THE DIRICHLET PROBLEM FOR A LINEAR ELLIPTIC EQUATION IN A HALF SPACE WITH L^2-BOUNDARY DATA

J. Chabrowski

Let $R_n^+ = \{x \; ; \; x \in R_n \; , \; x_n > 0\}$. We denote point $x \in R_n^+$ by $x = (x', x_n)$, where $x' = (x_1, x_2, \ldots, x_{n-1}) \in R_{n-1}$.

We consider the Dirichlet problem for the elliptic equation of the form

$$(1) \qquad Lu = - \sum_{i,j=1}^{n} D_i(a_{ij}(x) \; D_j u) + \sum_{i=1}^{n} b_i(x) \; D_i u + c(x) \; u = f(x)$$

in R_n^+ . We make the following assumptions about the operator L :

(A) L is uniformly elliptic in R_n^+ , i.e., there exists a positive constant δ such that

$$\delta|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \; \xi_i \; \xi_j$$

for all $x \in R_n^+$ and $\xi \in R_n$, moreover $a_{ij} \in L^\infty(R_n^+)$ $(i,j = 1, \ldots, n)$.

(B) (i) There exist positive constants K and $0 < \alpha < 1$ such that

$$|a_{nn}(x', x_n) - a_{nn}(x', \bar{x}_n)| \leq K|x_n - \bar{x}_n|^\alpha$$

for all $x' \in R_{n-1}$ and all x_n , $\bar{x}_n \in [0,\infty)$.

(ii) $a_{in} \in C^1(R_n^+)$ and $|D_k \; a_{in}(x)| \leq K_1 \; x_n^{-\beta}$ for all $x \in R_{n-1} \times (0,b]$, where K_1 , b and β are positive constants,

$0 \le \beta < 1$, and moreover $D_k a_{in} \in L^\infty(R_{n-1} x[b,\infty))$ $(k,i = 1, \dots, n)$.

(iii) $b_i \in L^\infty(R_n^+)$ $(i = 1, \dots, n)$ and $c \in L^\infty(R_n^+) + L^n(R_n^+)$.

(C) $$\int_{R_n^+} f(x)^2 \min(1, x_n)\, dx < \infty .$$

A function u is said to be a weak solution of the equation (1) if $u \in W^{1,2}_{loc}(R_n^+)$ and u satisfies

$$(2) \quad \int_{R_n^+} \left[\sum_{i,j=1}^n a_{ij}(x)\, D_j u\, D_i v + \sum_{i=1}^n b_i(x)\, D_i u \cdot v + c(x)\, u \cdot v \right] dx = \int_{R_n^+} f(x) v\, dx$$

for every $v \in W^{1,2}(R_n^+)$ with compact support in R_n^+ .

Let $\Phi \in L^2(R_{n-1})$ and assume that there is a function $\Phi_1 \in W^{1,2}(R_n^+)$ such that $\Phi_1(x',0) = \Phi(x')$ on R_{n-1} in the sense of trace. A weak solution in $W^{1,2}(R_n^+)$ of the equation (1) is a solution of the Dirichlet problem with the boundary condition $u(x',0) = \Phi(x')$ on R_{n-1} if $u - \Phi_1 \in W^{1,2}_0(R_n^+)$.

In the above definition it is assumed that the boundary data Φ is a trace of some function belonging to $W^{1,2}(R_n^+)$. This condition is rather restrictive, because not every function in $L^2(R_{n-1})$ is the trace of some function in $W^{1,2}(R_n^+)$. It is clear that the Dirichlet problem with L^2-boundary data requires a new definition.

Theorems 1, 2 and 3 below justify our approach to the Dirichlet problem with L^2-boundary data.

Let $\tilde{W}^{1,2}_{loc}(R_n^+) = \{u \,;\, u \in W^{1,2}_{loc}(R_n^+)$ and $\int_{R_n^+} u(x)^2\, dx < \infty\}$

<u>THEOREM 1</u>. *Let* $u \in \tilde{W}^{1,2}_{loc}(R_n^+)$ *be a solution of* (1) *in* R_n^+ . *Then the following conditions are equivalent:*

(I) *there exists* $T > 0$ *such that*

$$\sup_{0<x_n<T} \int_{R_{n-1}} u(x', x_n)^2 \, dx' < \infty ,$$

(II)
$$\int_{R_n^+} |Du(x)|^2 \min(1, x_n) dx < \infty .$$

Since bounded sets in $L^2(R_{n-1})$ are weakly compact, we deduce from Theorem 1, that if one of the conditions (I) or (II) holds, then there exists a function $\Phi \in L^2(R_{n-1})$ such that

$$\lim_{\delta \to 0} \int_{R_{n-1}} u(x', \delta) \, \Psi(x') \, dx' = \int_{R_{n-1}} \Phi(x') \, \Psi(x') \, dx'$$

for every $\Psi \in L^2(R_{n-1})$. Using the fact that u satisfies (1) one can show that $u(\cdot, \delta) \to \Phi$ in $L^2(R_{n-1})$. Namely, we have

<u>THEOREM 2</u>. *Let* $u \in \tilde{W}^{1,2}_{loc}(R_n^+)$ *be a solution of* (1) . *Suppose that one of the conditions* (I) *or* (II) *holds. Then there exists a function* $\Phi \in L^2(R_{n-1})$ *such that*

$$\lim_{\delta \to 0} \int_{R_{n-1}} \left[u(x', \delta) - \Phi(x') \right]^2 dx' = 0 .$$

Theorem 2 suggests the following definition of the Dirichlet problem.

Let $\Phi \in L^2(R_{n-1})$. A weak solution $u \in \tilde{W}^{1,2}_{loc} (R_n^+)$ of (1) is a solution of the Dirichlet problem with the boundary condition

(3) $$u(x', 0) = \Phi(x') \quad \text{on} \quad R_{n-1}$$

if $$\lim_{\delta \to 0} \int_{R_{n-1}} \left[u(x', \delta) - \Phi(x') \right]^2 dx' = 0 .$$

To establish the existence of a solution of the Dirichlet problem (1) , (3) we need the energy estimate for the equation

(1') $$Lu + \lambda u = f \quad \text{in} \quad R_n^+ ,$$

where λ is a real parameter.

THEOREM 3. *Let* $u \in \tilde{W}^{1,2}_{loc}(R_n^+)$ *be a solution of the Dirichlet problem* (1') , (3) . *Then there exist positive constants* d , λ_0 *and* C , *independent of* u , *such that if* $\lambda \geq \lambda_0$,

(4) $$\int_{R_n^+} |Du(x)|^2 \min(1,x_n)dx + \sup_{0<\delta\leq d} \int_{R_{n-1}} u(x',\delta)^2 dx' + \int_{R_n^+} u(x)^2 \min(1,x_n)dx \leq C\Big[\int_{R_{n-1}} \phi(x')^2 dx' + \int_{R_n^+} f(x)^2 \min(1,x_n)dx \Big] .$$

Using the energy estimate and the result of [1] one can establish the following existence theorems.

THEOREM 4. *Let* $\lambda \geq \lambda_0$. *Assume that* $b_i \in L^n(R_n^+) \cap L^\infty(R_n^+)$ $(i = 1, \ldots, n)$ *and that* $c \in L^{n/2}(R_n^+) \cap L^n(R_n^+) + L^\infty(R_n^+)$. *Then for every* $\Phi \in L^2(R_{n-1})$ *there exists a unique solution of the Dirichlet problem* (1') , (3) *in* $\tilde{W}^{1,2}_{loc}(R_n^+)$.

THEOREM 5. *Suppose that the assumptions of Theorem 4 hold and moreover* $c(x) \geq \text{Const} > 0$ *on* R_n^+ . *Then for every* $\Phi \in L^2(R_{n-1})$ *there exists a unique solution* u *to the Dirichlet problem* (1) , (3) *in* $\tilde{W}^{1,2}_{loc}(R_n^+)$ *satisfying the following estimate*

$$\int_{R_n^+} |Du(x)|^2 \min(1,x_n)dx + \sup_{0<\delta\leq d} \int_{R_{n-1}} u(x',\delta)^2 dx' + \int_{R_n^+} u(x)^2 \min(1,x_n)dx \leq C\Big[\int_{R_{n-1}} \Phi(x')^2 dx' + \int_{R_n^+} f(x)^2 \min(1,x_n)dx \Big] .$$

To establish the existence of a solution of the problem (1) , (3) we have assumed that $c \geq \text{Const} > 0$. If the coefficient c is non-negative one can also construct a solution but belonging to a different function space.

Namely, denote by $D(R_n^+)$ the completion of $C_0^\infty(R_n^+)$ with respect to the norm $\left[\int_{R_n^+} |Du(x)|^2 dx\right]^{\frac{1}{2}}$. By Sobolev's inequality $D(R_n^+) \subset L^{2^*}(R_n^+)$ with $1/2^* = 1/2 - 1/n$ and $D(R_n^+) \quad L^2_{loc}(R_n^+)$.

THEOREM 6. *Suppose that* $f \in L^2(R_n^+)$ *and* $\Phi \in L^2(R_{n-1})$ *and moreover assume that* $b_i \in L^n(R_n^+)$ $(i = 1, \ldots, n)$, $c \in L^{n/2}(R_n^+)$ *and* $c \geq 0$ *on* R_n^+ .

Then there exists a solution u *to the problem* (1) , (3) *belonging to the space* $\tilde{W}^{1,2}_{loc}(R_n^+) + D(R_n^+)$.

Here the condition (3) *is understood in the following sense : for every* $R > 0$.

$$\lim_{\delta\to 0} \int_{|x'|<R} \left[u(x',\delta) - \Phi(x')\right]^2 dx' = 0 .$$

Theorem 6 is a consequence of Theorem 5 and the results of M. Chicco [3] . The full details of this paper will appear in [2] .

<u>REFERENCES</u>

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Department of Mathematics
University of Queensland
St Lucia Qld 4067
AUSTRALIA