THE BOUNDARY INTEGRAL METHOD FOR PDE's.

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The last ten years have seen the development of the boundary integral method as an important tool in practical engineering computations The early work in aeronautical fluid flow (see Hess [1975]) elastostatics (see Brebbia [1978], Cruse and Rizzo [1968]) and potential theory (see Jaswon and Symm [1977]) has lead to the sophisticated techniques and varied applications reported more recently in Brebbia, Futagami, and Tanaka [1983], Brebbia, Telles and Wrobel [1983], Liggett and Liu [1983], and Butter et al [1983] . Here we give an introduction to these ideas.

1. POTENTIAL PROBLEMS

The easiest application of the boundary integral method is to the two dimensional problems of classical potential theory (Kellog [1929], Mikhlin [1970]). We are given an open bounded region $\Omega \subset \mathbb{R}^2$ whose boundary Γ is smooth except for a finite number of corners. We need the solution U of the partial differential equation

$$(1) \qquad \Delta U(x) = \frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} = 0 \qquad , \; x = (x_1, x_2) \in \Omega$$

with the boundary conditions

$$(2.1) \qquad U(x) = g(x) \; , \qquad x \in \Gamma_0$$

$$(2.2) \qquad U_\nu(x) = h(x) \; , \qquad x \in \Gamma_1$$

Here for any $x \in \Gamma$ (except a corner point) $\nu(x)$ is the outward normal and $U_\nu(x) = \nabla U(x).\nu(x)$ is the normal derivative. Γ_0 and Γ_1 are the disjoint components of the boundary on which Dirichlet data g and Neumann data h are given. (For the Neumann problem with $\Gamma_1 = \Gamma$ we need $\int_\Gamma h = 0$ and impose the extra condition $\int_\Gamma U = 0$, on the solution). These are the simplest models used in heat transfer and fluid flow calculations.

To apply the boundary integral method the *fundamental solution* (or Green's Function) must be known : in this case

$$G(x,\xi) = \frac{1}{2\pi} \ell n \frac{1}{|x-\xi|} \quad . \tag{3}$$

Using the notation

$$G_\nu(x,\xi) = \nabla_\xi G(x,\xi).\nu(\xi), \qquad \xi \in \Gamma ,$$

Greens third identity states that for any harmonic function U (i.e. any function satisfying (1)),

$$= U(x) \qquad x \in \Omega \tag{4.1}$$

$$\int_\Gamma G(x,\cdot)U_\nu - G_\nu(x,\cdot)U = \tfrac{1}{2}U(x) \qquad x \in \Gamma^0 \tag{4.2}$$

$$= 0 \qquad x \in \mathbb{R}\backslash\overline{\Omega} \tag{4.3}$$

(Kellog [1929]. Γ^0 denotes Γ less the exceptional points. The *exceptional points* are the corners of Γ together with the points on Γ at which Γ_0 and Γ_1 join). However we need a converse to this identity.

If $u,v : \Gamma \to \mathbb{R}$ *satisfy the identity*

$$\int_\Gamma G(x,\cdot)v - G_\nu(x,\cdot)u = \tfrac{1}{2}u(x) \quad , \qquad x \in \Gamma^0 , \tag{5}$$

then the potential U *defined by*

$$U(x) = \int_\Gamma G(x,\cdot)v - G(x,\cdot)u \quad , \qquad x \in \Omega , \tag{6}$$

satisfies

$$\Delta U(x) = 0 \qquad x \in \Omega \tag{7.1}$$

$$U(x) = u \qquad x \in \Gamma^0 \tag{7.2}$$

$$U_\nu(x) = v \qquad x \in \Gamma^0 \quad . \tag{7.2}$$

This can be immediately applied to the Dirichlet problem (when (2) becomes $U(x) = g(x)$, $x \in \Gamma$). Suppose we can find a function v to satisfy

$$\int_\Gamma G(x,\cdot)v = \tfrac{1}{2}g(x) + \int_\Gamma G_\nu(x,\cdot)g , \qquad x \in \Gamma^0 , \tag{8}$$

then the potential

$$U(x) = \int_\Gamma G(x,\cdot)v - G_\nu(x,\cdot)g \quad , \qquad x \in \Omega$$

satisfies

$$\Delta U(x) = 0 , \qquad x \in \Omega ,$$

$$U(x) = g(x) , \; U_\nu(x) = v(x) \qquad x \in \Gamma^0$$

That is, U is the required solution to the original differential equation and as a bonus v is the unknown U_ν (which is often the quantity required). Similarly the integral equation for the Neumann problem is

$$(8.1) \qquad \tfrac{1}{2}u(x) + \int_\Gamma G_\nu(x,\cdot)u = \int_\Gamma G(x,\cdot)h \qquad x \in \Gamma^0$$

The same idea works for the general mixed problem (1) - (2). We need u and v so that

$$(9.1) \qquad \int_\Gamma G(x,\cdot)v - G_\nu(x,\cdot)u = \tfrac{1}{2}u(x), \qquad x \in \Gamma^0$$

and

$$(9.2) \qquad u(x) = g(x) \qquad x \in \Gamma_0$$

$$(9.3) \qquad v(x) = h(x) \qquad x \in \Gamma_1$$

Equivalently using (9.2) and (9.3), (9.1) become the system of integral equations.

$$(10.1) \qquad \int_{\Gamma_0} G(x,\cdot)v - \int_{\Gamma_1} G_\nu(x,\cdot)u = \tfrac{1}{2}g(x) - \int_{\Gamma_1} G(x,\cdot)h + \int_{\Gamma_0} G_\nu(x,\cdot)g, \; x \in \Gamma_0 \cap \Gamma^0$$

$$(10.2) \qquad \int_{\Gamma_0} G(x,\cdot)v - \tfrac{1}{2}u(x) - \int_{\Gamma_1} G_\nu(x,\cdot)u = -\int_{\Gamma_1} G(x,\cdot)h + \int_{\Gamma_0} G_\nu(x,\cdot)g \; , \; x \in \Gamma_1 \cap \Gamma^0$$

with unknowns $v|_{\Gamma_1}$ and $u|_{\Gamma_0}$.

This exposition makes light of some difficulties. The purely mathematical ones are the conditions to be imposed on u and v for (7) to be valid. This must be settled before the converse Greens identity can be proved. A problem of practical concern is that the integral equations may not determine the unknowns uniquely. For example the unknown in (8) is U_ν and hence we expect $\int_\Gamma v = 0$. For most regions Ω this property can be derived from (8) and there is a unique solution to the equation (i.e. the reformulation is well posed.). For special regions, the Γ - contours (see Jaswon and Symm [1977] p.52), this derivation will fail, and (8) may have multiple solutions. However only one of these will satisfy the additional constraint.

(11) $$\int_{\Gamma} v = 0$$

Thus (8) + (11) (or more generally (10) + (11)) will always have a unique solution. In practice numerical solutions of (10) alone may appear to fail. This can be corrected by adding (11), (see Christiansen [1983]) or by rescaling the problem so that Γ is no longer a Γ contour. The second procedure is more convenient but imposing (11) on the numerical solution is a necessary technique for exterior problems, (See Christiansen [1983]) and Neumann problems (8.1).

To be concise the boundary integral equations can be written in the form

(12) $$\rho(x) = Lw(x) - f(x) = 0 \quad ,$$

where w is the unknown, and f is calculated from the given data. We assume that the boundary is parameterized as

$$\Gamma = \{\gamma(s) \quad : \quad 0 \le s \le L\} \quad ,$$

with the function γ smooth except at corners. We occasionally write $w(s)$ for $w(\gamma(s))$.

2. NUMERICAL SOLUTIONS

Integral equation reformulations of potential problems have been known since Green in 1828 (See Elliott [1980] and the references there). They first became important in Fredholm's proof of the existence of solutions for the Dirichlet and Neumann problems. However interest declined when more general proofs were provided using the differential equations themselves. The recent revival has come because the finite element methods developed for differential equations were found to give competative solutions when applied to the integral equations: giving the boundary element method.

To construct the finite elements choose a sequence of mesh points $\{x_i : i=1,\ldots n\}$ $x_i = \gamma(s_i) \in \Gamma$, and let $\Gamma_i = \{\gamma(s) : s_{i-1} < s < s_i\}$ denote the segment of Γ between x_{i-1} and x_i. The set $\{x_i\}$ should include all the exceptional points.

The unknown w is to be approximated by $w_n \in S_n$; where S_n, the set of *basis functions*, is the set of functions ϕ_n satisfying

(1) on each segment Γ_i , $\phi_n(s)$ is a polynomial of degree r , and

(2) if $\nu_i \geq 0$, $\phi_n(s)$ is continuous at s_i, and the tangential derivatives $\phi_n^{(1)}, \ldots \phi_n^{(\nu_i)}$ are continuous.

The degree r and continuities $\{\nu_i\}$ may be varied to give different classes of basis functions. (If $\nu_i = -1$, ϕ_n is allowed to be discontinuous at x_i. It is usual to impose fewer continuity restrictions if x_i is an exceptional point.) As the segments become smaller more accurate approximations become possible. This approximation is usually also used for the given data when calculating the right hand side f.

As the approximation to w we choose the basis function which satisfies the integral equation most nearly. There are two main methods:-

(1) the Galerkin method, where w_n satisfies

$$\int_\Gamma (Lw_n - f)\phi = 0 \quad , \quad \forall\, \phi \in S_n \, . \tag{13}$$

(2) the collocation method, where w_n satisfies

$$(Lw_n - f)(x_{ij}) = 0 \quad , \quad \forall x_{ij} \tag{14}$$

where $x_{ij} \in \Gamma_i$ are collocation points carefully chosen.

Both (13) and (14) reduce to a system of linear equations. This will usually be too large to be solved by the standard LU decomposition methods. A combination of direct and iterative methods may be needed using multi-grid ideas (Atkinson [1976], Hackbusch [1981], Schippers [1984]).

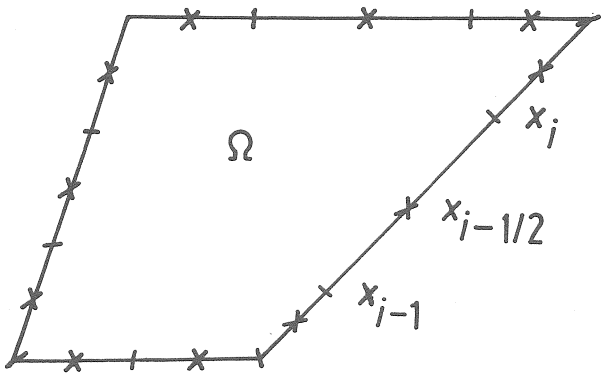

Fig. 1

Returning to the simplest example of the Dirichlet problem (8) , let S_n be the piecewise constant functions and choose the collocation points $x_{i-\frac{1}{2}} = \frac{1}{2}(x_{i-1}+x_i)$ (Fig. 1). Then writing $v_{n,i}$ for $v_n(x_{i-\frac{1}{2}})$, the collocation equations become

$$A[v_{n,i}] = b \quad ;$$

where

$$A_{ij} = \frac{1}{2\pi}\int_{\Gamma_j} \ell n \frac{1}{|x_{i-\frac{1}{2}}-\xi|}\, d\xi \quad ,$$

and

$$b_i = \tfrac{1}{2}g(x_{i-\frac{1}{2}}) - \frac{1}{2\pi}\sum_j \int_{\Gamma_j} \frac{(x_{i-\frac{1}{2}}-\xi).\nu(\xi)}{|x_{i-\frac{1}{2}}-\xi|^2}\, g(x_{j-\frac{1}{2}})\, d\xi \quad .$$

We can now foresee the advantages and disadvantages of the boundary integral equation reformulation, compared to a finite element or difference solution of the original differential equation. The *advantages* are:-

A1) A two dimensional problem has been replaced by a one dimensional problem. This reduces overheads, simplifies programming and mesh generation, and reduces storage.

A2) If the region is intricate (e.g. thin or having reentrant corners) an appropriately graded mesh is easily constructed to give better approximations where the solution is rapidly varying.

A3) Exterior problems are as easy as interior problems; no special techniques are needed.

A4) The physically interesting quantities $U|_\Gamma$ and $U_{\nu|\Gamma}$ are directly available.

A5) ∇U is found more accurately in the interor.

A6) Empirical evidence suggests coarser meshes can be used.

The *disadvantages* are:-

D1) Not all problems have a known fundamental solution; non-homogeneous media or non-linear equations severely complicate any integral equation reformulation.

D2) Repeated evaluation of the potential (6) is expensive if U is required throughout the interior.

D3) It is difficult to solve a problem with source terms. If equation (1) is replaced by the Poisson equation

$$\Delta U(x) = f(x) \quad , \quad x \in \Omega \ ;$$

the solution can be expressed as

$$U = U_0 + F \quad , \quad F(x) = \int_\Omega G(x,\cdot) f \quad . \tag{15}$$

U_0 is found as the solution of

$$\Delta U_0 = 0$$

and

$$U_0(x) = g(x) - F(x) \qquad x \in \Gamma_0$$

$$U_{0,\nu}(x) = h(x) - F_\nu(x) \qquad x \in \Gamma_1 \ .$$

But the multiple integral for F must be repeatedly evaluated (Brebbia et al [1984]).

D4) Setting up the collocation equations requires many integrations, which are often performed numerically (and must be performed numerically in 3 dimensional problems). Because of the singularities in the Green's function this is more difficult than for domain methods. It is unclear just how accurately this need be done.

It is important minimize D3. Whenever non-linear problems are solved the non-linearities are included as a source term. Presently the domain is divided into elements in order to calculate the multiple integrals; which is undesirable. Recently Young *et al* [1983] have applied capacitance matrix techniques to D2 - D3. Capacitance matrix methods (Proskurowski and Windlund [1976]) use the derivation of the boundary integral equations as an analogy to derive fast methods for solving the matrix equations in domain methods. They depend on the use of regular grids, and by themselves cannot cope well with non-smooth boundaries.

Disadvantage D4 is particularly acute if Galerkin methods are used, as these require double integrations

$$\int_\Gamma \int_\Gamma G(x,\xi)\phi_n(x)\psi_n(\xi)\,dx\,d\xi \qquad \phi_n,\psi_n \in S_n$$

to set up the matrix. They do have the advantage of a more extensive theoretical justification, and when basis functions with high continuity are used produce a more favourable distribution of the error within each element. This has lead Lamp *et al* [1982] to a careful investigation of the practical problems involved.

Since Γ is generally curved it will usually be necessary to approximate its parameterization by a more manageable function. Thus $\Gamma = \{\gamma(s):0\le s\le L\}$ is replaced by $\tilde{\Gamma} = \{\tilde{\gamma}(s):0\le s\le L\}$ where the components $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ of $\tilde{\gamma}$ are piecewise polynomials on the mesh $\{s_i\}$. For example, if $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ are continuous piecewise linear functions with $\tilde{\gamma}(s_i) = x_i$, then $\tilde{\Gamma}$ is the polygonal approximation to Γ illustrated in Fig 2. The boundary integral equation

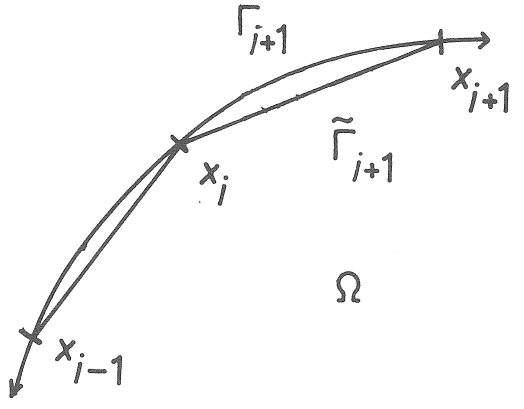

Fig. 2

is solved on $\tilde{\Gamma}$ rather than Γ. The effect of this approximation has been analysed mathematically in Nedelec [1976], McLean [1984], and in the finite element literature (Ciarlet [1977]). Although for some two dimensional problems the exact boundary could be used, the approximate boundary can be specified using only data at the mesh points (in the above example only the coordinates of x_i are required). It is also necessary if free or moving boundary problems are to be solved (Liggett and Liu [1983], Taib *et al* [1984]).

There is little convergence theory to justify the practically successful numerical methods outlined above. If the integral equation is a second kind Fredholm operator (e.g. (8.1)) and the boundary is smooth, the theory is fairly complete (Atkinson [1976]). But if the boundary contains corners or if the equations are of the first kind results are usually available only for Galerkin methods (Wendland [1983] Chandler [1984a,b] and the references given there) or for collocation methods with uniform grids (Atkinson and de Hoog [1984], Wendland [1983], and the references there). Because they are incomplete these results have had no influence in practical computations.

3. ELASTOSTATICS

The main application of the boundary integral method is elastostatics. This section outlines the solution of plane strain elastostatic problems (see Hartman [1982] for a thorough review).

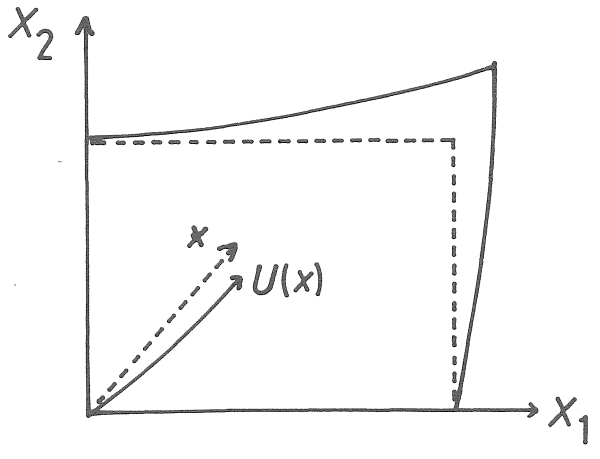

Fig. 3

Let $U(x) = (U_1(x), U_2(x))^T$ denote the displacement of the point x when an elastic body Ω is deformed by external forces (Fig.3). Then U may be sensibly modelled by the Cauchy-Navier equation

$$\Delta U + \frac{1}{1-2\nu}\nabla(\nabla.U) = 0 \quad , \tag{16}$$

where ν is Poisson's ratio for the medium. For $x \in \Gamma$ the *traction* is defined by

$$U_\tau(x) = 2(\nabla U)(x).\nu(x) + \frac{1}{1-2\nu}(\nabla.U)(x)\nu(x) + \nu(x) \times \nabla \times U(x).$$

Physically U_τ is the force per unit length experienced by a small segment of the boundary of Ω due to internal stresses. Mathematically U_τ behaves like a normal derivative. Boundary conditions for (16) are of the form

$$\begin{aligned} U(x) &= g(x) && x \in \Gamma_0 \\ U_\tau(x) &= h(x) && x \in \Gamma_1 \quad , \end{aligned} \tag{17}$$

where g and h are the prescribed (vector valued) data (more general boundary data are also possible). The fundamental solution for this system is the Kelvin solution

$$S(x,\xi) = G(x,\xi)\begin{bmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{bmatrix} + \gamma_2\begin{bmatrix}r_1^2 & r_1r_2\\ r_1r_2 & r_2\end{bmatrix} ,$$

where $r = |x-\xi|$, $r_j = \frac{\partial r}{\partial x_j}$ and

$$\gamma_1 = \frac{3-4\nu}{4(1-\nu)} , \quad \gamma_2 = \frac{1}{8\pi(1-\nu)} ,$$

The role of Green's identity is taken by the Somigliana identity.

$$\int_\Gamma S(x,\cdot)U_\tau - S_\tau(x,\cdot)^T U = \begin{cases} U(x) & x \in \Omega \\ \tfrac{1}{2}U(x) & x \in \Gamma^0 \\ 0 & x \in \mathbb{R}^2\setminus\overline{\Omega} \end{cases} \tag{18}$$

The strategy for converting the identity (18) to an integral equation is the same one used for potential problems. For the example of the assumed displacement problem with boundary conditions $U(x) = g(x)$, $x \in \Gamma$, the integral equation to be solved is

$$\int_\Gamma S(x,\cdot)v = \tfrac{1}{2}g(x) + \int_\Gamma S_\tau(x,\cdot)^T g \qquad x \in \Gamma^0 \tag{19}$$

The solution to the differential equation is then

$$U(x) = \int_\Gamma S(x,\cdot)v - \int_\Gamma S_\tau(x,\cdot)^T g \qquad x \in \Omega$$

The numerical solution of (19) is in principle no different than in the case of potential theory; the unknown is a vector valued function and its two components will be approximated by finite element functions. The additional difficulty comes from S_τ.

For

$$S_\tau(x,\xi)^T = G_\nu(x,\xi)M(x,\xi) + \frac{1}{r}r_\sigma\begin{bmatrix}0 & \gamma_3\\ -\gamma_S & 0\end{bmatrix} ;$$

where r_σ is the tangential derivative

$$r_\sigma = r_\sigma(x,\xi) = \nabla_\xi r(x,\xi).\sigma(\xi)$$

and $\sigma(\xi)$ is the unit tangent vector at ξ , where $\gamma_3 = (1-2\nu)/4\pi(1-\nu)$, and where M is the matrix

$$M(x,\xi) = \frac{1}{2(1-\nu)} \begin{bmatrix} (1-2\nu)+2\ r_1^{\ 2} & 2r_1r_2 \\ 2r_1r_2 & (1-2\nu)+2r_2^2 \end{bmatrix}$$

(r_j now denotes $r_j = \frac{\partial r}{\partial \xi_j}$), If the curve Γ is smooth, then G_ν and M are also smooth functions of x and ξ. But using the parameterization $\Gamma = \{\gamma(S)\}$ of Γ

$$\frac{1}{r}r_\sigma(\gamma(S),\gamma(6)) = \frac{1}{S-\sigma}\left[|\dot{\gamma}(S)|+0(|S-\sigma|)\right],$$

and hence the off diagonal terms of $S_\tau(x,\xi)$ are Cauchy principal value kernels. Thus in equation (18) the right hand side is now more difficult to calculate than in the analagous equation (8) of potential theory. Moreover, when the assumed traction problem is reformulated (i.e. when the boundary conditions are $U_\tau(x) = h(x), x\epsilon\Gamma^0$) we obtain a second kind equation with a Cauchy principal value kernel

$$(20) \qquad \tfrac{1}{2}u(x) + \int_\Gamma S_\tau(x,\cdot)^T u = \int_\Gamma S(x,\cdot)h \qquad x \in \Gamma^0 \quad .$$

(The analagous Neuman problem of potential theory produced a second kind equation with a smooth kernel.) Again this difficulty restricts the convergence results available (See Wendland [1983], Elliott [1983]).

4. REFINED NUMERICAL METHODS

The methods outlined in section 2 have been applied successfully to a range of elastostatic problems; for example the codes of Brebbia [1978] and their subsequent development are based on polygonal approximations to the boundary and piecewise constant approximation of the unknowns. But the stresses (i.e. gradients) in elastostatic problems are known to be singular near reentrant corners and cracks, and excessive grid refinement is needed to maintain accuracy. However the precise form of the singularities is known. By specifically including them amongst the basis functions, the numerical approximations are able to model more closely the true solution. This approach is well known in domain finite element methods (Strang and Fix [1973]). In solutions to boundary integral equations it has appeared in Jaswon and Symm [1977], Fairweather *et al*

[1979], Lamp *et al* [1982], and in the free surface problems of Liggett and Liu [1983]. Here we describe work of Watson [1982], which aims to combine the addition of singularities with high order piecewise polynomial representation in developing general software.

The basis functions are the Hermite cubic splines, the continuous piecewise cubic functions with continuous derivatives at the mesh points. Given the mesh $x_i = \gamma(s_i)$ on $\Gamma = \{\gamma(s)\}$, the approximate boundary $\tilde{\Gamma} = \{(\tilde{\gamma}_1(s), \tilde{\gamma}_2(s))\}$ is given by Hermite cubics $\tilde{\gamma}_k$ (which are however allowed to have discontinuous derivatives at corners) satisfying $\gamma(s_i) = \tilde{\gamma}(s_i)$ and $\dot{\gamma}(s_i) = \dot{\tilde{\gamma}}(s_i)$ (or $\dot{\gamma}(s_i\pm) = \dot{\tilde{\gamma}}(s_i\pm)$ if s_i is a corner). For smooth Γ the components of boundary functions u and v are also approximated by S_n For the assumed displacement problem (equation (19)) the known data g is approximated by $g_n \in S_n$ satisfying $\forall i$

$$g_n(s_i) = g(s_i) \quad \text{and} \quad \frac{\partial g_n}{\partial s}(s_i) = \frac{\partial g_n}{\partial s}(s_i) \quad , \tag{21}$$

To specify the approximation to the unknown v we impose conditions analagous to (21) on the residual. That is we require

$$\rho_n(s_i) = 0 \quad \text{and} \quad \frac{\partial \rho_n}{\partial s}(s_i) = 0 \quad , \tag{22}$$

for the residual vector

$$\rho_n(x) = \int_{\tilde{\Gamma}} S(x,\cdot) v_n - \tfrac{1}{2} g_n(x) + \int_{\tilde{\Gamma}} S_i(x,\cdot)^T g_n \quad .$$

The calculation of the tangential derivatives of ρ in (22) will require the evaluation of Hadamard principal value integrals when the off diagonal functions of S_τ are differentiated. But since g_n itself has continuous derivatives, this may be done unambiguously.

The equations (22) are an extension of the collocation equations (14). Indeed, this particular collocation method has been provided with a complete convergence theory in Arnold and Wendland [1983]: for the equations (22) may be shown to be equivalent to a Galerkin method for which a full analysis is available.

Suppose that the boundary is now allowed to contain corners. For example suppose that near $x_i = \gamma(s_i)$ the boundary is a corner with an interior angle of $2\chi\pi$, $\chi > \frac{1}{2}$ (Fig.4).

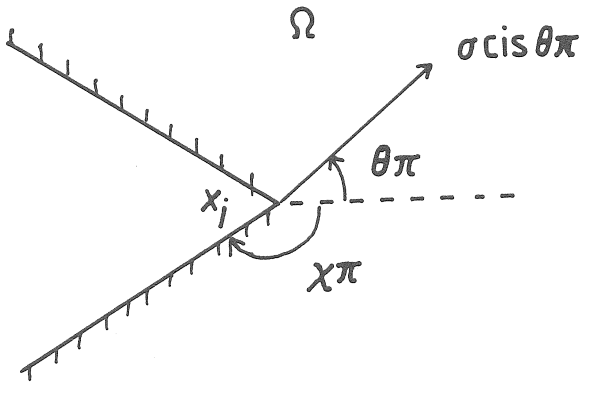

Fig. 4

Then establishing polar coordinates about x_i, $x - x_i = \sigma \operatorname{cis} \theta\pi$ so that Ω is the region

$$\sigma \geq 0, \quad -\chi \leq \theta \leq \chi \quad ;$$

it is known from Williams [1952] that in the neighbourhood of x_i

(23) $$U(\sigma \operatorname{cis} \theta\pi) = K_1\sigma^{\lambda_1}\Phi_1(\theta) + K_2\sigma^{\lambda_2}\Phi_2(\theta) + \text{"smoother terms"}.$$

Here Φ_1 and Φ_2 are respectively symmetric and anti-symmetric functions of θ; $\lambda_1, \lambda_2 \in (\frac{1}{2}, 1)$ are known from χ and the material properties of the medium ; and K_1, K_2 are unknown numbers (the *stress intensity factors*) which will only be known when the solution is found. The"smoother terms" denotes functions whose gradients at x_i are bounded. Therefore the boundary functions are of the form.

(24) $$u(\sigma) = K_1|\sigma|^{\lambda_1}\phi_1 + K_2|\sigma|^{\lambda_2}\phi_2 + \text{"smoother terms"}$$

$$v(\sigma) = K_1|\sigma|^{\lambda_1 - 1}\psi_1 + K_2|\sigma|^{\lambda_2 - 1}\psi_2 + \text{"smoother terms"}$$

where ϕ_k and ψ_k are vectors depending only on the sign of σ and the given data.

But it is easy to see that $|\sigma|^{\lambda_k}$ cannot be approximated well by polynomials. Thus functions of this form are added to the set of basis functions. For example, again in the case of the assumed displacement problem (equation (19)), define for $k = 1,2$

$$\chi_{ik}(s) = |s-s_i|^{\lambda_k}\psi_k - \pi_k(s) \quad , \qquad s \in \Gamma_i \cup \Gamma_{i+1}$$

$$= 0 \quad , \qquad \text{otherwise}$$

where π_k is a piecewise polynomial with $\pi_k(s_i) = \pi_k^{(\nu)}(s_i) = 0$ selected so that $\chi_{ik}(s_j) = \chi_{ik}^{(1)}(s_j) = 0$, $j = i-1$, $i+1$.

The set of basis functions is now taken as $S_n + \{\chi_{i1},\chi_{i2}\}$, where i runs through the exceptional points. At each exceptional point two extra degrees of freedom have been added to the approximation. Hence the collocation equations (22) are modified at exceptional points by imposing additional constraints on the normal derivatives of ρ. (see Watson [1982])

Variants of this idea are possible (see the references already cited), and problems could be expected as the grid is refined. The explicit inclusion of singlarities increases overheads and makes any numerical integrations required to set up the collocation equations more difficult. Nevertheless Watson [1982] reports "engineering accuracy" for a number of test problems with a number of test problems with a small number of boundary elements. This reduces storage and simplifies input. We can also remark that since the collocation matrices for the boundary element method are not sparse, the explicit inclusion of singular functions will not make the linear algebra more difficult.

5. CONCLUSION

The main contribution of mathematics to the boundary integral method is the classical mathematics used in reformulation of the problem. More recently there is the description of the singularities of the solution. Numerical analysis can take credit for the numerical linear algebra and various quadrature techniques; although new methods have been developed (by practitioners) to utilise the special features of the problem. The theoretical analysis of the

solution of the integral equations themselves has had little impact. In this respect the development of boundary integral methods parallels the development of the finite element method.

6. REFERENCES

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