B(p,M)$ given by $\exp_p$.

v_1, v_2 . By Lemma 2.3.1

$$|v_1 - v_2| \cdot \frac{\sin(t\kappa)}{t\kappa} \leq d(c_1(t), c_2(t))$$

as long as $c_1(t), c_2(t) \in B(p,M)$.

Therefore, on $B(p,M)\backslash B(q,\varepsilon)$, with the help of (2.2.4)

$$d(c_1(t), c_2(t)) \geq \min\left(d(c_1(\varepsilon), c_2(\varepsilon)), |v_1 - v_2| \cdot \frac{\sin(2M\kappa)}{2M\kappa}\right) .$$

Consequently, there exists $\varepsilon_0 > 0$ with the property that $B_0 := B(q,\varepsilon) \cap B(p,M)$ and $B_1 := B(p,M)$ satisfy the assumptions of Lemma 4.10.2 for every $q \in B(p,M)$ and every $\varepsilon \leq \varepsilon_0$. Lemma 5.2.2 then implies that for each $x \in \Omega$ there exists a sufficiently small $\rho > 0$ with the property that

$$h(B(x,\rho) \cap \Omega) \subset B(q,\varepsilon)$$

for some $q \in B(p,M)$. ρ depends on ε , λ , $i(\Omega)$, the energy of h (which is bounded by the energy of $\bar{g}$), and the modulus of continuity of g .

Therefore, Lemma 4.10.2 implies the continuity of h . Higher regularity then follows as in chapter 4.

q.e.d.

5.3 THE EXISTENCE THEOREM OF LEMAIRE AND SACKS-UHLENBECK

We are now in a position to attack the general existence problem for harmonic maps between surfaces.

For this purpose, let Σ_1 and Σ_2 denote compact surfaces, $\partial\Sigma_2 = \emptyset$, but Σ_1 possibly having nonempty boundary. Let $\phi : \Sigma_1 \to \Sigma_2$ be a continuous map with finite energy. We denote by $[\phi]$ the class of all continuous maps which are homotopic to ϕ and coincide with ϕ on $\partial\Sigma_1$, in case $\partial\Sigma_1 \neq \emptyset$.

We choose $s = \frac{1}{3}\min(i(\Sigma_2), \pi/2\kappa)$, where $\kappa^2 \geq 0$ is an upper curvature bound on Σ_2 , and $i(\Sigma_2)$ is the injectivity radius of Σ_2 .

Let $\delta_0 < \min(1, i(\Sigma_1)^2, 1/\lambda^2)$ ($-\lambda^2$ being a lower bound for the curvature of Σ_1) satisfy

$$(5.3.1) \qquad 2\pi \cdot E(\phi)^{\frac{1}{2}} (\log 1/\delta_0)^{-\frac{1}{2}} \leq s/2 ,$$

where $E(\phi)$ is the energy of ϕ , and

$$(5.3.2) \qquad d(x_1,x_2) \leq \sqrt{\delta_0} \Rightarrow d(\phi(x_1), \phi(x_2)) \leq s/2 \qquad \text{for } x_1,x_2 \in \partial\Sigma_1 .$$

Let $0 < \delta \leq \delta_0$. There exists a finite number of points $x_i \in \Sigma_1$, $i = 1,\ldots,m = m(\delta)$, for which the discs $B(x_i,\delta/2)$ cover Σ_1 .

We let u_n be a continuous energy minimizing sequence in $[\phi]$, $E(u_n) \leq E(\phi)$ w.l.o.g. for all n .

Applying Lemma 5.2.2 and using (5.3.1) and (5.3.2), for every n , we can find $r_{n,1}$, $\delta < r_{n,1} < \sqrt{\delta}$, and $p_{n,1} \in \Sigma_2$ with the property that

$$(5.3.3) \qquad u_n(\bar{\partial}B(x_1,r_{n,1})) \subset B(p_{n,1},s)$$

where we defined $\bar{\partial}B(x,r) = \partial(B(x,r) \cap \Sigma)$.

We now have two possibilities:

either

1) There exists some δ , $0 < \delta \leq \delta_0$, with the property that for any $x \in \Sigma_1$, some r (depending on x and n) with $\delta < r \leq \sqrt{\delta}$ and with $u_n(\bar{\partial}B(x,r)) \subset B(p,s)$ for some $p \in \Sigma_2$, and every sufficiently large n , $u_n|B(x,r)$ is homotopic to the solution of the Dirichlet problem

$$(5.3.4) \qquad \begin{aligned} &g : B(x,r) \to B(p,s) \\ &g|\bar{\partial}B(x,r) = u_n|\bar{\partial}B(x,r) \end{aligned} \qquad \text{harmonic and energy minimizing}$$

(The existence of g is ensured by Lemma 5.2.3; g is actually unique by Thm. 4.11.1, but this is not needed in the following constructions.)

or

2) Possibly choosing a subsequence of the u_n, we can find a sequence of points $x_n \in \Sigma_1$, and radii $r_n > 0$, $x_n \to x_0 \in \Sigma_1$, $r_n \to 0$, with $u_n(\bar{\partial} B(x_n, r_n)) \subset B(p_n, \varepsilon_n)$ for some $p_n \in \Sigma_2$, $p_n \to p \in \Sigma_2$, $\varepsilon_n \to 0$ (using Lemma 5.2.2), but for which $u_n|B(x_n, r_n)$ is not homotopic to the solution of the Dirichlet problem (5.3.4).

In case 1), we replace u_n on $B(x_1, r_{n,1})$ by the solution of the Dirichlet problem (5.3.4) for $x = x_1$ and $r = r_{n,1}$. We can assume $r_{n,1} \to r_1$ and, using the interior modulus of continuity estimates for the solution of (5.3.4) (cf. Lemma 5.2.3) that the replaced maps, denoted by u_n^1, converge uniformly on $B(x_1, \delta-\eta)$, for any $0 < \eta < \delta$. By Lemma 5.2.1

$$E(u_n^1) \le E(u_n) \;. \tag{5.3.5}$$

By the same argument as above, we then find radii $r_{n,2}$, $\delta < r_{n,2} < \sqrt{\delta}$, with

$$u_n^1(\partial B(x_2, r_{n,2})) \subset B(p_{n,2}, s)$$

for points $p_{n,2} \in \Sigma_2$.

Again, we replace u_n^1 on $B(x_2, r_{n,2})$ by the solution of the Dirichlet problem (5.3.4) for $x = x_2$ and $r = r_{n,2}$. We denote the new maps by u_n^2. Again, w.l.o.g., $r_{n,2} \to r_2$.

If we take into consideration that, by the first replacement step, u_n^1 in particular converges uniformly on $B(x_2, r_2) \cap B(x_1, \delta-\eta/2)$, if $0 < \eta < \delta$, we see that the boundary values for our second replacement step converge uniformly on $\bar{\partial} B(x_2, r_{n,2}) \cap B(x_1, \delta-\eta/2)$.

Using the estimates for the modulus of continuity for the solution of (5.3.4) at these boundary points (cf. Lemma 5.2.3) we can assume that the maps u_n^2 converge uniformly on $B(x_1,\delta-\eta) \cup B(x_2,\delta-\eta)$, if $0 < \eta < \delta$.

Furthermore, by Lemma 5.2.1 again and (5.3.5)

$$E(u_n^2) \le E(u_n^1) \le E(u_n) .$$

In this way, we repeat the replacement argument, until we get a sequence $u_n^m =: v_n$, with

$$E(v_n) \le E(u_n) \tag{5.3.6}$$

which converges uniformly on all balls $B(x_i,\delta/2)$, $i = 1,\dots,m$, and hence on all of Σ_1 , since these balls cover Σ_1 .

We denote the limit of the v_n by u . By uniform convergence, u_n is homotopic to ϕ .

Since $E(v_n) \le E(\phi)$ by (5.3.6), the v_n converge also weakly in H_2^1 to u , and by lower semicontinuity of the energy w.r.t. weak H_2^1 convergence and since the v_n are a minimizing sequence by (5.3.6), u minimizes energy in its homotopy class.

In particular, u minimizes energy when restricted to small balls, and hence it is harmonic and regular by Lemma 5.2.1 and Lemma 5.2.3. Observing that if $\pi_2(\Sigma_2) = 0$, any two maps from a disc into Σ_2 are homotopic, we obtain

THEOREM 5.3.1 *Suppose* Σ_1 *and* Σ_2 *are compact surfaces,* $\partial\Sigma_2 = \emptyset$ *, and* $\pi_2(\Sigma_2) = 0$. *If* $\phi : \Sigma_1 \to \Sigma_2$ *is a continuous map with finite energy, then there exists a harmonic map* $u : \Sigma_1 \to \Sigma_2$ *which is homotopic to* ϕ *, coincides with* ϕ *on* $\partial\Sigma_1$ *in case* $\partial\Sigma_1 \neq \emptyset$ *and is energy minimizing among all such maps.*

Theorem 5.3.1 is the fundamental existence theorem due to Lemaire ([L1], [L2]) and Sacks-Uhlenbeck ([SkU], in case $\partial\Sigma_1 = \emptyset$) .

A different proof was given by Schoen-Yau [SY2]. The present proof was taken from [J6].

In the case of the Dirichlet problem, it is actually not necessary that Σ_2 is compact, but only that it it homogeneously regular in the sense of Morrey [M2], cf. [L2], since the boundary values prevent a minimizing sequence from disappearing at infinity.

Furthermore, the image can be of arbitrary dimension, not necessarily 2, for Thm. 5.3.1 to hold. This is also easily seen from the present proof. Finally, if one does not prescribe the homotopy class of u , the existence of a harmonic map was already proved by Morrey [M2].

5.4 THE DIRICHLET PROBLEM IF THE IMAGE IS HOMEOMORPHIC TO S^2 . TWO SOLUTIONS FOR NONCONSTANT BOUNDARY VALUES

In this section, we want to show the following result of Jost [J7] and Brezis and Coron [BC2] (in the latter paper, only simply connected domains are treated).

THEOREM 5.4.1 *Suppose* Σ_1 *is a compact two-dimensional Riemannian manifold with nonempty boundary* $\partial\Sigma_1$ *, and* Σ_2 *is a Riemannian manifold homeomorphic to* S^2 *(the standard 2-sphere), and* $\psi : \partial\Sigma_1 \to \Sigma_2$ *is a continuous map, not mapping* $\partial\Sigma_1$ *onto a single point and admitting a continuous extension to a map from* Σ_1 *to* Σ_2 *with finite energy. Then there are at least two homotopically different harmonic maps* $u : \Sigma_1 \to \Sigma_2$ *with* $u|\partial\Sigma_1 = \psi$ *, and both mappings minimize energy in their respective homotopy classes.*

Proof We first investigate more closely case 2) of section 5.3. W.l.o.g.

$B(p_n,\varepsilon_n) \subset B(p,2\varepsilon_n)$ and $\varepsilon_n \le s/2$ for all n, and thus the solution g of (5.3.4) for $x = x_n$, $r = r_n$ is contained in $B(p,2\varepsilon_n)$ by Lemma 5.2.1.

Since $u_n|B(x_n,r_n)$ is not homotopic to g, it has to cover $\Sigma_2\backslash B(p,2\varepsilon_n)$. If we define

$$\tilde{u}_n = \begin{cases} u_n & \text{on } \Sigma_1\backslash B(x_n,r_n) \\ g & \text{on } B(x_n,r_n) \end{cases}$$

then we see that

$$\begin{aligned} (5.4.1) \qquad \lim E(u_n) &\ge \lim E(u_n|\Sigma_1\backslash B(x_n,r_n)) + \lim E(u_n|B(x_n,r_n)) \\ &\ge \lim E(\tilde{u}_n) + \mathrm{Area}(\Sigma_2) , \end{aligned}$$

since $E(g) \to 0$ as $n \to \infty$, because

$$\int_0^{2\pi} |g_\theta(r_n,\theta)|^2 \, d\theta \to 0$$

as $n \to \infty$ (cf. (5.2.3)).

(Furthermore, by Lemma 5.1.2

$$E(v;B) \ge \mathrm{Area}(v(B)) ,$$

and equality holds if and only if v is conformal.)

We now define

$$E_\alpha := \inf\{E(v) : v \in \alpha\}$$

for a homotopy class α of maps with $v|\partial\Sigma_1 = \psi$, and

$$E := \min_\alpha E_\alpha .$$

We first show the existence of a minimizing harmonic map in any homotopy class α with

$$(5.4.2) \qquad E_\alpha < E + \mathrm{Area}(\Sigma_2) .$$

We choose a minimizing sequence u_n in α with

$$E(u_n) < E + \mathrm{Area}(\Sigma_2) .$$

Assuming that 2) holds, we define $\tilde{u}_n$ as above. Since clearly

$$E(\tilde{u}_n) \geq E ,$$

this would contradict (5.4.1), however. Therefore, as shown above, we obtain an energy minimizing harmonic map in α (cf. [BC1] for a similar argument). Now let $\tilde{\alpha}$ be a homotopy class with

$$E_{\tilde{\alpha}} = E ,$$

and let $\tilde{u}$ an energy minimizing map in $\tilde{\alpha}$, i.e. $E(\tilde{u}) = E$. We want to construct a map v in some homotopy class $\alpha \neq \tilde{\alpha}$ with

$$E(v) < E(\tilde{u}) + \mathrm{Area}(\Sigma_2) . \tag{5.4.3}$$

Then the arguments above show that we can find a harmonic map of minimal energy in α . In order to complete the proof, it thus only remains to construct v .

By Thm. 5.5.1 below, the metric on Σ_2 is conformally equivalent to the standard metric on S^2 , and thus, we can use S^2 as a parameter domain for the image. Since ψ is not a constant map, also $\tilde{u}$ is not a constant map, and hence we can find a point x_0 in the interior of Σ_1 for which $d\tilde{u}(x_0) \neq 0$. Rotating S^2 , we can assume that $\tilde{u}(x_0)$ is the south pole p_0 . We introduce local coordinates on the image by stereographic projection $\pi : S^2 \to \mathbb{C}$ from the south pole p_0 . $d\pi(p_0)$ then is the identity map up to a conformal factor. By Taylor's theorem, $\pi\circ\tilde{u}|\partial B(x_0,\varepsilon)$ is a linear map up to an error of order $O(\varepsilon^2)$, i.e.

$$\left|\pi\circ\tilde{u}(x) - d(\pi\circ\tilde{u})(x_0)(x - x_0)\right| = O(\varepsilon^2) \tag{5.4.4}$$

for $x \in \partial B(x_0,\varepsilon)$.

We now look at conformal maps of the form

$$w = az + b/z \;, \qquad a,b \in \mathbb{C} \;, \qquad a = a_1 + ia_2 \;, \qquad b = b_1 + ib_2 \;.$$

The restrictions of such a map to a circle $\rho(\cos\theta + i\sin\theta)$ in $\mathbb{C}$ is given by

$$u = \left(a_1\rho + \frac{b_1}{\rho}\right)\cos\theta + \left(\frac{b_2}{\rho} - a_2\rho\right)\sin\theta$$

$$v = \left(a_2\rho + \frac{b_2}{\rho}\right)\cos\theta + \left(a_1\rho - \frac{b_1}{\rho}\right)\sin\theta \;,$$

where $w = u + iv$.

Therefore, we can choose a and b in such a way that w restricted to this circle coincides with any prescribed nontrivial linear map. This map is nonsingular if

$$\rho^4 \neq \frac{b_1^2 + b_2^2}{a_1^2 + a_2^2}$$

W.l.o.g.

(5.4.5) $$\rho^4 \le \frac{b_1^2 + b_2^2}{a_1^2 + a_2^2}$$

(otherwise we perform an inversion at the unit circle).

Hence w can be extended as a conformal map from the interior of the circle $\rho(\cos\theta + i\sin\theta)$ onto the exterior of its image. (If equality holds in (5.4.5), then this image is a straight line covered twice, and the exterior is the complement of this line in the complex plane.)

We are now in a position to define v .

On $\Sigma_1 \setminus B(x_0,\varepsilon)$ we put $v = \tilde{u}$.

On $B(x_0,\varepsilon-\varepsilon^2)$ we choose a conformal map w as above which coincides on the boundary with the linear map $\frac{1}{1-\varepsilon}\cdot d(\pi\circ\tilde{u})(x_0)$, and put $v = \pi^{-1}\circ w$.

On $B(x_0,\varepsilon)\backslash B(x_0,\varepsilon-\varepsilon^2)$ we interpolate in the following way. We introduce polar coordinates r,ϕ and define

$$f(\phi) := (\pi\circ\tilde{u})(\varepsilon,\phi)$$

$$g(\phi) := d(\pi\circ\tilde{u})(x_0)(\varepsilon,\phi) = \frac{1}{1-\varepsilon}\, d(\pi\circ\tilde{u})(x_0)(\varepsilon-\varepsilon^2,\phi)$$

and

$$t(r,\phi) := (f(\phi) - g(\phi))\cdot\frac{r}{\varepsilon^2} + \frac{1}{\varepsilon}\,(g(\phi) - (1-\varepsilon)\, f(\phi)) .$$

Thus $t(r,\phi)$ coincides with $f(\phi)$ and $g(\phi)$, resp. for $r = \varepsilon$ and $r = \varepsilon-\varepsilon^2$, resp.

The energy of $t(r,\phi)$ on the annulus $B(x_0,\varepsilon)\backslash B(x_0,\varepsilon-\varepsilon^2)$ is given by

$$E(t) = \int_{r=\varepsilon-\varepsilon^2}^{\varepsilon}\int_{\phi=0}^{2\pi}\left(\frac{1}{\varepsilon^4}\,|f(\phi) - g(\phi)|^2 + \frac{1}{r^2}\,\left|\left(\frac{r}{\varepsilon^2} - \frac{1-\varepsilon}{\varepsilon}\right) f'(\phi) + \left(\frac{1}{\varepsilon} - \frac{r}{\varepsilon^2}\right) g'(\phi)\right|^2\right) r\,dr\,d\phi .$$

Using (5.4.4) and $|f'(\phi)| = O(\varepsilon)$, $|g'(\phi)| = O(\varepsilon)$, we calculate

$$E(t) = O(\varepsilon^3) ,$$

and hence also

$$E(\pi^{-1}\circ t) = O(\varepsilon^3) ,$$

We put $v = \pi^{-1}\circ t$ on the annulus $B(x_0,\varepsilon)\backslash B(x_0,\varepsilon-\varepsilon^2)$. Therefore

$$\begin{aligned} E(v) &= E(\tilde{u}|\Sigma_1\backslash B(x_0,\varepsilon)) + E(\pi^{-1}\circ w|B(x_0,\varepsilon-\varepsilon^2)) + E(\pi^{-1}\circ t|B(x_0,\varepsilon)\backslash B(x_0,\varepsilon-\varepsilon^2)) \\ &\le E(\tilde{u}) - O(\varepsilon^2) + \mathrm{Area}(\Sigma_2) + O(\varepsilon^3) , \end{aligned}$$

since $E(\tilde{u}|B(x_0,\varepsilon)) = O(\varepsilon^2)$, because $d\tilde{u}(x_0) \neq 0$, and the energy of $\pi^{-1}\circ w$ is the area of its image, as π and w and hence also $\pi^{-1}\circ w$ are conformal. Thus, for sufficiently small $\varepsilon > 0$, (5.4.3) is satisfied, and the proof is complete.

5.5 CONFORMAL DIFFEOMORPHISMS OF SPHERES. THE RIEMANN MAPPING THEOREM

THEOREM 5.5.1 *Suppose* Σ *is a compact two-dimensional Riemannian manifold diffeomorphic to* S^2 *. Then there is a conformal (and hence harmonic) diffeomorphism* $h : S^2 \to \Sigma$.

This is of course well-known. We want to provide a variational proof of Theorem 5.5.1, in order to illustrate on one hand how one can overcome the difficulties arising from the noncompactness of the action of the conformal group on S^2 , and on the other hand the idea to minimize energy in an a priori suitably restricted subclass of mappings.

Proof of Thm. 5.5.1 We choose three different points z_1 , z_2 , z_3 in S^2 and three different points p_1 , p_2 , p_3 in Σ . Let $\mathcal{D}$ be the class of all diffeomorphisms $v : S^2 \to \Sigma$ satisfying

$$(5.5.1) \qquad v(z_i) = p_i \qquad (i = 1,2,3) ,$$

and let $\bar{\mathcal{D}}$ be the weak H^1_2-closure of $\mathcal{D}$.

We now claim that a sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$ converging weakly in H^1_2 is equicontinuous. For each $z \in S^2$ and $\varepsilon > 0$, by Lemma 5.2.2 we can find $\delta > 0$ and for each $n \in \mathbb{N}$ then some $r_n \in (\delta, \sqrt{\delta})$ for which

$$\operatorname{diam}(v_n(\partial B(x,r_n)) \le \varepsilon .$$

Here, δ is independent of z and n , since the energy of a weakly convergent sequence is uniformly bounded. We can choose δ so small that $B(z,\sqrt{\delta})$ contains at most one of the points z_1 , z_2 , z_3 . Now $v_n(\partial B(z,r_n))$ divides Σ into two parts, one of them being $v_n(B(z,r_n))$, since v_n is a diffeomorphism. If ε is chosen small enough, then the smaller part, i.e. the one having diameter at most ε , contains at most one of the points p_1 ,

p_2 , p_3 and hence has to coincide with $v_n(B(z,r_n))$. In particular,

$$\operatorname{diam}(v_n(B(z,\delta)) \le \varepsilon ,$$

and the v_n are equicontinuous as claimed.

We now choose an energy minimizing sequence in $\mathcal{D}$. A subsequence then converges weakly in H_2^1 towards some $v \in \bar{\mathcal{D}}$. Since the energy is lower semicontinuous with respect to weak H_2^1 convergence, v minimizes energy in $\bar{\mathcal{D}}$. We also can find a sequence of diffeomorphisms $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$ converging weakly to v . Since the v_n are equicontinuous as shown above, they converge uniformly to v . In particular, v is continuous and homotopic to the v_n . (We can of course assume that all the v_n are homotopic.)

Moreover, if we have a sequence of diffeomorphisms $(w_n)_{n\in\mathbb{N}}$ from S^2 onto Σ , not necessarily satisfying (5.5.1), and converging uniformly and weakly in H_2^1 towards some w , then we still have

$$E(v) \le E(w) \tag{5.5.2}$$

since the normalization (5.5.1) can always be achieved by composing w_n with a Möbius transformation, i.e. a conformal automorphism of S^2 , without changing $E(w_n)$ (cf. Lemma 1.3.3).

Hence, if $\sigma_t : S^2 \to S^2$ is a family of diffeomorphisms, depending smoothly on t , with $\sigma_0 = \mathrm{id}$, then

$$\frac{d}{dt} E(v\circ\sigma_t)\Big|_{t=0} = 0 , \tag{5.5.3}$$

since $v\circ\sigma_t$ is the uniform and weak H_2^1-limit of $v_n\circ\sigma_t$.

We introduce local coordinates $z = x+iy$ on S^2 by stereographic projection and put

$$E = |v_x|^2 , \qquad F = \langle v_x, v_y \rangle , \qquad G = |v_y|^2$$

(E, F, G are defined almost everywhere, since $v \in H^1_2$) ,

$$\sigma_t = \xi + i\eta$$

(5.5.4)
$$\left.\frac{\partial \sigma_t}{\partial t}\right|_{t=0} = \nu + i\omega .$$

Using Lemma 1.3.2, the energy is given by

$$E(v) = \frac{1}{2} \int_{\mathbb{C}} (E + G) \, dx \, dy$$

and

$$E(v \circ \sigma_t) = \frac{1}{2} \int_{\mathbb{C}} \{E(\xi_y^2 + \eta_y^2) - 2F(\xi_x \xi_y + \eta_x \eta_y) + G(\xi_x^2 + \eta_x^2)\}(\xi_x \eta_y - \xi_y \eta_x)^{-1} \, dx \, dy$$

Since $\sigma_0(z) = z$ and hence for $t = 0$ $\xi_x = \eta_y = 1$, $\xi_y = \eta_x = 0$, (5.5.3) then implies

$$\int_{\mathbb{C}} \{(E - G)(\nu_x - \omega_y) + 2F(\nu_y + \omega_x)\} \, dx \, dy = 0 ,$$

Putting $\phi := E - G - 2iF$, this becomes

$$\mathrm{Re} \int_{\mathbb{C}} \phi(\nu + i\omega)_{\bar{z}} \, dx \, dy = 0 .$$

Replacing $\nu + i\omega$ by $\omega - i\nu$, we see that the imaginary part likewise vanishes, and thus

(5.5.5)
$$\int_{\mathbb{C}} \phi(\nu + i\omega)_{\bar{z}} \, dx \, dy = 0 .$$

Given ν and ω , we can always find a family of diffeomorphisms (for small t) satisfying (5.5.4), for example

$$\sigma_t(z) = x + t\nu(x,y) + i(y + t\omega(x,y)) .$$

Hence (5.5.5) implies

(5.5.6)
$$\phi_{\bar{z}} \equiv 0 ,$$

i.e. that ϕ is holomorphic.

Since ϕ represents a quadratic differential on S^2, in stereographic projection we have $\phi(\infty) = 0$. Hence

$$\phi \equiv 0$$

by Liouville's Theorem, i.e. v satisfies the conformality relations

$$|v_x|^2 \equiv |v_y|^2 \qquad (5.5.7)$$

$$\langle v_x, v_y \rangle \equiv 0$$

almost everywhere.

For notational convenience, we introduce local coordinates (v^1, v^2) on Σ. We want to exploit that v is weakly (anti)conformal and the uniform limit of diffeomorphisms in order to show that the Jacobian $v_x^1 v_y^2 - v_y^1 v_x^2$ of v has the same sign almost everywhere in S^2 (cf. 9.3.7 [M3]). Here, additional difficulties arise from the fact that v so far is only known to be of class $C^0 \cap H_2^1$, but these problems can be overcome with the arguments of Lemmata 9.2.4, 9.2.5 of [M3].

DEFINITION 5.5.1 *Suppose* G *is a plane domain of class* C^1, $\phi \in C^1(G, \mathbb{R}^2)$, $z \notin \phi(\partial G)$.

Then $m(z, \phi(\partial G))$ *is defined to be the winding number of the curve* $\phi(\partial G)$ *w.r.t. z.*

If only $\phi \in C^0(G, \mathbb{R}^2)$, *then*

$$m(z, \phi(\partial G)) := \lim_{n\to\infty} m(z, \phi_n(\partial G))$$

for any sequence $\phi_n \in C^1(\partial G, \mathbb{R}^2)$ *which converges uniformly to* ϕ *on* ∂G.

That $m(z, \phi(\partial G))$ is well defined, follows from elementary properties of winding numbers (cf. e.g. [Fe]).

LEMMA 5.5.1 G *a plane domain,* $\phi \in C^0 \cap H_2^1(G,\mathbb{R}^2)$. *Then for every* $x_0 \in G$, *there exists a set* $C(x_0)$ *with* $H^1(C(x_0)) = 0$, *where* H^1 *is 1-dimensional Hausdorff measure, such that for all* $R \notin C(x_0)$

$$\int_{B(x_0,R)} J(\phi)\ dx = \int_{\phi(B(x_0,R))} m(z,\phi(\partial B(x_0,R))\ dz$$

if $B(x_0,R) \subset\subset G$

$$(J(\phi) := \phi_x^1\phi_y^2 - \phi_y^1\phi_x^2)$$

Proof We can find a sequence $\phi_n \in C^1(D)$, $D \subset\subset G$, converging uniformly and strongly in H_2^1 to ϕ, so that $\phi_n \to \phi$ strongly in $H_2^1(\partial B(x_0,R))$ on $\partial B(x_0,R)$, if $R \notin C(x_0)$, $H^1(C(x_0)) = 0$.

Since $H_2^1(\partial B(x_0,R))$ functions are absolutely continuous, and the lengths of $\phi_n(\partial B(x_0,R))$ and $\phi(\partial B(x_0,R))$ are uniformly bounded, the two-dimensional measure of $\phi(\partial B(x_0,R))$ vanishes ($R \notin C(x_0)$). Consequently, $z \notin \phi(\partial B(x_0,R))$ for almost all z, and thus

$$m(z,\phi_n(\partial B(x_0,R)) \to m(z,\phi(\partial B(x_0,R)) \qquad \text{for these } z. \tag{5.5.8}$$

Now

$$\lim_{n\to\infty} \int_{\phi_n(B(x_0,R))} m(z,\phi_n(\partial B(x_0,R))\ dz = \lim_{n\to\infty} \int_{B(x_0,R)} J(\phi_n)\ dx = \int_{B(x_0,R)} J(\phi)\ dx$$

Since

$$\int_I m(z,\phi_n(\partial B(x_0,R))\ dz \le \left(\frac{\text{meas } I}{\pi}\right)^{\frac{1}{2}} \text{length } (\phi_n(\partial B(x_0,R)),$$

for any measurable set I, we can integrate (5.5.8), and the result follows.

q.e.d.

LEMMA 5.5.2 *We suppose that* $\phi_n : S^2 \to \Sigma$ *are diffeomorphisms, converging uniformly and weakly in* H_2^1 *to* ϕ.

Then $J(\phi)$ *has the same sign almost everywhere.*

Proof We introduce coordinates on S^2 by stereographic projection. Let $B(x_0,R)$, $R \notin C(x_0)$ satisfy the assumptions of Lemma 5.5.1

$$\varepsilon_n := \max_{x\in\partial B(x_0,R)} |\phi_n(x) - \phi(x)|$$

$$V_n := \{z : d(z,\phi(\partial B(x_0,R)) > \varepsilon_n\} .$$

For $z \in V_n$, $m(z,\phi_n(\partial B(x_0,R)) = m(z,\phi(\partial B(x_0,R))$.

Lemma 5.5.1 therefore implies

$$\lim_{n\to\infty} \int_{\phi_n^{-1}(V_n)\cap B(x_0,R)} J(\phi_n) = \int_{B(x_0,R)} J(\phi) . \tag{5.5.9}$$

Since we can assume w.l.o.g., $J(\phi_n) \geq 0$ in $B(x_0,R)$ for all n , and (5.5.9) holds for almost all discs $B(x_0,R)$, the result follows.

q.e.d.

Thus, v is a weak solution of the corresponding Cauchy-Riemann equations, i.e.

$$v_x^2 = -g_{22}^{-1}(g_{12}v_x^1 + k\sqrt{g}\ v_y^1) \tag{5.5.10}$$

$$v_y^2 = g_{22}^{-1}(k\sqrt{g}\ v_x^1 - g_{12}v_y^1)$$

$(g = g_{11}g_{22} - g_{12}^2)$, where $k = \pm 1$ is constant by Lemma 5.5.2. Since (5.5.10) is a linear first-order elliptic system, v is regular.

LEMMA 5.5.3 v *is a homeomorphism.*

Proof We assume that v is not a homeomorphism. The v is not injective, i.e. there must exist two points z_1 , z_2 , $z_1 \neq z_2$ with $v(z_1) = v(z_2)$. We choose a shortest segment γ_n joining $v_n(z_1)$ and $v_n(z_2)$. Since v_n is a homeomorphism, $\tilde{\gamma}_n := v_n^{-1}(\gamma_n)$ is a curve joining z_1 and z_2.

If $p_{n,\delta}$ is a point on $\partial B(z_1,\delta) \cap \tilde{\gamma}_n$, then for $n \to \infty$ we can find a subsequence of $(p_{n,\delta})$ converging to some point p_δ on $\partial B(z_1,\delta)$. Since the v_n converge uniformly to v, we see that $v(p_\delta) = v(z_1) = v(z_2)$. Thus, a whole continuum is mapped onto the single point $v(z_1) = v(z_2)$ by v.

At interior points, we can choose again local coordinates v^1, v^2. From (5.5.10) we conclude that v^1 and v^2 are harmonic, e.g.

(5.5.11) $$\Delta v^1 + \Gamma^1_{11}(v^1_x v^1_x + v^1_y v^1_y) + 2\Gamma^1_{12}(v^1_x v^2_x + v^1_y v^2_y) + \Gamma^1_{22}(v^2_x v^2_x + v^2_y v^2_y) = 0 .$$

From (5.5.10) and (5.5.11) we obtain

(5.5.12) $$|v^1_{z\bar{z}}| \le c|v^1_z| \le K$$

since $v \in C^2(B)$.

We now use the following result of Hartman-Wintner [HtW] (a proof of the version presented here can also be found in [J8]).

LEMMA 5.5.4 *Suppose* $u \in C^{1,1}(G,\mathbb{R})$, G *a plane domain,* $z_0 \in G$, *and*

(5.5.13) $$|u_{z\bar{z}}| \le K(|u_z| + |u|) ,$$

where K *is a fixed constant.*

If

(5.5.14) $$u(z) = o(|z - z_0|^n)$$

for some $n \in \mathbb{N}$ *in a neighbourhood of* z_0, *then*

$$\lim_{z\to z_0} u_z \cdot (z - z_0)^{-n}$$

exists. If (5.5.14) holds for all $u \in \mathbb{N}$, *then*

$$u \equiv 0 .$$

We continue the proof of Lemma 5.5.3.

If now $v_z^1(z_0) = 0$ for some $z_0 \in S^2$, Lemma 5.5.4 gives the asymptotic representation

(5.5.15) $$v_z^1 = a(z - z_0)^n + o|z - z_0|^n)$$

for some $a \in \mathbb{C}$, $z \neq 0$, and some positive integer n, unless $v_z^1 \equiv 0$ in a neighbourhood of z_0. The latter is not possible, however, since it implies that the set where $v_x^1 v_y^2 - v_y^1 v_x^2 = 0$ is nonvoid and open in S^2, and therefore $v_x^1 v_y^2 - v_x^1 v_y^2 \equiv 0$ in S^2 in contradiction to the fact that v is a surjective $C^{2,\alpha}$ map onto Σ. We can choose the local coordinates in such a way that

(5.5.16) $$g_{ij}(v(z_0)) = \delta_{ij} .$$

Using (5.5.16), (5.5.11) and integrating (5.5.15), we infer

$$w(z) := v^1 + iv^2 = \rho(z - z_0)^{n+1} + \sigma(\bar{z} - \bar{z}_0)^{n+1} + o(|z - z_0|^{n+1}) + w_0 ,$$

where $\rho , \sigma \in \mathbb{R}$, $|\rho| + |\sigma| \neq 0$, $w_0 = (v^1 + iv^2)(z_0)$, in a neighbourhood of z_0.

Without loss of generality, by performing homeomorphic linear transformations, we can assume $\rho = 1, \sigma > 0, z_0 = w_0 = 0$, i.e.

(5.5.17) $$w(z) = z^{n+1} + \sigma\bar{z}^{n+1} + o(|z|^{n+1}) .$$

This, however, is in contradiction to the consequence we have obtained from the assumption that v is not injective, namely that a whole continuum of points is mapped to a single point. This proves the lemma. (The application of the Hartman-Wintner formula in the above argument is due to E. Heinz [Hz2]).

LEMMA 5.5.5 v *is a diffeomorphism.*

Proof We want to show that since v is a homeomorphism by Lemma 5.5.3, (5.5.17) cannot hold with $n \geq 1$.

Assume on the contrary, (5.5.17) holds for $n \geq 1$. Then

$$v^1(re^{i\theta}) = (1+\sigma)\, r^{n+1} \cos((n+1)\theta) + o(r^{n+1}) ,$$

and in particular

$$v^1(re^{i\pi k/n+1}) = (1+\sigma)\, r^{n+1} (-1)^k + o(r^{n+1}) \tag{5.5.18}$$

for $k = 0,1,\ldots,2n+1$.

For sufficiently small $\varepsilon > 0$ and $r \leq \varepsilon$, the sign of the left hand side of (5.5.18) is therefore $(-1)^k$.

If z traverses a Jordan curve in $\{z : z \neq 0, |z| \leq \varepsilon\}$, then $v^1(z)$ hence has to change sign at least $2n+2$ times. On the other hand, for sufficiently small $\delta > 0$, since v is a homoemorphism, the preimage of $\{|w| = \delta\}$ is such a curve, but here v^1 changes sign exactly twice. Hence $n = 0$, and the Jacobian of v does not vanish, and the lemma is proved.

q.e.d.

This also finishes the proof of Thm. 5.5.1.

COROLLARY 5.5.1 *Let* Σ *be a surface homeomorphic to* S^2 *with metric tensor given in local coordinates by bounded measurable functions* g_{ij} *, satisfying*

$$g_{11}g_{22} - g_{12}^2 \geq \lambda > 0 \qquad \text{almost everywhere .} \tag{5.5.19}$$

Then there is a homeomorphism $h : S^2 \to \Sigma$ *satisfying the conformality relations*

$$g_{ij} \frac{\partial h^i}{\partial x} \frac{\partial h^j}{\partial x} = g_{ij} \frac{\partial h^i}{\partial y} \frac{\partial h^j}{\partial y} \tag{5.5.20}$$

$$g_{ij} \frac{\partial h^i}{\partial x} \frac{\partial h^j}{\partial y} = 0$$

almost everywhere.

If $(g_{ij}) \in C^\alpha$ *, then* h *is a diffeomorphism of class* $C^{1,\alpha}$ *,* *satisfying (5.5.20) everywhere.*

Proof We let (g^n_{ij}) be a sequence of $C^{2,\alpha}$ metrics converging to (g_{ij}) pointwise almost everywhere. We denote the corresponding surfaces by Σ^n and let $h_n : S^2 \to \Sigma^n$ be a conformal diffeomorphism constructed in Thm. 5.5.1.

Since the h_n satisfy a system of the type of (5.5.10), elliptic regularity theory implies uniform C^α as well as H^1_2 estimates. Hence a subsequence converges uniformly and weakly in H^1_2 towards a weak solution h of (5.5.10).

Furthermore, since the h_n are diffeomorphisms, their inverses satisfy a system of the same type, namely

(5.5.21)
$$\begin{aligned} y^n_{v^1} &= \frac{g^n_{12}}{\sqrt{g^n}} x^n_{v^1} - \frac{g^n_{11}}{\sqrt{g^n}} x^n_{v^2} \\ y^n_{v^2} &= \frac{g^n_{22}}{\sqrt{g^n}} x^n_{v^1} - \frac{g^n_{12}}{\sqrt{g^n}} x^n_{v^2} , \end{aligned}$$

where $g^n = g^n_{11} g^n_{22} - (g^n_{12})^2$.

Therefore, also h^{-1}_n satisfies a uniform Hölder estimate by elliptic regularity theory, and thus we see that the limit map h has to be invertible, i.e. a homeomorphism.

In case $\Sigma \in C^{1,\alpha}$, the metrics (g^n_{ij}) can be chosen to converge with respect to the C^α-norm to (g_{ij}). From (5.5.14) we infer that the h^{-1}_n then satisfy uniform $C^{1,\alpha}$ estimates, and consequently the limit map h is a diffeomorphism.

Thus we have found the desired conformal representation of Σ , and the proof of Cor. 5.5.1 is complete.

q.e.d.

We can also derive the following version of the Riemann mapping theorem (cf. e.g. [AB]):

COROLLARY 5.5.2 *Let* S *be a compact surface with boundary, homeomorphic to the unit disc* D *, and a metric tensor* (g_{ij}) *satisfying the assumptions of Cor. 5.5.1.*

Then there is a conformal representation $h : D \to S$ *, satisfying the same conclusions as in Cor. 5.5.1.*

Proof Let S' be an isometric copy of S with opposite orientation; let $i : S \to S'$ be the isometry. Identifying s with $i(s)$ for $s \in \partial S$ gives a surface Σ to which we can apply Cor. 5.5.1 and find a conformal homeomorphism $h : S^2 \to \Sigma$. Then $i \circ h$ is another conformal homeomorphism, and we can find a conformal automorphism k of S^2 satisfying $h \circ k = i \circ h$. (This is clear for smooth metrics on Σ , since then $h^{-1} \circ i \circ h$ is a conformal diffeomorphism of S^2 . The general case follows again by approximation.) The fixed point set of k then is a circle and hence bounds a disc which is conformally equivalent to S .

q.e.d.

Note that our proof immediately yields the one-to-one-correspondence of the boundaries, first proved by Osgood and Caratheodory.

We can again normalize the conformal map by e.g. prescribing the images on ∂S of three distinct points on ∂D .

The preceding result is due to Lichtenstein [Li] (for C^{α}-metrics), Lavrent'ev [Lv] (for continuous metrics), and Morrey [M1].

In a future publication, I shall demonstrate that the preceding methods can also yield conformal representations of surfaces of higher genus. This approach can considerably simplify a large portion of the uniformization theory.

5.6 EXISTENCE OF HARMONIC DIFFEOMORPHISMS, IF THE IMAGE IS CONTAINED IN A CONVEX BALL

THEOREM 5.6.1 *Assume* $u : D \to B(p,M)$ *is an injective harmonic map, where* D *is the unit disc and* $B(p,M)$ *is a disc on some surface with* $M < \frac{\pi}{2\kappa}$ *, where* κ^2 *again is an upper curvature bound. Assume that* $g := u|\partial D$ *is a* C^2*-diffeomorphism onto* $g(\partial D)$ *satisfying*

(5.6.1) $$0 < b \le \left|\frac{dg(\phi)}{d\phi}\right| \qquad \text{for all} \quad \phi \in \partial D .$$

Assume furthermore that $g(\partial D)$ *is strictly convex w.r.t.* $u(D)$ *, the geodesic curvature* κ_g *satisfying*

(5.6.2) $$0 < a_1 \le \kappa_g(g(\partial D))(g(\phi)) \le a_2 \qquad \text{for all} \quad \phi \in \partial D .$$

Then the functional determinant $J(u(x))$ *satisfies for all* $x \in D$

(5.6.3) $$|J(u(x))| \ge \delta_1^{-1} ,$$

where $\delta_1 = \delta_1(\omega,\kappa,M,a_1,a_2,b,|g|_{C^{1,\alpha}})$.

Without assuming (5.6.1) and (5.6.2), on each disc $B(0,r)$ *,* $0 < r < 1$ *,*

$$|J(u(x)| \ge \delta_2^{-1} \qquad \text{for} \quad x \in B(0,r)$$

$$\delta_2 = \delta_2(\omega,\kappa,M,r, \text{meas } u(D), |g|_{C^\alpha}) ,$$

or

$$|J(u(x)| \ge \delta_3^{-1} \qquad \text{for} \quad x \in B(0,r)$$

where δ_3 *depends on* ω *,* κ *,* M *,* r *,* meas $u(D)$ *,* $E(u)$ *, and on some kind of normalization like fixing the images of three boundary points or of one interior point.*

We omit the proof which can be found in [JK1]. Whereas the boundary estimate basically follows by applying the maximum principle to $d^2(u(x), g(\partial D))$, the interior estimate depends on deep estimates of E. Heinz ([Hz5]).

We can now prove the main result of [J3].

THEOREM 5.6.2 *Suppose* Ω *is a compact domain with* C^2 *boundary* $\partial\Omega$ *on some surface, and that* Σ *is another surface. We assume that* $\psi : \bar{\Omega} \to \Sigma$ *maps* $\bar{\Omega}$ *homeomorphically onto its image, that* $\psi(\partial\Omega)$ *is contained in some disc* $B(p,M)$ *with radius* $M < \frac{\pi}{2\kappa}$ *(where* $\kappa^2 \geq 0$ *is an upper curvature bound on* $B(p,M)$ *) and that the curves* $\psi(\partial\Omega)$ *are of class* C^2 *and convex w.r.t.* $\partial(\Omega)$.

Then there exists a harmonic mapping $u : \Omega \to B(p,M)$ *with the boundary values prescribed by* ψ *which is a homeomorphism between* $\bar{\Omega}$ *and its image, and a diffeomorphism in the interior.*

Moreover, if $\psi|\partial\Omega$ *is even a* C^2*-diffeomorphism then* u *is a diffeomorphism up to the boundary.*

Theorems 5.6.2 and 4.11.1 imply

COROLLARY 5.6.1 *Under the assumptions of Thm. 5.6.2, each harmonic map which solves the Dirichlet problem defined by* ψ *and which maps* Ω *into a geodesic disc* $B(p,M)$ *with radius* $M < \frac{\pi}{2\kappa}$ *, is a diffeomorphism in* Ω .

Proof of Theorem 5.6.2 First of all, $\partial\Omega$ is connected. Otherwise, $\psi(\partial\Omega)$ would consist of at least two curves, both of them convex w.r.t. $\psi(\Omega)$. Therefore, we could find a nontrivial closed geodesic γ in $\psi(\Omega) \subset B(p,M)$ with an easy Arzela-Ascoli argument. Since a geodesic can be considered as a special case of a harmonic map and $M < \frac{\pi}{2\kappa}$, Lemmata 1.7.1 and 2.3.2 imply that γ has to be a point, which is a contradiction. Therefore, $\partial\Omega$ is connected, and since Ω is homeomorphic to $\psi(\Omega)$, we conclude that Ω is a disc, topologically.

Therefore, we have to prove the theorem only for the case where Ω is the plane unit disc D, taking the existence (cf. Cor. 5.5.2) of a conformal map $k : D \to \Omega$ and the composition property Lemma 1.3.3 into account.

For the rest of this section, we assume that $\psi : \partial D \to \psi(\partial D)$ is a C^2-diffeomorphism between curves of class $C^{2,\alpha}$, that $\psi(\partial\Omega)$ is not only convex, but strictly convex, and that we have the following quantitative bounds

$$(5.6.4)\qquad \left|\frac{d^2}{d\phi^2}\psi(\phi)\right| \le b_1$$

and $\qquad\qquad\qquad\qquad\qquad$ for $\phi \in \partial D$

$$(5.6.5)\qquad \left|\frac{d}{d\phi}\psi(\phi)\right| \ge b_2^{-1}$$

and

$$(5.6.6)\qquad 0 < a_1 \le \kappa_g(\psi(\partial D)) \le a_2 .$$

These assumptions can be removed later on by approximation arguments which we shall indicate below.

By virtue of Cor. 5.5.2 again, there is a conformal map $k : D \to \psi(D)$. By a variation of boundary values, we now want to deform this conformal map into a harmonic diffeomorphism u.

Without loss of generality, we may assume that the boundary value map preserves the orientation of ∂D. Now let γ be the parametrization of the boundary curve of $\psi(D)$ by arc length. We set

$$(5.6.7)\quad \omega(\phi,\lambda) := \gamma(\lambda\gamma^{-1}(k(\phi)) + (1-\lambda)\gamma^{-1}(\psi(\phi))) , \qquad \phi \in \partial D , \ \lambda \in [0,1] .$$

ω deforms the boundary values of k into the boundary values prescribed by ψ .

Since we assumed that (5.6.4) and (5.6.5) hold and that $\psi(\partial D) \in C^{2,\alpha}$, well-known regularity properties of conformal maps imply that

$$(5.6.8) \qquad \omega(\phi,\lambda) \, , \quad \frac{\partial}{\partial\phi}\omega(\phi,\lambda) \quad \text{and} \quad \frac{\partial^2}{\partial\phi^2}\omega(\phi,\lambda)$$

are continuous functions of λ ,

$$(5.6.9) \quad \frac{\partial}{\partial\phi}\omega(\phi,\lambda) \quad \text{does not vanish for any } \phi \in \partial D \text{ and } \lambda \in [0,1] \, .$$

Let now u_λ denote the harmonic map from D to $B(p,M)$ with boundary values $\omega(\cdot,\lambda)$, (the existence of u_λ follows from Lemma 5.2.3) and let $\lambda_n \in [0,1]$ be a sequence converging to some $\lambda \in [0,1]$.

By Thm. 4.9.1, the Arzela-Ascoli Theorem and the uniqueness theorem 4.11.1, u_{λ_n} converges to the harmonic map u_λ in the $C^{1,\beta}$-topology, $0 < \beta < \alpha$. In particular,

$$p(\lambda) := \inf_{x\in D} \left| J(u_\lambda)(x) \right|$$

depends continuously on λ ($J(u_\lambda)$ denotes the Jacobian of u_λ). We define $L := \{\lambda \in [0,1] : p(\lambda) > 0\}$. By Cor. 5.5.2, $0 \in L$ (u_0 is the conformal map k), and therefore L is not empty. Since we assumed (5.6.5) and (5.6.6), which implied (5.6.8) and (5.6.9) we can apply Thm. 5.6.1 to the extent that

$$(5.6.10) \qquad p(\lambda) \ge p_0 > 0 \quad \text{for } \lambda \in L \, .$$

Since $p(\lambda)$ depends continuously on λ , (5.6.10) implies $L = [0,1]$. Thus, u_1 is a local diffeomorphism and a diffeomorphism between the boundaries of D and $u_1(D)$, and consequently a global diffeomorphism by the homotopy lifting theorem.

Thus, the proof of Thm. 5.6.2 is complete, except for the approximation arguments.

So far, we have assumed that the boundary of the image is strictly convex, and, in addition, that the boundary values are a diffeomorphism of class C^2. We now have to prove the theorem also for the case that the boundary is only supposed to be convex and that the boundary values are only supposed to induce a homeomorphism of the boundaries.

We shall present only the first approximation argument. It is a modification of the corresponding one given by E. Heinz in [Hz4], pp.178-183. The reasoning for the second case can be taken over from [Hz3], pp.351-352, in case $\partial\psi(D) \in C^{2,\alpha}$.

Therefore, let us suppose that the boundary of the image $\psi(D)$ is only convex, while the boundary values ψ are still assumed to be a diffeomorphism of class C^2. Then we argue in the following way:

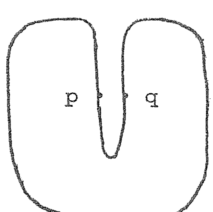
Given a metric g_{ij} on the image with respect to which the boundary of $A := \psi(D)$ is convex, there is a sequence $\{g^n_{ij}\}$ of metrics on A such that ∂A is even strictly convex with respect to g^n_{ij}, according to [Hz4], §4. Moreover, $\{g^n_{ij}\}$ can be chosen to converge uniformly to g_{ij} on A together with their first and second derivatives, as $n \to \infty$. Keeping the boundary values ψ fixed, we consider the map $u_n(x)$ which is harmonic in the metric g^n_{ij} and which solves the Dirichlet problem with boundary values ψ. The existence of u_n is guaranteed by the arguments given above — at least for large values of n, when g^n_{ij} is so close to g_{ij} that the geometric conditions are satisfied.

By virtue of Thm. 5.6.1, on each disc $B(0,r)$, $r < 1$, there is an a-priori bound of the functional determinant of $u_n(x)$ from below. Moreover, by virtue of Thm. 4.9.1, we can choose a subsequence of the functions $u_n(x)$ which converges uniformly on D together with the first derivatives to a map $u(x)$. In particular, the u_n converge to u strongly in H_2^1. Therefore, u is a weakly harmonic map w.r.t. the metric g_{ij}, i.e. a weak solution of the corresponding Euler equations. Since u is also of class C^1, linear elliptic regularity theory implies that u is a classical solution, i.e. harmonic. Moreover, u is a local diffeomorphism in the interior, and since it is the uniform limit of diffeomorphisms, it is a diffeomorphism in the interior.

q.e.d.

Remarks 1) Actually, using a further approximation argument, we do not even have to assume that the boundary values are homeomorphic. We need only that they are continuous and monotonic, i.e. a uniform limit of homeomorphisms. The corresponding harmonic solution of the Dirichlet problem still remains a diffeomorphism in the interior.

2) In the case where both Ω and $\psi(\Omega)$ are bounded simply connected domains in the plane, the assertion of Thm. 5.6.2 was already obtained by Radó and Kneser [Rd], [Kn1], and Choquet [Cq]. Choquet also showed that the convexity of the boundary of the image is necessary for Thm. 5.6.2 to hold. The reason is the following. Suppose the image has the depicted shape. If



the boundary values $\psi(\partial\Omega)$ are concentrated near p and q, then by the mean value property of harmonic functions, some points of Ω will be mapped onto points between p and q not belonging to $\psi(\Omega)$.

This is in essential contrast to the case of conformal maps where convexity of the image is not necessary to guarantee that the solution is a diffeomorphism (cf. Cor. 5.5.2). Note that a conformal map is a solution of a free boundary value problem instead of a Dirichlet problem.

5.7 EXISTENCE OF HARMONIC DIFFEOMORPHISMS BETWEEN CLOSED SURFACES

The main result of this section is

THEOREM 5.7.1 *Suppose that* Σ_1 *and* Σ_2 *are compact surfaces without boundary, and that* $\phi : \Sigma_1 \to \Sigma_2$ *is a diffeomorphism. Then there exists a harmonic diffeomorphism* $u : \Sigma_1 \to \Sigma_2$ *homotopic to* ϕ *. Furthermore,* u *is of least energy among all diffeomorphisms homotopic to* ϕ *.*

Thm. 5.7.1 was proved by Jost-Schoen [JS], but it was first claimed by Shibata [Sh] in 1963. His proof contained several mistakes, however, and was therefore rejected.

H. Sealey then carefully examined Shibata's paper in his thesis [Se] and was able to correct some (but not all) of the mistakes. The proof of [JS], however, proceeds along completely different lines than the Shibata-Sealey approach and depends in an essential way on Thm. 5.6.2.

Thms. 5.7.1 and 4.11.1 immediately imply the following corollary, proved by Schoen-Yau [SY1] and Sampson [Sa].

COROLLARY 5.7.1 *If under the assumptions of Thm. 5.7.1,* Σ_2 *has nonpositive curvature, then every harmonic map homotopic to a diffeomorphism is itself diffeomorphic.*

Furthermore, we have

COROLLARY 5.7.2 *Suppose that* Σ_1 *and* Σ_2 *are compact surfaces without boundary, and that* $\psi : \Sigma_1 \to \Sigma_2$ *is a covering map, i.e. a local diffeomorphism. Then there exists a harmonic covering map* $u : \Sigma_1 \to \Sigma_2$ *, homotopic to* ψ *.*

Proof of Corollary 5.7.2 We can pull back the metric ds^2 of Σ_2 via ψ to obtain a surface Σ_2', diffeomorphic to Σ_1 and with metric $\psi^* ds^2$. Then $\psi : \Sigma_2' \to \Sigma_2$ is a local isometry. By Thm. 5.7.1, there is a harmonic diffeomorphism $u' : \Sigma_1 \to \Sigma_2'$, homotopic to the identity. $u := \psi \circ u$ then is the desired harmonic covering map.

Proof of Theorem 5.7.1 (following [JS]) If Σ_1 and Σ_2 are homeomorphic to S^2, then we can find a conformal (and hence harmonic) diffeomorphism homotopic to ψ by Thm. 5.5.1. The case where Σ_1 and Σ_2 are homeomorphic to the real projective space is similarly handled by passing to two-sheeted coverings. Thus we can assume w.l.o.g. that $\pi_2(\Sigma_i) = 0$ $(i = 1,2)$.

We let $\mathcal{D}$ be the class of diffeomorphisms from Σ_1 onto Σ_2 homotopic to ϕ. Since $\pi_2(\Sigma_2) = 0$ a homotopically trivial Jordan curve separates Σ_2 into two topologically different parts, one being a disc and the other one having higher connectivity. Therefore, the argument in the proof of Thm. 5.5.1 gives equicontinuity of a weakly convergent sequence in $\mathcal{D}$ even without a normalization.

We again let $\bar{\mathcal{D}}$ be the weak H_2^1-closure of $\mathcal{D}$, and choose an energy minimizing sequence in $\bar{\mathcal{D}}$. A subsequence then converges weakly in H_2^1 towards some $u_0 \in \bar{\mathcal{D}}$, and u_0 minimizes energy in $\bar{\mathcal{D}}$ by lower semicontinuity again. We also can find a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$ converging weakly in H_2^1 to u_0. Since the u_n are equicontinuous, they converge also uniformly to u_0, and hence u_0 is continuous and homotopic to ϕ. The u_n, since converging weakly, have uniformly bounded energy,

$$E(u_n) \le K, \quad \text{say} .$$

We want to show that u_0 is a harmonic diffeomorphism. We consider an arbitrary point $x_0 \in \Sigma_1$ and define

$$B_\sigma := \mathring{B}(u_0(x_0), \sigma)$$

i.e. the open disc in Σ_2 centred at $u_0(x_0)$ with radius σ.

We restrict ourselves in the sequel to values of σ which are smaller than the injectivity radius of Σ_2 and smaller than $\pi/2\kappa$, where κ^2 again is an upper bound for the curvature of Σ_2. We define

$$\Omega_0 := u_0^{-1}(B_\sigma)$$

$$\Omega_n := u_n^{-1}(B_\sigma) \qquad (n \in \mathbb{N})$$

W.l.o.g., we can assume $x_0 \in \Omega_n$ for all n, since the u_n converge uniformly to u_0. Let D be the unit disc in the complex plane and

$$F_n : D \to \bar{\Omega}_n$$

be a conformal mapping which maps 0 to x_0.

The proof of the existence of F_n is the same as that of Cor. 5.5.2 since instead of fixing three boundary points, we can fix an interior point (and a tangent direction at this point, but that is not necessary for the proof) in order to guarantee the equicontinuity of a minimizing sequence as in 5.5.

Since $\Gamma_n := \partial\Omega_n$ is a Jordan curve of class C^1 (because u_n is a diffeomorphism), F_n is a homeomorphism of D onto $\bar{\Omega}_n$, and therefore $u_n \circ F_n$ maps ∂D homeomorphically onto ∂B_σ. By Thm. 5.6.2 and Cor. 5.6.1, there exists a unique harmonic mapping $v_n : D \to B_\sigma$ which assumes the boundary values prescribed by $u_n \circ F_n$, and v_n minimizes energy in its homotopy class and is a diffeomorphism.

In particular

(5.7.1) $$E_D(v_n) \le E_D(u_n \circ F_n) = E_{\Omega_n}(u_n) \le K$$

by Lemma 1.3.2 ($E_S(f)$ is the energy of the mapping f over the set S). Since the u_n converge uniformly to u_0, we can assume that $u_n \circ F_n(0)$ stays in an arbitrarily small neighbourhood of $u_0(x_0)$. Therefore, we can again apply the argument of section 5.5 to show that the maps $u_n \circ F_n$ are equicontinuous on D. In particular, the boundary values of v_n, namely $u_n \circ F_n|\partial D$, are equicontinuous. By Thms. 4.9.1 and 4.7.1, we can therefore assume that the v_n converge uniformly on D to a map v_0 which is harmonic in the interior of D. Using Thm. 5.6.1, we see furthermore that v_0 is a diffeomorphism in the interior of D.

We define now

$$\tilde{u}_n = \begin{cases} v_n \circ F_n^{-1} & \text{in } \Omega_n \\ u_n & \text{in } \Sigma_1 \backslash \Omega_n . \end{cases}$$

Clearly, $\tilde{u}_n$ is a Lipschitz map and lies in H_2^1 and $E(\tilde{u}_n) \le K$. We can also assume w.l.o.g. (by approximation) that the u_n are of class $C^{1,\alpha}$. Then, for each n, the functional determinant of $\tilde{u}_n$ is defined and bounded from below on $\Sigma_1 \backslash \Omega_n$ by Thm. 5.6.1. It is easily seen by an approximation argument that $\tilde{u}_n \in \bar{\mathcal{D}}$.

Using Lemma 5.2.2 as before, we can assume again w.l.o.g. that the $\tilde{u}_n$ converge on Σ_1 weakly in H_2^1 and uniformly to a map $\tilde{u}_0 \in \bar{\mathcal{D}}$ and that the F_n converge uniformly on compact subsets to a conformal map F. Since $E_D(F_n) = \text{Area}(\Omega_n) \le \text{Area}(\Sigma_1)$, F maps $\mathring{D}$ diffeomorphically onto some open set $\Omega \subset \Sigma_1$, and 0 is mapped to x_0. F is not necessarily smooth on ∂D, but that does not affect the following arguments.

$u_0 \circ F$ is the uniform limit of $u_n \circ F_n$ and thus extends continuously to D. Since $u_n \circ F_n$ and v_n coincide on ∂D, it follows that also $u_0 \circ F$ and

v_0 coincide there, and since v_0 is harmonic and therefore energy minimizing (by Theorem 4.11.1) in its homotopy class,

$$E_D(v_0) \le E_D(u_0 \circ F) .$$

Since conformal maps preserve energy by Lemma 1.3.2, this implies

$$E_\Omega(\tilde{u}_0) \le E_\Omega(u_0) . \tag{5.7.2}$$

We now want to show that

$$E_{\Sigma_1 \setminus \Omega}(\tilde{u}_0) = E_{\Sigma_1 \setminus \Omega}(u_0) . \tag{5.7.3}$$

For this, it is sufficient to show that u_0 and $\tilde{u}_0$ coincide almost everywhere outside Ω . We claim that

$$\Sigma_1 \setminus \Omega \subset u_0^{-1}(\Sigma_2 \setminus B_\sigma) . \tag{5.7.4}$$

We define

$$\rho_n(x) := d(u_n(x),\ u_0(x_0))$$

$$\rho_0(x) := d(u_0(x),\ u_0(x_0))$$

for $x \in \Sigma_1$. Let $x \in \Sigma_1 \setminus \Omega$. If

$$\rho_0(x) = \lim_{n\to\infty} \rho_n(x) \ge \sigma ,$$

then

$$x \in u_0^{-1}(\Sigma_2 \setminus B_\sigma) .$$

Since the $\rho_n \circ u_n \circ F_n$ are equicontinuous and equal to σ on ∂D , $\rho_0(x) < \sigma$ implies that

$$d(F_0^{-1}(x),\ \partial D) \ge \delta > 0$$

for sufficiently large n .

Since on the other hand, the F_n converge uniformly to F on compact subsets of D, this would imply $x \in F(D) = \Omega$ which contradicts the assumption $x \in \Sigma_1 \sim \Omega$. This proves (5.7.4).

We also have

$$u_0^{-1}(\Sigma_2 \backslash B_\sigma) = u_0^{-1}(\partial B_\sigma) \cup u_0^{-1}(\Sigma_2 \backslash \bar{B}_\sigma) ,$$

and since the sets $u_0^{-1}(\partial B_\sigma)$ cover a neighbourhood of x_0 and are disjoint, we can assume w.l.o.g. that the two-dimensional measure of $u_0^{-1}(\partial B_\sigma)$ vanishes for our chosen σ. If

$$x \in u_0^{-1}(\Sigma_2 \backslash \bar{B}_\sigma) ,$$

then

$$\lim_{n\to\infty} \rho_n(x) = \rho_0(x) > \sigma$$

and because of the equicontinuity of the functions ρ_n, there exists an open neighbourhood U of x such that $\rho_n|U > \sigma$ for sufficiently large n. This implies

$$\tilde{u}_0 = \lim_{n\to\infty} \tilde{u}_n = \lim_{n\to\infty} u_n = u_0 \quad \text{on } U .$$

Therefore $u_0 = \tilde{u}_0$ almost everywhere on $u_0^{-1}(\Sigma_2 \backslash B_\sigma)$, and (5.7.3) now follows from (5.7.4). By the choice of u_0, we have on the other hand

$$E_{\Sigma_1}(u_0) \le E_{\Sigma_1}(\tilde{u}_0) .$$

Thus, we conclude from (5.7.2) and (5.7.3) that

$$E_\Omega(\tilde{u}_0) = E_\Omega(u_0)$$

and consequently

$$E_D(v_0) = E_D(u_0 \circ F) .$$

Since v_0 and $u_0 \circ F$ coincide on ∂D, we conclude from the uniqueness of energy minimizing maps (Thms. 4.11.1 and Lemma 5.2.3) that v_0 and $u_0 \circ F$ coincide on D. Therefore $u_0 \circ F$ and consequently also u_0 is a harmonic diffeomorphism, the latter in Ω , which is a neighbourhood of an arbitrarily chosen point $x_0 \in \Sigma_1$. This finishes the proof of Theorem 5.7.1.

q.e.d.

With the same method, we can also improve Thm. 5.6.2.

THEOREM 5.7.2 *Let* $\Omega \subset \Sigma_1$ *be a two-dimensional domain with nonempty boundary* $\partial\Omega$ *consisting of* C^2 *curves, and let* $\psi : \bar{\Omega} \to \Sigma_2$ *be a homeomorphism of* $\bar{\Omega}$ *onto its image* $\psi(\bar{\Omega})$ *, and suppose that the curves* $\psi(\partial\Omega)$ *are of class* C^2 *and convex with respect to* $\psi(\Omega)$.

Then there exists a harmonic diffeomorphism $u : \Omega \to \psi(\Omega)$ *which is homotopic to* ψ *and satisfies* $u = \psi$ *on* $\partial\Omega$ *. Moreover,* u *is of least energy among all diffeomorphisms homotopic to* ψ *and assuming the same boundary values.*

This result is again taken from [JS]. The case of non-positive image curvature was solved in [SY1].

Proof We assume first that $\partial\Omega$ and $\psi(\partial\Omega)$ are of class $C^{2+\alpha}$ and that ψ gives rise to a diffeomorphism between $\partial\Omega$ and $\psi(\partial\Omega)$ and that $\psi(\partial\Omega)$ is strictly convex with respect to $\psi(\Omega)$.

In this case, the proof proceeds along the lines of the proof of Theorem 5.7.1 with an obvious change of the replacement argument at boundary points involving the first estimate of Thm. 5.6.1. The general case now follows by approximation arguments as in 5.6.

q.e.d.

5.8 SOME REMARKS

We want to indicate briefly which of the results of this chapter can be generalized to higher dimensions.

Prop. 5.1.1 was extended to arbitrary dimensions by Wood [W2], Karcher-Wood [KW], and Schoen-Uhlenbeck [SU2]. This result can be used to prove complete boundary regularity of energy minimizing maps, cf. [SU2] and [JM].

As was observed by Morrey (cf. [ES]), the minimum of energy is attained in no nontrovial homotopy class for maps from S^n onto itself, if $n \geq 3$.

It is not known whether Prop. 5.1.3 can be generalized, i.e. whether for example there is a harmonic map of degree 1 from the three-dimensional torus onto S^3 or not.

As already pointed out the existence question becomes quite different in higher dimensions, and thus it is not likely that Thm. 5.3.1 can be fully generalized. For known existence results beyond those of chapters 3 and 4, see [SU1], [SU2], [E], [J6]. An interesting non-existence result was derived by Baldes [Ba].

Thm. 5.7.1 fails in higher dimensions; even Cor. 5.7.1 does not extend, as was pointed out by Eells-Lemaire in [EL2], based on a result of Calabi [Ca].

There are, however, some interesting results about harmonic diffeomorphisms between certain classes of Kähler manifolds, cf. [Si] and [JY].

For a more complete guide to the literature on harmonic maps, we refer to the excellent survey articles by Eells and Lemaire [EL1-4]).