BOX MAXIMAL FUNCTIONS*

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The theory of Hardy spaces is closely intertwined with the study of partial differential equations. Properties of analytic and harmonic functions, and temperatures, are key ingredients in proving basic results concerning Hardy spaces. These results in turn establish new principles which can be used in the study of p.d.e.'s. It is also often the case that, in spite of their name, harmonic analysts prefer proofs concerning the Hardy spaces independent of properties of analytic or harmonic functions.

We will illustrate some of these remarks by presenting an elementary proof of an estimate involving box maximal functions. Our strategy will be to view this result as an imbedding inequality of Hardy and Littlewood type and adapt to this setting the techniques introduced by Calderón and Torchinsky [2, Lemma 2.6] and Jawerth and Torchinsky [7]. Our result will extend an estimate which appears in recent work of Chanillo and Wheeden [3], [4] as a step in obtaining weighted Poincaré and Sobolev inequalities.

First some background. While discussing a basic result in the theory of Hardy spaces of several real variables, namely the passage to arbitrary approximate identities in the definition of the $H^p(R^n)$ spaces, C. Fefferman and Stein [6] introduced the following box maximal function. Let $T(x,y) = \{(y,s) \in R^{n+1}_+ : |x-y| \le h \ , \ 0 < s \le h\}$ denote the box over x of height h ; and let f be defined on R^n and u its Poisson integral in the upper half-space. Then set

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* Research conducted at the Australian National University under the sponsorship of a grant from the Centre for Mathematical Analysis.

$$N_{\lambda,r}(f,x) = \sup_{h>0} \frac{\left(\iint_{T(x,h)} t^{\lambda n} |u(y,t)|^r dy \frac{dt}{t} \right)^{1/r}}{\left(\iint_{T(x,h)} t^{\lambda n} dy \frac{dt}{t} \right)^{1/r}} .$$

They proved that if $f \in L^{p_0}(R^n)$ and $1 < p_0 < r$, $r/p_0 = 1 + \lambda$, then the mapping $f \to N_{\lambda,r}f$ is of weak type (p_0,p_0) and of type (p,p) if $p_0 < p \le \infty$.

Muckenhoupt and Wheeden [9] used this result to complete the study initiated by C. Fefferman concerning the weak type behavior of the Littlewood-Paley function g_λ^*. Their work also covered some weighted Hardy spaces $H_w^p(R^n)$, $0 < p \le 1$. More recently Barker [1] and Torchinsky [11] gave a simple proof of all these results using the concept of Carleson measure. In fact the proof given in [11] holds for arbitrary, rather than harmonic, functions $u(y,t)$ defined in the upper half-space and weights in the A_∞ class. In this case one replaces f by the non-tangential maximal function $M(u,x)$ of u, where

$$M(u,x) = \sup_{|x-y| \le t} |u(y,t)| .$$

Chanillo and Wheeden [3], [4] extended these results, removing the assumption that the weights be in A_∞. Their proof also relies on Carleson measures and because of the applications they had in mind (to weighted Poincaré and Sobolev inequalities discussed by Fabes, Kenig and Serapioni [5]) they consider a "local" version as well. To state their result we need some notations. We say that a measure $w(x)dx$ is "doubling of order μ_w", or $w \in D_{\mu_w}$, $\mu_w \ge 1$, if for balls $B(x,s) = \{y \in R^n : |x-y| < s\}$ we have

$$\begin{aligned} w(B(x,h)) &\equiv \int_{B(x,h)} w(y)\,dy \\ &\le c(h/s)^{n\mu_w} w(B(x,s)) , \end{aligned}$$

where $0 < s \leq h$ and c is independent of x, s, h. When the order μ_w is unimportant we say that w is merely doubling, in this case $w(B(x,2s)) \leq cw(B(x,s))$. The result of Chanillo and Wheeden [4, Lemma 2.4] is then: let $v \in D_{\mu_v}(R^n)$, w doubling, $f(y,t)$ be measurable on R_+^{n+1}, $\lambda > 0$, $\varepsilon > 0$ and $0 < p_0 \leq p < \infty$. If

(i) $\quad n\mu_v\delta + \lambda > n\mu_v$, and

(ii) $\quad (s/h)^\lambda[v(B(y,s))/v(B(y,h)]^\delta \leq c[w(B(y,s))/w(B(y,h))]^{p/p_0}$,

$\quad 0 < s \leq h$, $y \in B(x,2h)$,

then

$$\left(\frac{h^{-\delta}}{v(B(x,h))^\delta}\iint_{T(x,h)} t^{\lambda-n-1}v(B(y,t))^\delta|f(y,t)|^p dy\, dt\right)^{1/p}$$

$$\leq c\left(\frac{1}{w(B(x,(1+\varepsilon)h)}\int_{B(x,(1+\varepsilon)h)} (M^h f(y))^{p_0} w(y)\,dy\right)^{1/p_0}$$

where $M^h f(y) = \sup_{|y-z|\leq t\leq h}|f(z,t)|$, is the truncated non-tangential maximal function of f.

It is the purpose of this note to show that for $p_0 < p$ the same conclusion holds under essentially no restrictions on v. Because the "local" result stated above readily follows from a "global" result we consider the case R^n first. Put $v(B(y,s)) = v(y,s)$, etc.

<u>PROPOSITION 1</u> *Assume* v, w *are positive measures in* R^n *with* w *doubling. If* $\lambda > 0$, $\delta > 0$, $p > p_0 > 0$, *and*

$$s^\lambda v(y,s)^\delta \leq kw(y,s)^{p/p_0}$$

then

$$\left(\iint_{R^{n+1}} s^{\lambda} v(y,s)^{\delta} |f(y,s)|^{p} dy \frac{ds}{s^{n+1}}\right)^{1/p}$$

$$\le ck^{1/p}\left(\int_0^{\infty} \xi^{p-1} w(\{Mf > \xi\})^{p/p_0} d\xi\right)^{1/p_0}$$

$$= ck^{1/p}\|Mf\|_{L^{p_0,p}(w)} \le ck^{1/p}\|Mf\|_{L^{p_0}(w)} .$$

<u>Proof</u> Let $\Omega_{\xi} = \{Mf > \xi\}$. It is well known and readily verified, see [1, Lemma 1] for instance, that there exists a pairwise disjoint family of Whitney cubes Q_j^{ξ} with center x_j^{ξ} and sidelength d_j^{ξ} with the property that

(i) $Q_j^{\xi} \subseteq \Omega_{\xi}$, and

(ii) there is a (dimensional) constant D_1 so that

$$\{(y,s) : |f(y,s)| > \xi\} \subseteq \bigcup_j (Q_j^{\xi} \times [0, D_1 d_j^{\xi}]) .$$

Since

$$I = \iint_{R_+^{n+1}} s^{\lambda} v(y,s)^{\delta} |f(y,s)|^{p} dy \frac{ds}{s^{n+1}}$$

$$= p\int_0^{\infty} \xi^{p-1} \iint_{\{|f|>\xi\}} s^{\lambda} v(y,s)^{\delta} dy \frac{ds}{s^{n+1}} \, d\xi$$

$$\equiv p\int_0^{\infty} \xi^{p-1}\phi(\xi)\,d\xi , \quad \text{say} ,$$

it is enough to show that

$$\phi(\xi) \le c\,k\,w(\{M>\xi\})^{p/p_0} .$$

From (ii) above we see that it suffices to estimate terms of the form

$$\phi_j(\xi) = \iint_{Q_j^\xi \times [0, D_1 d_j^\xi]} s^\lambda v(y,s)^\delta dy \frac{ds}{s^{n+1}} ,$$

and then sum over j . Because j and ξ are fixed for the time being we drop them from the notation, so we denote $x_j^\xi = x$, etc. Since there is a constant D_2 such that $Q \subseteq B(x, D_2 d)$, setting $D = D_1 + D_2$ we readily see that $Q \times [0, D_1 d] \subseteq T(x, Dd)$. Thus each $\phi_j(\xi)$ is dominated by

$$v(x,2Dd)^{\delta(1-p_0/p)} \iint_{T(x,Dd)} s^\lambda v(y,s)^{\delta p_0/p} dy \frac{ds}{s^{n+1}}$$

$$\leq v(x,2Dd)^{\delta(1-p_0/p)} k^{p_0/p} \iint_{T(x,Dd)} w(y,s) s^{\lambda(1-p_0/p)} dy \frac{ds}{s^{n+1}}$$

$$= k^{p_0/p} v(x,2Dd)^{\delta(1-p_0/p)} \int_{B(x,2Dd)} w(z) \iint_{T(x,Dd) \cap B(z,s)} s^{\lambda(1-p_0/p)} dy \frac{ds}{s^{n+1}} dz$$

$$\leq c k^{p_0/p} v(x,2Dd)^{\delta(1-p_0/p)} (Dd)^{\lambda(1-p_0/p)} w(x,2Dd)$$

$$\leq c k^{p_0/p} k^{1-p_0/p} w(x,2Dd)^{p/p_0(1-p_0/p)} w(x,2Dd)$$

$$= c k\, w(x,2Dd)^{p/p_0} \leq c k\, w(x,d)^{p/p_0} ,$$

since w is doubling. Thus

$$\phi(\xi) \leq \sum_j \phi_j(\xi) \leq c\ k \sum_j w(x_j^\xi, d_j^\xi)^{p/p_0}$$

$$\leq c\ k \left(\sum_j w(x_j^\xi, d_j^\xi) \right)^{p/p_0} \leq c k\, w(\{M > \xi\})^{p/p_0} ,$$

as we wanted to show. This proves the first inequality in the conclusion. Moreover since $\xi^{p_0} w(\{M > \xi\}) \leq \|Mf\|_{L_w^{p_0}}^{p_0}$, we see that

$$I \le c\,k \int_0^\infty \xi^{p_0-1} w(\{Mf>\xi\})\,d\xi \;\sup_\xi \left(\xi^{p_0} w(\{M>\xi\})\right)^{(p-p_0)/p_0}$$

$$\le c\,k\, \|Mf\|_{L_w^{p_0}}^{p_0} \|Mf\|_{L_w^{p_0}}^{p-p_0} ,$$

which gives the second inequality of the conclusion and completes the proof.

Now we consider the local version of the result. Let

$$M_a^h f(x) = \sup_{|x-y|\le at,\, t\le h} |f(y,t)|$$

denote the truncated non-tangential maximal function of f with opening a. We then have

<u>PROPOSITION 2</u> *Assume* v , w *are positive measures and* w *doubling. If* $\lambda > 0$, $\delta > 0$, $\varepsilon > 0$, $p > p_0$ *and*

$$(s/h)^\lambda (v(y,s)/v(x,h))^\delta \le c(w(y,s)/w(x,h))^{p/p_0} , \quad 0 < s \le h ,$$

then

$$\left(\frac{h^{-\lambda}}{v(x,h)^\delta} \iint_{T(x,h)} t^{\lambda-n-1} |f(y,t)|^p dy\, dt \right)^{1/p}$$

$$\le c \left(\frac{1}{w(x,(1+\varepsilon)h)} \int_{B(x,(1+\varepsilon)h)} (M^h f)^{p_0}(y) w(y)\, dy \right)^{1/p_0} .$$

<u>Proof</u> Let $f_1(y,t) = \chi_{T(x,h)}(y,t) f(y,t)$, $v_1(y) = v(y)/v(x,h)$, $w_1(y) = w(y)/w(x,h)$. The assumption on the measures can be rewritten as $s^\lambda v_1(y,s)^\delta \le k\, w_1(y,s)^{p/p_0}$, where $k = c\,h^\lambda$. Now choose a small enough so that $M_a^h f_1(y) = 0$ unless $|x-y| \le (1+\varepsilon)h$. But Proposition 1 applied to f_1 , v_1 and w_1 and M_a^h

gives

$$\left(\frac{1}{v(x,h)^{\delta}}\iint_{T(x,h)} t^{\lambda-n-1}v(y,t)^{\delta}|f(y,t)|^{p}dy\,dt\right)^{1/p}$$

$$\le c\,h^{\lambda/p}\left(\frac{1}{w(x,h)}\int_{B(x,(1+\varepsilon)h)}(M_a^h f)^{p_0}(y)w(y)\,dy\right)^{1/p_0}$$

$$\le c\,h^{\lambda/p}\left(\frac{1}{w(x,(1+\varepsilon)h)}\int_{B(x,(1+\varepsilon)h)}(M^h f)^{p_0}(y)w(y)\,dy\right)^{1/p_0},$$

where the last inequality follows since w is doubling and a can be assumed to be ≤ 1. This completes the proof.

In addition to the application of Proposition 1 given in [4], we point out another one along the lines of the original Hardy-Littlewood imbedding result, i.e. fractional integrals. Some of the results of [10] concerning fractional integrals, as well as extensions of those results, follow from Proposition 1; see for instance the work of Macías and Segovia [8].

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