THE MOTION OF A WETTING FRONT FOR A GREEN-AMPT MODEL OF INFILTRATION INTO A CRACKED SOIL

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1. INTRODUCTION

The presence of worm holes, root holes or cracks in soils can have an important effect on the vertical transfer of water from the surface (Bevan and Germann [2]). Here, a two-dimensional homogeneous soil containing regularly spaced vertical cracks which are open to the surface and have length a and spacing 2b (Figure 1), is considered. Because of symmetry, we consider only the shaded region bounded by a single crack. Rectangular coordinates (x,z) are chosen with the positive z axis directed downward.

Initially, the moisture content is taken to be uniform. From time $t = 0$ onward, it is assumed that free water is supplied to the surface and that the crack is completely filled with water, the pressure therein being hydrostatic. The front (assumed to be sharp in the Green-Ampt model) separating the wetted from the unwetted region, advances away from the top surface and the crack.

A numerical procedure (similar to that used by Longuet-Higgins and Cokelet [6], and more recently by Davidson [3]) is described which traces the motion of the front by progressively solving an integral equation for the velocity at points on the front at each time step. A detailed account of the modelling assumptions and the physics of the problem is given elsewhere (Davidson [4]).

2. GOVERNING EQUATIONS

In terms of potential ϕ (the sum of the pressure (Ψ) and gravitational potentials), Darcy's law for flow in the wetted region is

(2.1) $\underset{\sim}{u} = -K\nabla\phi$

where $\phi = \Psi - z$ (see e.g. Philip [7]), $\underset{\sim}{u}$ is the flow velocity, and K is the hydraulic conductivity. In the Green-Ampt model, both the pressure Ψ_c just behind the front and K are assumed to be constant.

At any instant, flow within the wetted region is given by the solution of

(2.2) $\nabla^2\phi = 0$ (by continuity)

subject to

(2.3) $\phi = \Psi_c - z$ on the front (C),

(2.4) $\phi = 0$ for $z = 0$ and $x = 0$, $0 \leqslant z \leqslant a$,

(2.5) $\dfrac{\partial\phi}{\partial x} = 0$ for $x = b$ and $x = 0$, $z > a$ (symmetry) .

A point (x,z) on C moves according to

(2.6)
$$\frac{dx}{dt} = -U_N \sin\chi ,$$
$$\frac{dz}{dt} = U_N \cos\chi ,$$

where $U_N = -\dfrac{K}{f}\dfrac{\partial\phi}{\partial N}$ is the normal velocity of the front assuming that the velocity ahead of it is small. Here, N is the normal to C directed from the wetted to the dry region, χ is the angle between N and the positive z axis, and f is the difference in moisture contents behind and ahead of the front.

It is convenient to transform the flow region by considering the following transformation linking the complex variables $\Omega = x + iz$ and $\omega = p + iq$:

$$\Omega = \frac{ib}{\pi}\left(\log\left(\frac{\cos\pi\omega/2d}{\sin\pi/2d} + \left(1 - \frac{\sin^2\pi\omega/2d}{\sin^2\pi/2d}\right)^{\frac{1}{2}}\right) - \log\left(\frac{\cos\pi\omega/2d}{\sin\pi/2d} - \left(1 - \frac{\sin^2\pi\omega/2d}{\sin^2\pi/2d}\right)^{\frac{1}{2}}\right)\right) \tag{2.7}$$

where $\sin\pi/2d = \tanh\pi a/2b$.

In each case, the square root and log are interpreted as that branch of the function obtained by cutting the plane of its argument along the negative real axis. In the ω plane, the transformed flow region is periodic with the shaded region in Figure 2 corresponding to that in Figure 1, and the boundary sections A'B'H'D'E' mapping onto ABHDE. The transformed wetting region and front are denoted by R' and C', respectively.

The flow equation remains

$$\nabla^2\phi = 0 \quad \text{in } R' \tag{2.8}$$

and the transformed boundary conditions are

$$\phi = 0 \quad \text{on } q = 0\ , \tag{2.9}$$

$$\phi = \Psi_c - z(p,q) \quad \text{on } C' \tag{2.10}$$

$$\frac{\partial\phi}{\partial p} = 0 \quad \text{when } p = 0 \text{ or } p = d\ . \tag{2.11}$$

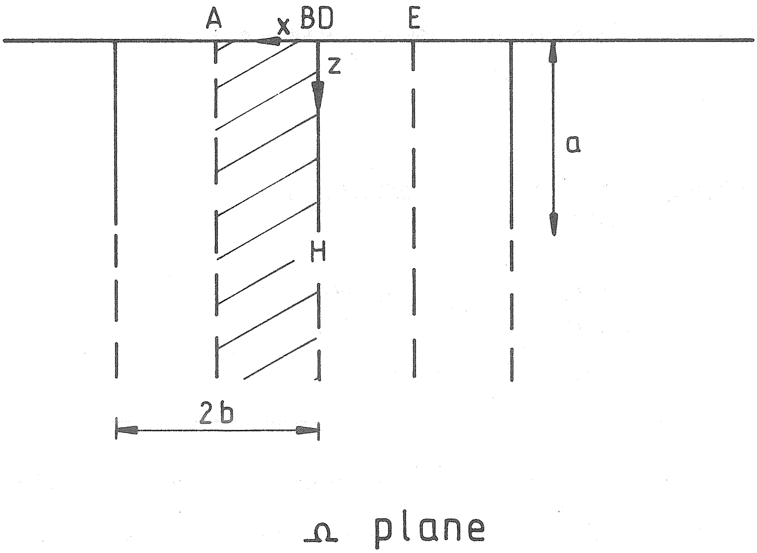

Figure 1. Representation in the physical (Ω) plane of a soil containing regularly spaced vertical cracks of depth a and spacing 2b. The broken lines are lines of symmetry and the width of the shaded region is half a period.

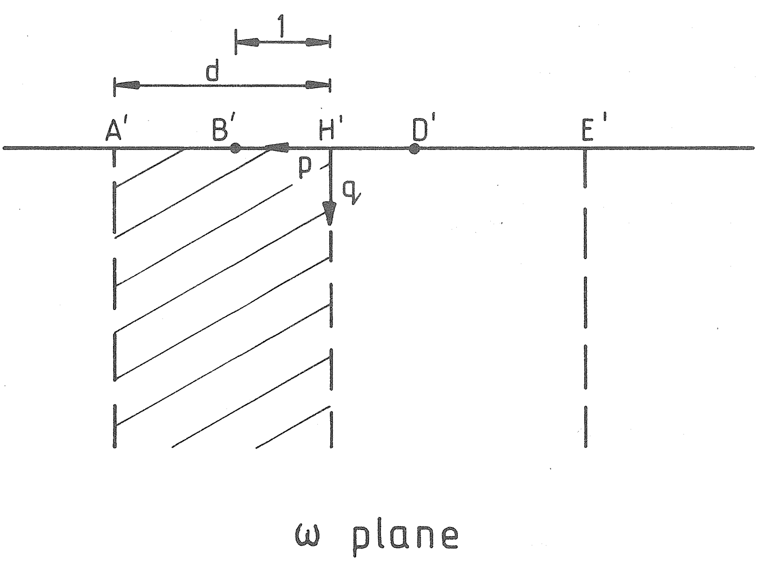

Figure 2. Representation in the transformed (ω) plane of a soil which contains regularly spaced vertical cracks in the physical plane. The broken lines are line of symmetry and the width of the shaded region is half a period.

Equations (2.6) become

$$\frac{dp}{dt} = -U_n \sin\theta \qquad (2.12)$$

$$\frac{dq}{dt} = U_n \cos\theta$$

$$(2.13) \quad \text{where } U_n = \frac{-K}{f}\,\frac{\partial\phi}{\partial n} \Big/ \left|\frac{d\Omega}{d\omega}\right|^2 ,$$

n is the outward normal (from R') to C' and θ denotes the angle between n and the positive q axis. When C' is described by a single valued function q(p,t), the kinematic condition (2.12) may be expressed equivalently as

$$(2.14) \quad \partial q/\partial t = U_n/\cos\theta$$

along lines of constant p.

3. THE INTEGRAL EQUATION

The Green's function

$$G(p,q;\zeta,\eta) = \frac{1}{4\pi}\log\left(\cosh\frac{\pi}{d}(q-\eta)-\cos\frac{\pi}{d}(p+\zeta)\right)\left(\cosh\frac{\pi}{d}(q-\eta)-\cos\frac{\pi}{d}(p-\zeta)\right)$$

$$(3.1) \quad -\frac{1}{4\pi}\log\left(\cosh\frac{\pi}{d}(q+\eta)-\cos\frac{\pi}{d}(p+\zeta)\right)\left(\cosh\frac{\pi}{d}(q+\eta)-\cos\frac{\pi}{d}(p-\zeta)\right)$$

satisfies the homogeneous boundary conditions (2.9) and (2.11). By applying Green's theorem over the region R' together with equations (2.8)-(2.11), Davidson [4] derived the following integral equation for $\partial\phi_1/\partial n$ on C' (where $\phi = \phi_1 + \Psi_c - z$):

$$(3.2) \qquad \int_{C'} G(P;Q) \frac{\partial \phi_1}{\partial n(Q)} \, dS(Q) = z(P) - \frac{qb}{d} - \Psi_c$$

where points P = (p,q) and Q = (ζ, η) both lie on C', and S is arc length. An alternative form, which avoids computational difficulties associated with the singularity in the kernel at P = Q, is

$$\frac{\partial \phi_1}{\partial n(P)} \left(\int_{C'} (\eta - q) \frac{\partial G}{\partial n(Q)} \, dS(Q) - q \right) = \int_{C'} \left(\cos\theta(Q) \frac{\partial \phi_1}{\partial n(P)} - \cos\theta(P) \frac{\partial \phi_1}{\partial n(Q)} \right) G \, dS(Q)$$

$$(3.3) \qquad + \cos\theta(P) \left(z(P) - \frac{qb}{d} - \Psi_c \right) .$$

The integrands in equation (3.3) are now zero when P = Q.

4. TIME STEPPING

Equations (2.14) and (3.3) are solved along equally spaced lines $p = p_j = j\Delta p$ $(j = 0,1,\ldots,N)$. On each line $p = p_j$, the Adams-Bashforth-Moulton scheme (see Hamming [5]) is applied to equation (2.14), to advance through time with local errors of the order $(\Delta t)^5$, where Δt is the time step. Thus,

$$q_* = q_o + \frac{\Delta t}{24} (55 q_o' - 59 q_{-1}' + 37 q_{-2}' - 9 q_{-3}')$$

and
$$q_{**} = q_o + \frac{\Delta t}{24} (9 q_*' + 19 q_o' - 5 q_{-1}' + q_{-2}')$$

where ' denotes $\partial/\partial t$, q_k denotes $q(p_j, t+k\Delta t)$ and q_*, q_{**} denote predicted and corrected values of q_1. A fourth order Runge-Kutta formula is used to start the process.

At each time step, quadrature of the integral in equation (3.3) (in terms of variable p) using Simpson's rule followed by collocation at $p = p_j$ gives a set of simultaneous linear algebraic equations having error of the order $(\Delta p)^4$ for the values of $\partial\phi_1/\partial n$ and hence U_n. In this evaluation, the slopes of C' are required; these are calculated at each time step by cubic spline fitting $q(p_j)$ (Ahlberg et al. [1]).

At time t = 0, the front corresponds with the p axis and its velocity is infinite. Thus we need to begin the numerical calculation from a frontal configuration corresponding to some non-zero time. This point may be derived by approximating the front at small times. On C'

$$\theta \approx 0 \quad \text{and} \quad \frac{\partial\phi}{\partial n} \approx \frac{\partial\phi}{\partial q} \approx \frac{\Psi_c - z}{q}$$

for small t. Thus, from equations (2.13) and (2.14)

$$(4.1) \qquad t = \frac{f}{K} \int_0^{q} \frac{\bar{q}\left|\frac{d\Omega}{d\omega}\right|^2}{z(p,\bar{q})-\Psi_c}\, d\bar{q}$$

which may be solved (Newton iteration) for the displacement q of the front at given p and t. The displacement and hence the error in the approximation increases as p approaches 0 (i.e. the tip of the crack).

From Green's theorem and the boundary conditions, it can be shown that

$$(4.2) \qquad \int_{C'} \left(q\,\frac{\partial\phi}{\partial n} + \cos\theta\,(z-\Psi_c)\right) dS = 0 \quad .$$

This provides a useful check on the accuracy of the calculation at each time step.

At small times, some resolution of the front is lost in the corner region of the Ω plane near the origin, corresponding to the point (1,0) in the ω plane, because $|d\Omega/d\omega|$ is large there; in that case, small distances in the ω plane correspond to much larger ones in the Ω plane. Since an accurate resolution of the corner at small times would require prohibitively small values of the interval spacing Δp, the curve in this region is simply interpolated in the ω plane in preparation for plotting in the physical (Ω) plane.

The starting position for the calculation was taken at time $\frac{Kt}{af} = 10^{-4}$ and the p axis was subdivided 40 times. Reducing the dimensionless starting time to 10^{-5} to improve the initial approximation had a negligible effect on the calculated front in a test case. Similarly, doubling the number of subdivisions of the p axis produced little effect, other than increasing the resolution in the corner.

In the absence of any theory on which to base our choice of time step Δt (ideally it should be the maximum permitted by accuracy and stability considerations), we adopt the practical criterion of requiring that the distance moved by the front along lines of constant p in time Δt be less than Δp. If the distance moved is greater than Δp, then Δt is halved; if for example, it is less than 0.2 Δp, then Δt is doubled.

Results of a typical calculation are shown in Figure 3. The time dependence of cumulative infiltration and infiltration rate can also be calculated; if the model is to be useful, it must be able to predict these integral properties accurately.

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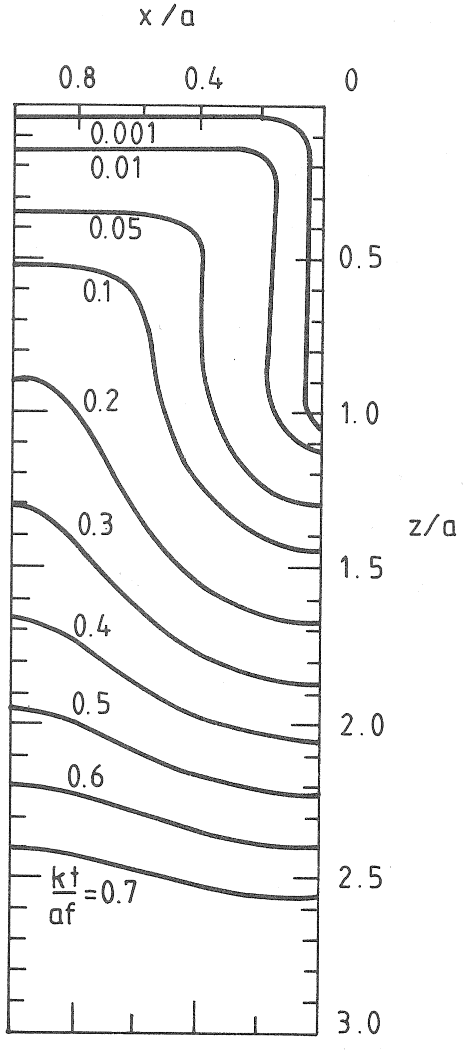

Figure 3. Development of the wetting front with dimensionless time Kt/af when $\Psi_c/a = -1.0$ and $b/a = 1.0$.

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