# CHAPTER 1

# INTRODUCTION

## 1.1 A SHORT HISTORY OF VARIATIONAL PRINCIPLES

Among the first persons to realize the importance of variational problems and the physical significance of their solutions was G.W. Leibniz (1646-1716). In his work, however, mathematical and physical reasoning was closely interwoven with philosophical and theological arguments. One of the aims of his philosophy was to solve the problem of theodizee, i.e. to reconcile the evil in the world with God's goodness and almightiness (cf. (Lz]). Leibniz' answer was that God has chosen from the innumerable possible worlds the best possible, but that a perfect world is not possible. (This infinite multitude can only be conceived by an infinite understanding, which provided a proof of the existence of God for Leibniz.) This best possible world is distinguished by a pre-established harmony between itself, the realm of nature, on one hand and the heavenly realm of grace and freedom on the other hand. Through this the effective causes unite with the purposive causes. Thus bodies move due to their own internal laws in accordance with the thoughts and desires of the soul. In this way, the contradiction between the predetermination of the physical world following strict laws and the constantly experienced spontaneity and freedom of the individual is removed. The best possible world must here obey specific laws since an ordered world is better than a chaotic one. This proves therefore the necessity of the existence of natural laws. The contents of the natural laws, however, are not completely determined as is the case for geometric laws but are only determined in a moral sense, since they must satisfy the criteria of beauty and simplicity in the best of all possible worlds. This leads Leibniz even to variational principles. This is because if a physical process did not yield an extreme value, a maximum or minimum, for a particular energy or action integral, the world could be improved and would therefore not be the best possible one. Conversely, Leibniz also uses the beauty and simplicity of natural laws as evidence for his thesis of pre-established harmony. (The notion that we live in the best possible world was frequently rejected and even ridiculed by subsequent critics, in particular Voltaire, on account of the apparent flaws of this world, but Leibniz' point that a perfectly good world is not possible was beyond reach of these arguments.)

Leibniz, however, did not elaborate his argument concerning variational principles in his publications, but only in a private letter. Thus, it happened that a principle of least (and not only stationary) action was later rediscovered by Maupertuis (1698-1759), without knowing of Leibniz' idea. When S. König (1712-1757) then claimed priority for Leibniz on account of his letter that he was not able to show however to the Prussian Academy of Sciences (whose president was Maupertuis) this led to one of the most famous priority controversies in scientific history in which even Voltaire, Euler, and Frederick the Great became involved. It was also pointed out that Maupertuis' principle of least action should be replaced by a principle of stationary action since physical equilibria need only be stationary points but not necessarily minima of variational problems.

## 1.2 THE CONCEPT OF GEODESICS

One of the variational problems of most physical importance and mathematical interest was the problem of geodesics, i.e. to find the shortest (or at least locally shortest) connections between two points in a metric continuum, e.g. a Riemannian manifold. Geodesics are critical points of the length integral

$$\int_0^1 \left|\frac{\partial}{\partial t} c\right| dt$$

where $c : [0,1] \to N$ is the parametrization, as well as, if they are parametrized proportionally to arclength, of the energy integral

$$\int_0^1 \left|\frac{\partial}{\partial t} c\right|^2 dt .$$

Here, unfortunately, we find some ambiguity of terminology, since the mathematical term "energy" corresponds to the physical concept of "action", while in physics "energy" has a different meaning.

Because of the many applications of geodesics, it was rather natural to generalize this concept. While minimal surfaces are critical points of a twodimensional analogue of the length integral, namely the area integral, the generalization of the energy integral for maps between Riemannian manifolds led to the concept of harmonic maps. They are critical points of the corresponding integral where the squared norm of the gradient or energy density has to be defined in terms intrinsic to the geometry of the domain and target manifold and the map between them.

## 1.3 DEFINITION AND SOME ELEMENTARY PROPERTIES OF HARMONIC MAPS

Suppose that $X$ and $Y$ are Riemannian manifolds of dimensions $n$ and $N$, resp., with metric tensors $(\gamma_{\alpha\beta})$ and $(g_{ij})$, resp., in some local coordinate charts $x = (x^1,\dots,x^n)$ and $f = (f^1,\dots,f^N)$ on $X$ and $Y$, resp. Let $(\gamma^{\alpha\beta}) = (\gamma_{\alpha\beta})^{-1}$. If $f : X \to Y$ is a $C^1$-map, we can define the energy density

$$e(f) := \frac{1}{2}\gamma^{\alpha\beta}(x)\, g_{ij}(f) \frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^j}{\partial x^\beta}$$

where we use the standard summation convention (greek minuscules occurring twice are summed from 1 to $n$, while latin ones are summed from 1 to $N$) and express everything in terms of local coordinates. Then the energy of $f$ is simply

$$E(f) = \int_X e(f)\,dX \;.$$

If $f$ is of class $C^2$ and $E(f) < \infty$, and $f$ is a critical point of $E$, then it is called *harmonic* and satisfies the corresponding Euler-Lagrange-equations. These are of the form

$$(1.3.1) \qquad \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{\gamma}\, \gamma^{\alpha\beta} \frac{\partial}{\partial x^\beta} f^i\right) + \gamma^{\alpha\beta}\, \Gamma^i_{jk} \frac{\partial}{\partial x^\alpha} f^j \frac{\partial}{\partial x^\beta} f^k = 0$$

in local coordinates, where $\gamma = \det(\gamma_{\alpha\beta})$ and the $\Gamma^i_{jk}$ are the Christoffel symbols of the second kind on $Y$.

(1.3.1) is proved as follows. If $f$ is critical, then for all admissible variations $\phi$ (e.g. $\phi \in C^\infty_c(X)$, and $\phi|\partial X = 0$ if $\partial X \neq \emptyset$)

$$\frac{d}{dt} E(f+t\phi)_{|t=0} = 0 \;.$$

and thus

$$0 = \int_X \left(\gamma^{\alpha\beta}(x) g_{ij}(f(x)) \frac{\partial f^i}{\partial x^\alpha} \frac{\partial \phi^j}{\partial x^\beta} + \frac{1}{2} \gamma^{\alpha\beta} g_{ij,k} \phi^k \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^j}{\partial x^\beta}\right) \sqrt{\gamma}\, dx$$

$$= - \int_X \frac{\partial}{\partial x^\beta} \left(\sqrt{\gamma}\, \gamma^{\alpha\beta} \frac{\partial f^i}{\partial x^\alpha}\right) g_{ij}\, \phi^j\, dx - \int_X \gamma^{\alpha\beta}(x) \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^k}{\partial x^\beta} g_{ij,k} \phi^j \sqrt{\gamma}\, dx$$

$$+ \int_X \frac{1}{2} \gamma^{\alpha\beta}\, g_{ij,k}\, \phi^k \frac{\partial f^i}{\partial x^\beta} \frac{\partial f^j}{\partial x^\alpha} \sqrt{\gamma}\, dx$$

since $\phi$ is compactly supported

and from this, putting $\eta^i = g_{ij}\phi^j$, i.e. $\phi^j = g^{j\ell}\eta^\ell$, and using the symmetry of $\gamma^{\alpha\beta}$ in the second integral,

$$0 = -\int_X \frac{\partial}{\partial x^\beta}\left(\sqrt{\gamma}\,\gamma^{\alpha\beta}\frac{\partial f^i}{\partial x^\alpha}\right)\eta^i\,dx - \int_X \frac{1}{2}\gamma^{\alpha\beta} g^{\ell j}(g_{ij,k}+g_{kj,i}-g_{ik,j})\frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^k}{\partial x^\beta}\eta^\ell\sqrt{\gamma}\,dx$$

which implies (1.3.1) by the lemma of Du Bois-Raymond.

We thus obtain a nonlinear elliptic system of partial differential equations, where the principal part is the Laplace-Beltrami operator on $X$ and is therefore in divergence form, while the nonlinearity is quadratic in the gradient of the solution.

We now want to look at the definition of harmonic maps from a more intrinsic point of view. The differential $df$ of $f$, given in local coordinates by

$$df = \frac{\partial f^i}{\partial x^\alpha}\,dx^\alpha\,\frac{\partial}{\partial f^i}$$

can be considered as a section of the bundle $T^*X \otimes f^{-1}TY$. Then

$$e(f) = \frac{1}{2}\gamma^{\alpha\beta}\left\langle \frac{\partial f}{\partial x^\alpha}, \frac{\partial f}{\partial x^\beta}\right\rangle_{f^{-1}TY}$$

$$= \frac{1}{2}\left\langle df, df\right\rangle_{T^*X\otimes f^{-1}TY}$$

i.e. $e(f)$ is the trace of the pullback via $f$ of the metric tensor of $Y$. In particular, $e(f)$ and hence also $E(f)$ are independent of the choice of local coordinates and thus intrinsically defined. $f$ is harmonic, if

(1.3.2) $$\tau(f) = 0\ ,$$

where $\tau(f) = \text{trace}\,\nabla df$, and $\nabla$ here denotes the covariant derivative in the bundle $T^*X \otimes f^{-1}TY$ .

Let us quickly show, why (1.3.1) and (1.3.2) are equivalent (cf. [EL 4 ]).

$$\nabla_{\partial/\partial x^\beta}(df) = \nabla_{\partial/\partial x^\beta}\left(\frac{\partial f^i}{\partial x^\alpha}\, dx^\alpha\, \frac{\partial}{\partial f^i}\right)$$

$$= \frac{\partial}{\partial x^\beta}\left(\frac{\partial f^i}{\partial x^\alpha}\right) dx^\alpha\, \frac{\partial}{\partial f^i} + \left(\nabla^{T^*X}_{\partial/\partial x^\beta}\, dx^\alpha\right) \frac{\partial f^i}{\partial x^\alpha}\, \frac{\partial}{\partial f^i}$$

$$+ \left(\nabla^{f^{-1}TY}_{\partial/\partial x^\beta}\, \frac{\partial}{\partial f^i}\right) \frac{\partial f^i}{\partial x^\alpha}\, dx^\alpha$$

$$= \frac{\partial^2 f^i}{\partial x^\alpha \partial x^\beta}\, dx^\alpha\, \frac{\partial}{\partial f^i} - {}^X\Gamma^\alpha_{\beta\gamma}\, dx^\gamma\, \frac{\partial f^i}{\partial x^\alpha}\, \frac{\partial}{\partial f^i} + {}^Y\Gamma^k_{ij}\, \frac{\partial}{\partial f^k}\, \frac{\partial f^j}{\partial x^\beta}\, \frac{\partial f^i}{\partial x^\alpha}\, dx^\alpha \quad {}^{1)}$$

and thus, since $\tau(f) = \text{trace}\, \nabla df$ ,

$$\tau^k(f) = \gamma^{\alpha\beta}\, \frac{\partial^2 f^k}{\partial x^\alpha \partial x^\beta} - \gamma^{\alpha\beta}\, {}^X\Gamma^\gamma_{\alpha\beta}\, \frac{\partial f^k}{\partial x^\gamma} + \gamma^{\alpha\beta}\, {}^Y\Gamma^k_{ij}\, \frac{\partial f^i}{\partial x^\alpha}\, \frac{\partial f^j}{\partial x^\beta} \quad ,$$

and we see that (1.3.1) and (1.3.2) are equivalent.

From the preceding calculation, we see that the Laplace-Beltrami operator is the contribution of the connection in $T^*X$ , while the connection in $f^{-1}TY$ gives rise to the nonlinear term involving the Christoffel symbols of the image.

With the preceding notations, we can also calculate the Hessian of a harmonic map $f$ for vector fields $v$ , $w$ along $f$ (i.e. $v$ and $w$ are sections of $f^{-1}TY$) . For this purpose, we consider a two-parameter variation $f_{st}$ with

$$v = \frac{\partial f_{st}}{\partial s}\bigg|_{s,t=0} \quad , \quad w = \frac{\partial f_{st}}{\partial t}\bigg|_{s,t=0} \; .$$

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1) Here, we distinguish the Christoffel symbols of $X$ and $Y$ by the superscript $X$ or $Y$ , resp.

We then want to calculate

$$H_f(v,w) := \frac{\partial^2 E(f_{st})}{\partial s \partial t}\Big|_{s,t=0}$$

We have, writing $f$ instead of $f_{st}$, and taking scalar products $\langle \cdot , \cdot \rangle$ in $T^*X \otimes f^{-1}TY$, if not otherwise indicated,

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}\frac{1}{2}\left\langle \frac{\partial f}{\partial x^\alpha} dx^\alpha , \frac{\partial f}{\partial x^\beta} dx^\beta \right\rangle$$

$$= \frac{\partial}{\partial t}\left\langle \nabla_{\partial/\partial s} \frac{\partial f}{\partial x^\alpha} dx^\alpha , \frac{\partial f}{\partial x^\beta} dx^\beta \right\rangle$$

$$= \frac{\partial}{\partial t}\left\langle \nabla^{f^{-1}TY}_{\partial/\partial x^\alpha} \left(\frac{\partial f}{\partial s}\right) dx^\alpha , \frac{\partial f}{\partial x^\beta} dx^\beta \right\rangle$$

$$= \left\langle \nabla_{\partial/\partial t} \nabla^{f^{-1}TY}_{\partial/\partial x^\alpha} \left(\frac{\partial f}{\partial s}\right) dx^\alpha , \frac{\partial f}{\partial x^\beta} dx^\beta \right\rangle$$

$$+ \left\langle \nabla^{f^{-1}TY}_{\partial/\partial x^\alpha} \left(\frac{\partial f}{\partial s}\right) dx^\alpha , \nabla^{f^{-1}TY}_{\partial/\partial x^\beta} \left(\frac{\partial f}{\partial t}\right) dx^\beta \right\rangle$$

$$= \left\langle \nabla^{f^{-1}TY}_{\partial/\partial x^\alpha} \nabla_{\partial/\partial t} \left(\frac{\partial f}{\partial s}\right) dx^\alpha , \frac{\partial f}{\partial x^\beta} dx^\beta \right\rangle$$

$$+ \left\langle R^N \left(\frac{\partial f}{\partial x^\alpha} dx^\alpha , \frac{\partial f}{\partial t}\right) \frac{\partial f}{\partial s} , \frac{\partial f}{\partial x^\beta} dx^\beta \right\rangle$$

$$+ \left\langle \nabla^{f^{-1}TY}_{\partial/\partial x^\alpha} v \, dx^\alpha , \nabla^{f^{-1}TY}_{\partial/\partial x^\beta} w \, dx^\beta \right\rangle .$$

Now

$$\int_X \left\langle \nabla^{f^{-1}TY}_{\partial/\partial x^\alpha} \nabla_{\partial/\partial t} \frac{\partial f}{\partial s} dx^\alpha , \frac{\partial f}{\partial x^\beta} dx^\beta \right\rangle dX$$

$$= \int \frac{\partial}{\partial x^\alpha}\left( \gamma^{\alpha\beta} \left\langle \nabla_{\partial/\partial t} \frac{\partial f}{\partial s} , \frac{\partial f}{\partial x^\beta} \right\rangle_{f^{-1}TY} \sqrt{\gamma} \right) dx^1 \ldots dx^n$$

$$- \int \left\langle \nabla_{\partial/\partial t} \frac{\partial f}{\partial s} \, dx^{\alpha} \, , \quad \nabla_{\partial/\partial x^{\alpha}} \frac{\partial f}{\partial x^{\beta}} \, dx^{\beta} \right\rangle$$

$$= - \int \left\langle \nabla_{\partial/\partial t} \frac{\partial f}{\partial s} \, , \quad \gamma^{\alpha\beta} \, \nabla_{\partial/\partial x^{\alpha}} \frac{\partial f}{\partial x^{\beta}} \right\rangle_{f^{-1} TY}$$

by Stokes' Theorem

$$= 0 \, , \quad \text{since} \quad \gamma^{\alpha\beta} \, \nabla_{\partial/\partial x^{\alpha}} \frac{\partial f}{\partial x^{\beta}} = \text{trace} \;\; \nabla \, df = 0 \, , \quad \text{as} \;\; f \;\; \text{is harmonic.}$$

Thus

$$H_f(v,w) = \int_X \gamma^{\alpha\beta} \left\langle \nabla^{f^{-1}TY}_{\partial/\partial x^{\alpha}} v \, , \quad \nabla^{f^{-1}TY}_{\partial/\partial x^{\beta}} w \right\rangle_{f^{-1} TY}$$

$$- \int_X \gamma^{\alpha\beta} \left\langle R^N \left( \frac{\partial f}{\partial x^{\alpha}} \, , \, v \right) \frac{\partial f}{\partial x^{\beta}} \, , \, w \right\rangle_{f^{-1} TY}$$

$$= \int_X \left\langle \nabla^{f^{-1}TY} v \, , \quad \nabla^{f^{-1}TY} w \right\rangle_{f^{-1} TY}$$

$$- \int_X \text{trace}_X \left\langle R^N(df,v) \; df \, , \, w \right\rangle_{f^{-1} TY} \, .$$

For the preceding calculations cf. also [EL4].

We now want to look at the definition of harmonic maps from a somewhat different point of view. By the famous embedding theorem of Nash ([Na]), $Y$ can be isometrically embedded in some Euclidean space $\mathbb{R}^{\ell}$. We define the Sobolev space

$$W^1_2(X,Y) = \{ f \in W^1_2(X,\mathbb{R}^{\ell}) : f(x) \in Y \quad \text{a.e.} \}$$

Since $W^1_2(X,\mathbb{R}^{\ell}) = H^1_2(X,\mathbb{R}^{\ell})$ by a well-known theorem of Meyers and Serrin (cf. [MS], p.52; we can assume $X$ to be a compact manifold (possibly with boundary), since we always can localize the problem in the domain. Namely, if

$f$ is a critical point of $E$ on $X$, then it is also critical on any subdomain) every element in $W_2^1(X,Y)$ can be approximated with respect to the $W_2^1$ norm by smooth mappings, namely from $C^\infty(X,\mathbb{R}^\ell)$, although the corresponding equality $W_2^1(X,Y) = H_2^1(X,Y)$ does not hold in general, cf. [SU2]. In particular, if we compose an element from $W_2^1(X,Y)$ with a smooth mapping, we can apply a chain rule.

In this Sobolev space, we can still define the energy functional by

$$E(f) = \frac{1}{2}\int |df(x)|^2 \, dX(x)$$

and look for critical points of $E$ in $W_2^1(X,Y)$.

Assume that $f \in W_2^1(X,Y)$ is a critical point of $E$ which maps $X$ into a compact part $Y_0$ of $Y$. $Y_0$ has a uniform neighbourhood in $\mathbb{R}^\ell$ on which the projection $\pi$, mapping a point in $\mathbb{R}^\ell$ to the closest point in $Y$, is smooth.

Thus, if $\phi : X \to \mathbb{R}^\ell$ is smooth and $\phi|\partial X = 0$ and $t$ is sufficiently small, $(f+t\phi)(x)$ lies in this neighbourhood for a. a. $x \in X$. Since $f$ is critical

$$0 = \frac{\partial}{\partial t} E(\pi(f+t\phi))|t = 0$$

$$= \int_X \langle D^2\pi(f) \circ \phi \, D_\alpha f \, , \, d\pi(f) \, D_\alpha f \rangle \, dX$$

$$+ \int_X \langle d\pi(f) \, D_\alpha \phi \, , \, d\pi(f) \, D_\alpha f \rangle \, dX$$

applying the chain rule,

where $D_\alpha f = e_\alpha(f)$ and $e_\alpha$ is a moving orthonormal frame on $X$, $\alpha = 1,\ldots n$

$$= \int_X \langle D^2\pi(f) \circ \phi \, D_\alpha f \, , \, d\pi(f) \, D_\alpha f \rangle \, dX$$

$$+ \int_X \langle D_\alpha \phi \, , \, d\pi(f) \, D_\alpha f \rangle \, dX$$

since $\pi$ is a projection

$$= \int_X \langle D^2\pi(f)\cdot\phi\cdot D_\alpha f \ , \ D_\alpha f \rangle \ dX$$

$$+ \int_X \langle D_\alpha\phi \ , \ D_\alpha f \rangle \ dX$$

since $\pi \circ f = f$ and consequently $d\pi \cdot D_\alpha f = D_\alpha f$ by the chain rule. Thus, $f$ is a weak solution of

$$0 = \Delta f - D^2\pi(f)(df,df) \ , \tag{1.3.3}$$

where $\Delta$ is the Laplace-Beltrami operator on $X$ (cf. [SU1] for somewhat different calculations). (1.3.1) and (1.3.3) are equivalent, since they both are the Euler-Lagrange equations of the energy functional $E$ . The point of view leading to (1.3.3) was different, however. Here, the energy was minimized among all maps $u : X \to \mathbb{R}^\ell$ of class $H^1_2 \cap L^\infty(X,\mathbb{R}^\ell)$ satisfying a nonlinear constraint $u(x) \in Y_0$ (for almost all $x \in X$) . Since the Dirichlet integral is lower semicontinuous w.r.t. weak $H^1_2$-convergence we also get

LEMMA 1.3.1 *The energy integral is lower semicontinuous w.r.t. weak* $H^1_2$*-convergence.*

Finally, let $\Sigma_1$ and $\Sigma_2$ be surfaces with conformal metrics

$$\sigma^2 dz\, d\bar{z} \qquad\qquad (z=x+iy)$$

and

$$\rho^2 du\, d\bar{u} \qquad\qquad (u=u^1+iu^2) \ \text{ resp.}$$

For a $C^1$-map $f : \Sigma_1 \to Y$ , the energy is then given by

$$E(f) = \frac{1}{2}\int_{\Sigma_1} g_{ij}(u^i_x u^j_x + u^i_y u^j_y)\,dx\,dy$$

in those coordinates. Hence

LEMMA 1.3.2 *If* $k : \Sigma_0 \to \Sigma_1$ *is a conformal map between surfaces, then*

$$E(f \circ k) = E(f) \ .$$

This means that the energy is conformally invariant.

Moreover, the Laplace-Beltrami operator of $\Sigma_1$ in our coordinates is given by $\frac{1}{4\sigma^2} \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}$ , and (1.3.1) hence takes the form

$$\frac{1}{\sigma^2} u^i_{z\bar{z}} + \frac{1}{\sigma^2} \Gamma^i_{jk} \, u^j_z \, u^k_{\bar{z}} = 0$$

(where $u_z := \frac{1}{2} (\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y})$ , $u_{\bar{z}} := \frac{1}{2} (\frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y})$ ) .

In the case the image is the surface $\Sigma_2$ , this in turn reads as

$$\frac{1}{\sigma^2} u_{z\bar{z}} + \frac{1}{\sigma^2} \frac{2\rho_u}{\rho} u_z \, u_{\bar{z}} = 0 \ . \tag{1.3.4}$$

Thus, the harmonicity of $u$ does not depend on the special metric of $\Sigma_1$ , but only on its conformal structure, since we can simply multiply the equation by $\sigma^2$ . Hence

LEMMA 1.3.3 *Suppose* $u : \Sigma_1 \to Y$ *is harmonic, and* $k : \Sigma_0 \to \Sigma_1$ *is a conformal map between surfaces. Then* $u \circ k$ *is also harmonic. In particular, in two dimensions conformal mappings are harmonic.*

The harmonicity of $u$ does depend, however, on the image metric, unless $u_z \equiv 0$ or $u_{\bar{z}} \equiv 0$ , i.e. $u$ is conformal or anticonformal. (Note that this distinction is only meaningful for oriented surfaces.)

We also note the following

LEMMA 1.3.4 *If* $u : \Sigma_1 \to \Sigma_2$ *is a harmonic map between surfaces, then*

$$\phi = \left\{ |u_x|^2 - |u_y|^2 - 2i \langle u_x, u_y \rangle \right\} dz^2 \qquad (z = x+iy)$$

$$= 4\rho^2 u_z \, \bar{u}_z \, dz^2$$

*is a holomorphic quadratic differential.*

Proof  Multiplying (1.3.4) by the conformal factor $\sigma^2$, we obtain

$$\tilde{\tau}(u) := u_{z\bar{z}} + \frac{2\rho_u}{\rho} u_z u_{\bar{z}} = 0 .$$

Thus,

$$\phi_{\bar{z}} = 2\rho\rho_u u_{\bar{z}} u_z \bar{u}_z + 2\rho\rho_{\bar{u}} \bar{u}_{\bar{z}} u_z \bar{u}_z + \rho^2 u_{z\bar{z}} \bar{u}_z + \rho^2 u_z \bar{u}_{z\bar{z}}$$

$$= \rho^2 (\bar{u}_z \tilde{\tau}(u) + u_z \bar{\tilde{\tau}}(u)) = 0$$

q.e.d.

We also observe, that if $\phi$ is holomorphic then $\tau(u) = 0$ with the possible exception of points where $|\bar{u}_z| = |u_z|$, i.e. where the Jacobian $|u_z|^2 - |u_{\bar{z}}|^2$ vanishes. This was actually used by Gerstenhaber and Rauch [GR] as a definition of harmonic maps between surfaces.

We note moreover, that $\phi$ is just the (2,0) part of the differential form $u^*(4\rho^2(u)\,du d\bar{u})$, i.e. the pull-back of the image metric under $u$.

Finally, of course $\phi \equiv 0$ if and only if $u$ is conformal or anticonformal. Therefore, Lemma 1.3.4, together with the observation that by Liouville's Theorem $\phi \equiv 0$ is the only holomorphic quadratic differential on $S^2$, shows that any harmonic map from $S^2$ is conformal or anticonformal.

## 1.4 MATHEMATICAL PROBLEMS ARISING FROM THE CONCEPT OF HARMONIC MAPS

From 1.3, one sees that new mathematical difficulties arise compared to the case of geodesics. Here, critical points lead to systems of non-linear partial differential equations, while geodesics lead only to systems of ordinary differential equations. The natural space to look for critical points of $E$ is the Sobolev space $W^1_2(X,Y) \cap L^\infty(X,Y)$, since the equations for weak solutions of (1.3.1), namely

$$(1.4.1) \qquad 0 = \int \gamma^{\alpha\beta} \left( \frac{\partial f^i}{\partial x^\alpha} \frac{\partial \phi^i}{\partial x^\beta} - \Gamma^i_{jk} \frac{\partial f^j}{\partial x^\alpha} \frac{\partial f^k}{\partial x^\beta} \phi^i \right) dx$$

make sense only for test functions $\phi \in \mathring{W}^1_2(X, \mathbb{R}^N) \cap L^\infty(X, \mathbb{R}^N)$ .

From an analytical point of view, it is not surprising that the equations (1.3.1) turned out to be rather difficult to handle, since the nonlinearity is quadratic in the gradient of the solution. Such systems may have nonsmooth weak solutions. This phenomenon can even occur in the present situation. Namely, mapping the unit ball $D^n$ of dimensions $n \geqq 3$ onto its boundary via radial projection, can be interpreted as a weakly harmonic map (i.e. a solution of (1.4.1)) $f : D^n \to S^{n-1}$ , cf. [HKW3].

In order to verify this, we first show that $\frac{x}{|x|}$ has finite energy for $n \geqq 3$ .

For $x \in D^n$ , $f(x) = \frac{x}{|x|}$ ,and hence for $x \neq 0$

$$(1.4.2) \qquad \frac{\partial}{\partial x^\alpha} \frac{x}{|x|} = \frac{e_\alpha}{|x|} - \frac{x \cdot x^\alpha}{|x|^3}$$

(here, $e_\alpha$ is a unit vector, and $x = x^\alpha e_\alpha$ )

and

$$(1.4.3) \qquad \left| d \frac{x}{|x|} \right|^2 = \frac{(n-1)}{|x|^2}$$

(1.4.3) clearly implies that $\frac{x}{|x|}$ has finite energy for $n \geq 3$ (and also, that the energy is infinite for $n = 2$ ).
$\frac{x}{|x|}$ is smooth for $x \neq 0$ , and we shall verify now, that $\frac{x}{|x|}$ satisfies equation (1.3.3) for $x \neq 0$ .

We note $\pi(f) = \frac{f}{|f|}$ , and from (1.4.2) thus

$$\frac{\partial}{\partial f^\alpha} \pi(f) = \frac{e_\alpha}{|f|} - \frac{f}{|f|^3} f^\alpha$$

and moreover

$$(1.4.4) \qquad \frac{\partial^2}{\partial f^\alpha \partial f^\beta} \frac{f}{|f|} = -\frac{e_\alpha f^\beta}{|f|^3} - \frac{e_\beta f^\alpha}{|f|^3} - \frac{f\delta_{\alpha\beta}}{|f|^3} + \frac{3ff^\alpha f^\beta}{|f|^5} \ .$$

Since $|f|^2 = 1$ implies $f^\alpha \dfrac{\partial f^\alpha}{\partial x^\gamma} = 0 \quad (\gamma = 1,\ldots n)$ , (1.4.4) yields

$$(1.4.5) \qquad D^2\pi(f)(df,df) = \frac{\partial^2}{\partial f^\alpha \partial f^\beta}\left(\frac{f}{|f|}\right) \frac{\partial f^\alpha}{\partial x^\gamma} \frac{\partial f^\beta}{\partial x^\gamma} = -f\ |df|^2 \ .$$

Hence the equation for a harmonic map from $D^n$ into $S^{n-1}$ is by (1.3.3) and (1.4.5)

$$(1.4.6) \qquad \Delta f + f|df|^2 = 0 \ .$$

$f = \frac{x}{|x|}$ now satisfies this equation, since by (1.4.4)

$$\Delta \frac{x}{|x|} = \frac{-(n-1)x}{|x|^3}$$

and by (1.4.3)

$$\left|d \frac{x}{|x|}\right|^2 \frac{x}{|x|} = \frac{(n-1)x}{|x|^3} \ .$$

The following lemma then implies that $\frac{x}{|x|} : D^n \to S^{n-1}$ indeed is a weak solution of (1.4.1).

LEMMA 1.4.1 *If* $f : X \to Y$ *is a map of finite energy which is smooth and harmonic outside a subset of* $X$ *of capacity zero, then* $f$ *is weakly harmonic on* $X$ .

For simplicity, we shall show this only for $\dim X \geq 3$ and the case where $f$ is not smooth only at one isolated point. This suffices for our application.

We have to show that

$$\int (\gamma^{\alpha\beta} D_\alpha f^i D_\beta \phi^i - \gamma^{\alpha\beta} \Gamma^i_{jk} D_\alpha f^j D_\beta f^k \phi^i) \sqrt{\gamma}\, dx = 0$$

for all $\phi \in H^1_2 \cap L^\infty(X,Y)$ . Let us choose the local coordinates in such a way that $0$ is the singular point of $f$ . We define

$$\eta_m := \begin{cases} \frac{1}{2^{m-1}} \left(\frac{1}{|x|} - 2^{m-1}\right) & \text{if} \quad 2^{-m} \le |x| \le 2^{-m+1} \\ 0 & \text{if} \quad 2^{-m+1} \le |x| \\ 1 & \text{if} \quad |x| \le 2^{-m} \end{cases}$$

Clearly, $\eta_m$ is Lipschitz continuous.

We write

$$\phi = (1-\eta_m)\phi + \eta_m \phi \ .$$

Since $f$ is harmonic for $x \neq 0$ , $f \in H^1_2$ and $\phi \in H^1_2 \cap L^\infty$ , it suffices to show

$$\int \gamma^{\alpha\beta} D_\alpha f^i (D_\beta \eta) \phi^i \sqrt{\gamma}\, dx \to 0 \qquad \text{as} \quad m \to \infty \ . \tag{1.4.7}$$

However,

$$D_\beta \eta_m = \begin{cases} \frac{x^\beta}{|x|^3} \, 2^{1-m} & \text{for} \quad 2^{-m} \le |x| \le 2^{-m+1} \\ 0 & \text{otherwise} \ . \end{cases}$$

Hence

$$|D_\beta \eta_m| \le \frac{2}{|x|} \ ,$$

and (1.4.7) follows from Hölder's inequality, since we assumed $n \ge 3$ .

q.e.d.

It might be worth pointing out that the regularity problem for weakly harmonic maps actually has two inherent nonlinearities, one being the nonlinearity of the equations, i.e. arising from the local geometry of the image, and the other one coming from the fact that in general the target space itself does not have a linear structure, i.e. arising from the global topology of the image.

In these notes, we shall first be concerned with the local regularity problem for solutions of the equations, i.e. the first nonlinearity, in chapters 3 and 4, and then deal with the global topological difficulties only in two dimensions, where the regularity theory is easier.

## 1.5 SOME EXAMPLES OF HARMONIC MAPS

The variational problem for harmonic maps seems to be the most natural such problem one can pose for mappings between manifolds, and hence it is not surprising that many other canonical or natural maps turn out to be harmonic. In the sequel, we shall list some examples:

- isometries of Riemannian manifolds
- harmonic functions on Riemannian manifolds
- geodesics as maps $S^1 \to M$
- minimal immersions and parametric minimal surfaces
- Hopf maps $S^3 \to S^2$, $S^7 \to S^4$, $S^{15} \to S^8$
- conformal maps on two-dimensional domains (cf. Lemma 1.3.3) (in higher dimensions, they are in general not harmonic, however)
- holomorphic maps between Kähler manifolds (Holomorphic maps between arbitrary complex manifolds are in general not harmonic. This is not surprising, since the Kähler condition just means that the metric and the complex structure of the manifold agree. The definition of harmonic maps was given in terms of the metric structure, and when deriving the Euler-Lagrange equation for stationary points of the energy integral, we tacitly used the fact that the manifold is endowed with the Levi-Civita connection. Otherwise, as is already the case for geodesics, those two concepts - minimizing the energy or length

integral on one hand and being autoparallel on the other hand for geodesics - would not agree. On the other hand, holomorphic maps are defined in terms of the complex structure, and as mentioned above, the Kähler condition means that the complex connection, i.e. the unique torsionfree connection for which the complex structure is parallel, and the Levi-Civita connection, i.e. the unique torsionfree connection for which the metric is parallel, do agree.)

- Gauss maps of minimal submanifolds of Euclidean space, or more generally, of submanifolds with parallel mean curvature vector. This is a theorem of Ruh and Vilms [RV]. With the help of this theorem, one can prove Bernstein type theorems for minimal submanifolds of Euclidean space by proving Liouville type theorems for harmonic maps, since, if the Gauss map is constant, the submanifold has to be a linear subspace. We shall come back to this point in chapter 4.

## 1.6 SOME APPLICATIONS OF HARMONIC MAPS

We want to calculate for a harmonic map $f$

$$\Delta e(f)$$

i.e.

$$\Delta \frac{1}{2}\gamma^{\alpha\beta}(x)g_{ij}(f(x))f^i_{x^\alpha}f^j_{x^\beta} .$$

In order to do this, it will be convenient to introduce normal coordinates at the points $x$ and $f(x)$, i.e. $\gamma_{\alpha\beta}(x) = \delta_{\alpha\beta}$ and $g_{ij}(f(x)) = \delta_{ij}$ and all Christoffel symbols vanish at $x$ and $f(x)$, so that we only have to take derivatives of the Christoffel symbols into account which will yield curvature terms eventually.

First of all, we write the equation for harmonic maps in the form

$$(1.6.1) \qquad 0 = \gamma^{\alpha\beta}f^i_{x^\alpha x^\beta} - \gamma^{\alpha\beta}\,{}^X\Gamma^\eta_{\alpha\beta}\,f^i_{x^\eta} + \gamma^{\alpha\beta}\,{}^Y\Gamma^i_{k\ell}\,f^k_{x^\alpha}\,f^\ell_{x^\beta}.$$

Differentiating this equation at $x$ w.r.t. $x^\varepsilon$, we obtain

(1.6.2) $$f^i_{x^\alpha x^\alpha x^\varepsilon} = \frac{1}{2}(\gamma_{\alpha\eta,\alpha\varepsilon} + \gamma_{\alpha\eta,\alpha\varepsilon} - \gamma_{\alpha\alpha,\eta\varepsilon}) f^i_{x^\eta}$$

$$- \frac{1}{2}(g_{ki,\ell m} + g_{\ell i,km} - g_{k\ell,im}) f^m_{x^\varepsilon} f^k_{x^\alpha} f^\ell_{x^\alpha} ,$$

using of course that by our choice of coordinates all first derivatives of the metric tensors vanish, and the Christoffel symbols are given by, e.g. $\Gamma^i_{k\ell} = \frac{1}{2} g^{im}(g_{mk,\ell} + g_{m\ell,k} - g_{k\ell,m})$ .

Furthermore, in our coordinates

(1.6.3) $$\gamma^{\alpha\beta}{}_{,\sigma\sigma} = - \gamma_{\alpha\beta,\sigma\sigma}$$

and by the chain rule

(1.6.4) $$\Delta g_{ij}(f(x)) = g_{ij,k\ell} f^k_{x^\sigma} f^\ell_{x^\sigma} .$$

From (1.6.2) - (1.6.4) we obtain

(1.6.5) $$\Delta \frac{1}{2} \gamma^{\alpha\beta}(x) g_{ij}(f(x)) f^i_{x^\alpha} f^j_{x^\beta}$$

$$= f^i_{x^\alpha x^\sigma} f^i_{x^\alpha x^\sigma}$$

$$- (\gamma_{\alpha\beta,\sigma\sigma} + \gamma_{\sigma\sigma,\alpha\beta} - \gamma_{\sigma\alpha,\sigma\beta} - \gamma_{\sigma\alpha,\sigma\beta}) f^i_{x^\alpha} f^i_{x^\beta}$$

$$+ (g_{ij,k\ell} + g_{k\ell,ij} - g_{ik,j\ell} - g_{j\ell,ik}) f^i_{x^\alpha} f^j_{x^\alpha} f^k_{x^\sigma} f^\ell_{x^\sigma}$$

$$= f^i_{x^\alpha x^\sigma} f^i_{x^\alpha x^\sigma} + R^X_{\alpha\beta} f^i_{x^\alpha} f^i_{x^\beta} - R^Y_{ikj\ell} f^i_{x^\alpha} f^j_{x^\alpha} f^k_{x^\sigma} f^\ell_{x^\sigma} ,$$

where $R^X_{\alpha\beta}$ is the Ricci tensor of $X$ and $R^Y_{ikj\ell}$ is the curvature tensor of $Y$ .

In arbitrary coordinates, this formula is of course transformed into

$$\Delta e(f) = g_{ij}(f(x))\gamma^{\alpha\beta}(x)\gamma^{\sigma\eta}(x)\left(f^i_{x^\alpha x^\sigma} + \Gamma^i_{k\ell}\frac{\partial f^k}{\partial x^\alpha}\frac{\partial f^\ell}{\partial x^\sigma}\right)\left(f^j_{x^\beta x^\eta} + \Gamma^j_{mn}\frac{\partial f^m}{\partial x^\beta}\frac{\partial f^n}{\partial x^\eta}\right)$$

$$+ g_{ij}(f(x))R^X_{\alpha\beta}(x)f^i_{x^\alpha}f^j_{x^\beta} - \gamma^{\alpha\beta}(x)\gamma^{\sigma\eta}(x)R^Y_{ikj\ell}(f(x))f^i_{x^\alpha}f^j_{x^\beta}f^k_{x^\sigma}f^\ell_{x^\eta}$$

and in invariant notation, if $e_\alpha$ is an orthonormal frame at $x$,

$$\Delta e(f) = |\nabla df|^2 + \langle df\circ \mathrm{Ric}^X(e_\alpha), df\circ e_\alpha\rangle - \langle R^Y(df\circ e_\alpha, df\circ e_\beta)\, df\circ e_\alpha, df\circ e_\beta\rangle$$

(1.6.5) immediately yields the following

COROLLARY 1.6.1 ([ES]) *Suppose* $f : X \to Y$ *is a harmonic map,* $X$ *is compact,* $\mathrm{Ric}^X \geq 0$ *, and the sectional curvature of* $Y$ *is nonpositive.*

*Then* $f$ *is totally geodesic and has constant energy density. If the Ricci curvature of* $X$ *is positive at one point of* $X$ *at least, then* $f$ *is constant.*

*If the sectional curvature of* $Y$ *is negative, then* $f$ *is either constant or maps* $X$ *onto a closed geodesic of* $Y$ .

Proof Since $\int_X \Delta e(f)\,dX = 0$ , the integral over the right hand side of (1.6.5) has to vanish. Since the integrand is pointwise non-negative by assumption, it has to vanish identically. In particular, $|\nabla df| \equiv 0$ , and thus $f$ is totally geodesic. Furthermore $\Delta e(f) \equiv 0$ , and since harmonic functions on compact manifolds are constant, $e(f) \equiv$ const.

If at $x \in X$ , $R^X_{\alpha\beta}(x)$ is positive definite, then

$$R^X_{\alpha\beta}(x)f^i_{x^\alpha}\, f^i_{x^\beta} = 0$$

implies that at $x$ and hence everywhere $e(f) = 0$ , and $f$ is constant.

If $Y$ has negative sectional curvature, then in the same way we see that

$$\dim(df(T_x X)) \le 1 \qquad \text{for any} \quad x \in X .$$

If the dimension is zero somewhere, then $e(f) = 0$ at this point and hence everywhere. Otherwise, $f$ as a totally geodesic map has to map $X$ onto a closed geodesic.

We now want to apply Cor. 1.6.1 in conjunction with the following basic existence and uniqueness theorem of Eells-Sampson(existence) and Hartman (uniqueness) which will be proved in chapter 3 in order to reprove some well known theorems about nonpositively curved manifolds by using harmonic maps.

THEOREM 1.6.1 *If* $X$ *and* $Y$ *are compact Riemannian manifolds and* $Y$ *has nonpositive sectional curvature, then every homotopy class of maps from* $X$ *to* $Y$ *contains a harmonic map. If the curvature of* $Y$ *is negative, then this harmonic map is unique unless its image is a single point or contained in a closed geodesic in which case every other homotopic harmonic map can differ from the given one only by a rotation of this closed geodesic.*

We first deduce Preissmann's Theorem:

THEOREM 1.6.2 *If* $Y$ *is a compact Riemannian manifold of negative sectional curvature, then every Abelian subgroup of the fundamental group is cyclic.*

Proof Suppose $a$ and $b$ are commuting elements of $\pi_1(Y)$ . The homotopy between $ab$ and $ba$ allows us to construct a map $g$ from the twodimensional torus $T^2$ into $Y$ . By Thm. 1.6.1 $g$ is homotopic to a harmonic map $f : T^2 \to Y$ , and the image of $f$ is contained in a closed geodesic by Cor. 1.6.1. Hence both $a$ and $b$ are homotopic to some multiple of this geodesic.

q.e.d.

Furthermore, we can prove the following consequence of the Hadamard-Cartan theorem.

THEOREM 1.6.3 *If* $Y$ *is a nonpositively curved compact Riemannian manifold, then all homotopy groups* $\pi_m(Y)$ *vanish for* $m \geq 2$, *i.e.* $Y$ *is a* $K(\pi,1)$ *manifold.*

Proof We have to show that every map $g$ from a sphere $S^m$, $m \geq 2$, into $Y$ is homotopic to a constant. By Thm. 1.6.1, $g$ is homotopic to a harmonic map $f : S^m \to Y$, and $f$ is constant by Cor. 1.6.1.

q.e.d.

Finally, we deduce

THEOREM 1.6.4 *If* $Y$ *is a negatively curved Riemannian manifold, then every isometry of* $Y$ *homotopic to the identity coincides with the identity, and the isometry group of* $Y$ *is discrete.*

Proof This follows from the uniqueness part of Thm. 1.6.1, since isometries are harmonic.

q.e.d.

The preceding argument can be generalized to show that the larger the isometry group of a compact manifold is, the more restrictions exist for mappings of this manifold into negatively curved ones, since composing a harmonic map with an isometry again yields a harmonic map. Cf. [SY3] for more details.

While in the preceding part of this section, we have used harmonic maps to reprove some elementary theorems merely for the sake of illustration, we now want to briefly mention some more difficult applications most of which we shall not prove in these notes.

- One can prove rigidity theorems for certain classes of nonpositively curved Kähler manifolds, i.e. that the topological type already determines the complex structure, by showing that a suitable harmonic map is actually a holomorphic diffeomorphism. Such results were obtained by Siu [Si], Jost-Yau [JY], Jost-Mok-Yau.

- One can easily prove many results of Teichmüller theory using harmonic maps, for example that Teichmüller space is contractible or even a cell (details can be found in [EE], [Tr], and [J8].) Also, one can recover the Weil-Petersson metric of Teichmüller space from the second variation formula for harmonic maps.

- One can reduce boundary regularity for the minima of certain quadratic functionals to the nonexistence of nontrivial solutions for a certain Dirichlet problem for harmonic maps, cf. [JM] and [SU2].

- As was pointed out by Eells-Wood [EW], harmonic maps can provide an analytic proof of the Theorem of Kneser, that a continuous map $\phi$ between closed orientable surfaces $\Sigma_1$ and $\Sigma_2$ has to satisfy the inequality

$$|d(\phi)|\ \chi(\Sigma_2) \geq \chi(\Sigma_1)$$

between its degree and the Euler characteristics of $\Sigma_1$ and $\Sigma_2$, in case $\chi(\Sigma_2) < 0$ (cf. chapter 5).

- As we shall show in chapter 4, harmonic maps can be used to prove Bernstein type theorems.

## 1.7 COMPOSITION PROPERTIES OF HARMONIC MAPS

In this section, we shall display an elementary composition property which shall be useful in the sequel. First of all, if $u \in C^2(X,Y)$ is a map between Riemannian manifolds, and $h \in C^2(Y,\mathbb{R})$ is a function, then the following Riemannian chain rule is valid.

$$\Delta(h \circ u) = D^2 h(u_{e_\alpha}, u_{e_\alpha}) + \langle (\text{grad } h)\circ u, \tau(u)\rangle_Y\,, \qquad (1.7.1)$$

where $e^\alpha$ is an orthonormal frame on $X$. In particular, if $u$ is harmonic, i.e. $\tau(u) = 0$, this reads as

$$\Delta(h \circ u) = D^2 h(u_{e^\alpha}, u_{e^\alpha}) \tag{1.7.2}$$

or in local coordinates

$$\Delta(h \circ u) = \gamma^{\alpha\beta} D^2 h(u_{x^\alpha}, u_{x^\beta}) .$$

Thus

LEMMA 1.7.1 *If* $h$ *is a (strictly) convex function on* $Y$ *and* $u$ *is harmonic, then* $h \circ u$ *is a subharmonic function on* $X$.

We note the following consequence (cf. Gordon [Go]).

COROLLARY 1.7.1 *Suppose* $X$ *is a compact manifold, possibly with boundary, and* $u : X \to Y$ *is harmonic. If there exists a strictly convex function on* $u(X)$, *and* $u(\partial X)$ *is constant in case* $\partial X \neq \emptyset$, *then* $u$ *is a constant mapping.*

Proof From the maximum principle for subharmonic functions, it follows that $h \circ u$ is constant, and since $h$ has definite second fundamental form, (1.7.2) implies that $u$ itself is constant.

In section 2.3, we shall see that the assumptions of Cor. 1.7.1 are in particular satisfied, if $u(X)$ is contained in a ball $B(p,M)$ which is disjoint to the cut locus of $p$ and satisfies $M < \frac{\pi}{2\kappa}$, where $\kappa^2$ is an upper curvature bound on this ball, because in this case $d^2(\cdot,p)$ is strictly convex.

Another consequence is

COROLLARY 1.7.2 *Suppose* $X$ *is a compact manifold with* $\pi_1(X) = 0$ *and the* *sectional curvature of* Y *is nonpositive. Then any harmonic map* $u : X \to Y$ *is constant, provided* $u(\partial X)$ *is constant in case* $\partial X \neq 0$ .

Proof By the homotopy lifting theorem, we can lift $u$ to a harmonic map $\tilde{u} : X \to \tilde{Y}$ into the universal covering of $Y$ . The required strictly convex function is then $d^2(\cdot,p)$ , where $p$ is any point in $\tilde{Y}$ .

If instead of a real-valued function, $h$ is a map from $Y$ into some other Riemannian manifold, then instead if (1.7.1) we get

(1.7.3) $$\Delta(h\circ u) = \nabla dh(u_{e_\alpha}, u_{e_\alpha}) + (dh)\circ u \cdot \tau(u) .$$

In particular

LEMMA 1.7.2 *If* $h$ *is totally geodesic and* $u$ *is harmonic, then* $h\circ u$ *is again harmonic.*