BEST APPROXIMATION OPERATORS IN FUNCTIONAL ANALYSIS

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The purpose of this primarily expository talk is to introduce two classes of (generally non-linear) maps which occur naturally in functional analysis. Both classes consist of retracts from a given Banach space E onto a closed subspace M.

The first class is that of best approximation operators, or closest point maps. There is already an enormous amount known about this subject. Our coverage is necessarily brief, so for further information we refer to [4] and the references therein.

The second class is a generalization of the projections onto complemented subspaces. Since the orthogonal projection onto a subspace of a Hilbert space is also the best approximation operator, these two classes do have something in common.

Given $a \in E$, we let $P(a) = P_M(a)$ be the set of points in M which best approximate a. That is,

$$P(a) = \{x \in M; \|x - a\| = d(a, M)\} = M \cap B(a, d(a, M)) .$$

If $P(a)$ contains exactly (at least/at most) one element, for every

$a \in E$, then M is said to be a Chebyshev (proximinal/unicital) subspace of E. The set-valued map $P; E \to 2^M$ is called the metric projection. If M is proximinal, then a proximity map $\pi: E \to M$ is any selection (choice function) for the metric projection. If M is Chebyshev, there is only one proximity map, determined by $P(a) = \{\pi(a)\}$.

Two famous examples of Chebyshev subspaces are

(i) the subspace of polynomials of degree at most n, in the function space $C[0, 1]$

(ii) any closed subspace of a Hilbert space.

In the first example, the proximity map is continuous, and in the second, it is also linear. Behaviour as decent as this cannot be expected in general. First, we recall some elementary approximation theory [18].

PROPOSITION 1 (i). If M is reflexive, then it is proximinal.

(ii) E is reflexive if and only if every subspace is proximinal

(iii) E is strictly convex if and only if every subspace is unicital

(iv) Suppose E is reflexive, strictly convex and has the Radon-Riesz property (i.e. $x_n \to x$ whenever $x_n \to x$ weakly and $\|x_n\| \to \|x\|$). Then, not only is every subspace M Chebyshev, but also each proximity map is continuous.

PROPOSITION 2. Let $f \in E^*$, with $\|f\| = 1$ and put $M = \ker f$ and $D(f) = \{x \in E: f(x) = \|x\| = 1\}$. Then $P(a) = a - f(a)D(f)$ for every $a \in E$. Thus every proximinal hyperplane admits a linear proximity map.

Now we give some examples, showing how variable the size of $P(a)$ can be. We choose examples which are Banach algebras, just to show that being a Banach algebra is not much help in this context.

Being one-dimensional, $\mathbb{C}1$ is a proximinal subspace in any unital Banach algebra. It is probably part of the folklore that $\mathbb{C}1$ is Chebyshev in $B(E)$, whenever E is a uniformly convex Banach space. To see this, choose $T \in B(E)$ with $d(T, \mathbb{C}1) = 1$, and suppose $\lambda 1, \mu 1 \in P(T)$. Then $\frac{1}{2}(\lambda+\mu) \in P(T)$ and so $\|T - \frac{1}{2}(\lambda+\mu)\| = 1$. Choose a sequence of norm one vectors $x_n \in E$ so that $\|(T - \frac{1}{2}(\lambda+\mu))x_n\| \to 1$. Then

$$\|(T - \lambda)x_n\| \le 1 , \quad \|(T - \mu)x_n\| \le 1$$

and

$$\|(T - \lambda)x_n + (T - \mu)x_n\| \to 2 .$$

Uniform convexity then yields

$$\|(T - \lambda)x_n - (T - \mu)x_n\| \to 0 .$$

Thus $|\lambda - \mu| = 0$.

However, $\mathbb{C}1$ is not Chebyshev, in general.

EXAMPLE 1. There is a unital Banach algebra A with the property that $\mathbb{C}1$ is not a Chebyshev subspace. Moreover, A is commutative, semisimple and has an isometric involution.

PROOF. Take $A = \mathbb{C}^2$, with the usual pointwise operations, and the norm

$$\|(x, y)\| = \tfrac{1}{2}(|x| + |y| + |x - y|) .$$

To see that $\|\cdot\|$ is submultiplicative, note that

$$|ax| + |by| - |ay| - |bx| = (|a| - |b|)(|x| - |y|)$$

$$\leq |a - b|\cdot|x - y| .$$

Hence

$$4\|(ax, by)\| = 2|ax| + 2|by| + |(a - b)(x + y) + (a + b)(x - y)|$$

$$\leq 2|ax| + 2|by| + (|x| + |y|)|a - b| + (|a| + |b|)|x - y|$$

$$= (|ax|+|by|+(|x|+|y|)|a-b|+(|a|+|b|)|x-y|)+(|ax|+|by|)$$

$$\leq (|ax| + |by| + (|x| + |y|)|a - b| + (|a| + |b|)|x - y|)$$

$$+ (|a - b|\cdot|x - y| + |ay| + |bx|)$$

$$= (|a| + |b| + |a - b|)(|x| + |y| + |x - y|)$$

$$= 4\|(a, b)\|\cdot\|(x, y)\|$$

as required.

So A is a Banach algebra under $\|\cdot\|$. Routine calculations show that, for any $x = (a, b) \in A$ we have $d(x, \mathbb{C}1) = \frac{1}{2}|a - b|$, and that $P(x)$ is the line segment joining $a1$ to $b1$. //

A Banach space is said to be strictly convex if every norm one element is an extreme point of the unit ball. In such a Banach space, every finite dimensional subspace is Chebyshev. The next example shows that a unital Banach algebra can have this property. (A simple non-unital example is ℓ_p, for $1 < p < \infty$, under pointwise multiplication.)

<u>EXAMPLE 2</u>. There is a unital Banach algebra in which every finite dimensional subspace is Chebyshev. Once again, the algebra can be constructed to be commutative, semisimple and have an isometric involution.

<u>PROOF</u>. First of all take $A = \mathbb{C}^2$, with the usual operations, but the norm

$$\|(x, y)\| = \left\{\frac{|x|^2 + |y|^2}{2}\right\}^{\frac{1}{2}} + |x - y| .$$

It is easy to see that $\|\cdot\|$ is strictly convex, and that $\|(1, 1)\| = 1$. Submultiplicativity of $\|\cdot\|$ follows from the estimates

$$|ax - by| \leq \tfrac{1}{2}|a - b|(|x| + |y|) + \tfrac{1}{2}|x - y|(|a| + |b|)$$

$$\leq |a - b|\left\{\frac{|x|^2 + |y|^2}{2}\right\}^{\frac{1}{2}} + |x - y|\left\{\frac{|a|^2 + |b|^2}{2}\right\}^{\frac{1}{2}}$$

and

$$\left\{\frac{|ax|^2 + |by|^2}{2}\right\}^{\frac{1}{2}} - \left\{\frac{|a|^2 + |b|^2}{2}\right\}^{\frac{1}{2}}\left\{\frac{|x|^2 + |y|^2}{2}\right\}^{\frac{1}{2}}$$

$$= \frac{\frac{1}{4}(|a|-|b|)(|x|-|y|)(|a|+|b|)(|x|+|y|)}{\left\{\frac{|ax|^2+|by|^2}{2}\right\}^{\frac{1}{2}} + \left\{\frac{|a|^2+|b|^2}{2}\right\}^{\frac{1}{2}}\left\{\frac{|x|^2+|y|^2}{2}\right\}^{\frac{1}{2}}}$$

$$\leq |a - b| \cdot |x - y| \cdot \frac{\frac{1}{2}(|a|+|b|)}{\left\{\frac{|a|^2+|b|^2}{2}\right\}^{\frac{1}{2}}} \frac{\frac{1}{2}(|x|+|y|)}{\left\{\frac{|x|^2+|y|^2}{2}\right\}^{\frac{1}{2}}}$$

$$\leq |a - b| \; |x - y| \quad .$$

More generally, let B be any Banach algebra and give $A(B) = B \oplus B$ pointwise multiplication and the norm

$$\|(x, y)\| = \{\tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|y\|^2\}^{\frac{1}{2}} + \|x - y\| \; .$$

Virtually identical calculations show that $A(B)$ is a Banach algebra under this norm. Clearly the map $x \mapsto (x, x)$ is an isometric isomorphism of B into $A(B)$. If B is commutative/unital/strictly convex, then so is $A(B)$. Now define an increasing sequence $A_0 \subset A_1 \subset A_2 \ldots$ of Banach algebras by $A_0 = \mathbb{C}$, $A_{n+1} = A(A_n)$. Then A_n is strictly convex, and algebraically isomorphic to $\mathbb{C}^{2^n}$. Passing to subalgebras we see that, for every $n \in \mathbb{N}$, $\mathbb{C}^n$ has a strictly convex algebra norm. Finally, the inductive limit of $(A_n)_{n=1}^{\infty}$ is clearly an infinite dimensional, commutative, unital Banach algebra. We have not been able to decide whether it is strictly convex, although it obviously has a strictly convex dense subspace.

Let us remark that in any commutative, unital Banach algebra, every maximal ideal is proximinal. For if $J = \ker(f)$, where f is a character on A, then $\|f\| = f(1) = 1$ and thus $x - f(x)1 \in P(x)$ for every $x \in A$. Our final example shows that this fails if the algebra is not unital.

EXAMPLE 3. There is a commutative semisimple Banach algebra in which every maximal ideal is nonproximinal.

PROOF. Let $\|\cdot\|_\infty$ denote the sup-norm on the Banach algebra $C_0(\mathbb{R})$. We equip the algebra $A = C_0(\mathbb{R}) \cap L_1(\mathbb{R})$ with pointwise multiplication and the norm $\|f\| = \|f\|_\infty + \int_{-\infty}^{\infty}|f(t)|dt$. It is not difficult to show that $\|\cdot\|$ is complete and submultiplicative. Thus A is a commutative Banach algebra. For fixed $t \in \mathbb{R}$, define $\varphi \in A^*$ by $\varphi(f) = f(t)$. Then φ is a character on A, but $|\varphi(f)| < \|f\|$ whenever $f \neq 0$. It follows easily that the ideal $J = \ker(\varphi)$ is not proximinal.

It remains to show that the evaluation functionals are the only characters. Let φ be any character on A. We claim that $|\varphi(f)| \leq \|f\|_\infty$ for all $f \in A$. If not, there is an $f \in A$ with $\|f\|_\infty < 1$ and $\varphi(f) = 1$. If $g(t) = f(t)(1 - f(t))^{-1}$ then $g \in A$ and $g - fg = f$. But then $0 = \varphi(g) - \varphi(g)\varphi(f) = \varphi(f) = 1$, which is absurd.

Since A is a dense subalgebra of $C_0(\mathbb{R})$, φ extends uniquely to a functional $\tilde{\varphi} \in C_0(\mathbb{R})^*$, which must also be a character. But the characters on $C_0(\mathbb{R})$ are precisely the evaluation functionals. //

If M is a finite dimensional Chebyshev subspace, an easy compactness argument shows that its proximity map is automatically continuous. This is not true for infinite dimensional subspaces, as several examples show. Brown [3] constructed one for which E is strictly convex, while M has codimension two and is isometric to a Hilbert space. In spite of Propositions 1 and 2, its proximity map is not continuous.

In particular, Brown's example shows that the Radon-Riesz property cannot be dispensed with in Proposition 1 (iv). Another such example was considered by Deutsch and Lambert [5, section 5]. They show that a certain strictly convex reflexive space has at least one subspace with a discontinuous proximity map, by applying a result of Ošman [17]. They do not explicitly exhibit a subspace with discontinuous proximity map. Both of these examples are rather complicated.

A simple example of a strictly convex reflexive space without the Radon-Riesz property was constructed by Smith [19, example 2]. It would be interesting to know if every proximity map on this space is continuous, although it seems unlikely.

Holmes and Kripke [6, Theorem 3] showed that a proximity map which is continuously differentiable is already linear. What this really says is that continuously differentiable proximity maps are just as rare as linear ones. If every subspace of E is Chebyshev, with linear proximity map, then E is already a Hilbert space [18, Section II.5.1]. The next theorem gives more examples - again in Banach algebras - of non-linear proximity maps. Recall from the remarks preceding Example 1, and the Gelfand-Naimark theorem, that $\mathbb{C}1$ is a Chebyshev subspace in any unital

C^* algebra.

LEMMA 3. Let X be a compact Hausdorff space containing at least three points. Then the proximity map $\pi\colon C(X) \to \mathbb{C}1$ is not Lipschitz continuous.

PROOF. First note that, for each $f \in C(X)$, $\|f - \lambda 1\|$ is a minimum (i.e., $\pi(f) = \lambda 1$) when λ is the centre of the (unique) disc of smallest radius containing $f(X)$. Now fix $n \in \mathbb{N}$, and put $a = -n^3$, $b = n^3 - n^{-1} + in$ and $c = n^3 - n^{-1}$. Define $p\colon \mathbb{C} \to \mathbb{C}$ by

$$p(x + iy) = \min\{x, c\} + iy$$

(for $x, y \in \mathbb{R}$). A routine application of Tietze's theorem shows that there is a function $f \in C(X)$ whose range $f(X)$ contains, and is contained in the convex hull of, the set $\{a, -a, b\}$. Then $f(X)$ is contained in the disc $D(0, |a|)$, and simple plane geometry shows that $\pi(f) = 0$. Next put $g = p \circ f$. Then $\|f - g\| = n^{-1}$, and $g(X)$ contains, and is contained in the convex hull of, $\{a, b, c\}$. It follows that $g(X) \subseteq D(\frac{1}{2}(a + b), \frac{1}{2}|a - b|)$, and so

$$\pi(g) = \tfrac{1}{2}(a + b) = \tfrac{1}{2}(-n^{-1} + in) .$$

For any exponent $\alpha > 0$, we then have

$$\|\pi(f) - \pi(g)\|/\|f - g\|^{\alpha} \geq \tfrac{1}{2}n^{1+\alpha} .$$

Clearly π is not Lipschitz continuous. //

THEOREM 4. If A is any unital C^*-algebra, and $\pi: A \to \mathbb{C}1$ is the proximity map, then the following statements are equivalent.

(i) π is uniformly continuous.

(ii) π is Lipschitz continuous.

(iii) A is isomorphic to either $\mathbb{C}$, $\mathbb{C}^2$ or $M_2(\mathbb{C})$.

(iv) π is linear and contractive.

PROOF. (i) $\Rightarrow$ (ii) By hypothesis, there is a constant $\delta > 0$ such that, for any $x, y \in A$,

$$\|x - y\| \le \delta \Rightarrow \|\pi(x) - \pi(y)\| \le 1 .$$

Since π is homogeneous, it follows that

$$\|\pi(x) - \pi(y)\| \le \delta^{-1}\|x - y\| .$$

(ii) $\Rightarrow$ (iii) Let B be any maximal abelian $*$-subalgebra of A. Then $\pi|B$ is Lipschitz continuous. By Lemma 3, the dimension of B cannot exceed two. The argument of Ogasawara [16, Theorem 1] then tells us that $(\dim A) \le (\dim B)^2 \le 4$. Thus A is isomorphic to either $\mathbb{C}$, $\mathbb{C}^2$ or $M_2(\mathbb{C})$.

(iii) $\Rightarrow$ (iv) This part of the proof is due to A.G. Robertson. There is almost nothing to prove if $A = \mathbb{C}$ or $\mathbb{C}^2$, so suppose that

$A = M_2(\mathbb{C})$, and define $\varphi: A \to A$ by

$$\varphi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} .$$

It follows from the C^*-equality that the norm of any matrix is equal to the norm of its transpose. Thus

$$\left\| \varphi\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \right\|^2$$

$$= \sup\{ |d\lambda - c\mu|^2 + |-b\lambda + a\mu|^2 : |\lambda|^2 + |\mu|^2 \le 1\}$$

$$= \sup\{ |a\mu + b(-\lambda)|^2 + |c\mu + d(-\lambda)|^2 : |\mu|^2 + |-\lambda|^2 \le 1\}$$

$$= \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|^2 ,$$

and so φ is a linear isometry. It then follows that $\frac{1}{2}(I + \varphi)$ is a linear projection of A onto $\mathbb{C}1$, with

$$\|I - \tfrac{1}{2}(I + \varphi)\| \le 1 .$$

The latter condition then forces $\pi = \frac{1}{2}(I + \varphi)$.

(iv) $\Rightarrow$ (i) This is trivial. //

We remark that Stampfli [20] proved that, for all $a, x \in A$,

$$2\|\pi(x) - \pi(a)\| \leq \|x - a\| + \{\|x - a\|^2 + 8\|x - a\|\cdot\|a - \pi(a)\|\}^{\frac{1}{2}} .$$

Thus the proximity map in Theorem 4 is "almost" Lipschitz. If we restrict our attention to self-adjoint elements, the situation is quite different.

PROPOSITION 5. Let $Her(A)$ denote the set of self-adjoint elements of a unital C^*-algebra. Then $\mathbb{R}1$ is a Chebyshev subspace, and the proximity map $\pi: Her(A) \to \mathbb{R}1$ is non-expansive.

PROOF. Let $\sigma(\cdot)$ denote the spectrum of an element of A. It is easy to show that $\pi(a) = \frac{1}{2}(\min \sigma(a) + \max \sigma(a))$ for any $a \in Her(A)$. It follows that if $\pi(a) = 0$ and $\lambda \in \mathbb{R}1$, then $\|a + \lambda\| = \|a\| + |\lambda|$. Thus $\mathbb{R}1$ is a semisummand (defined later in the paper) and the result follows from [13, Corollary 1.13]. //

Next, we consider the case when M is merely a proximinal subspace of E. It is of interest to know whether it admits a continuous proximity map. Nonexistence of a continuous proximity map gives us the useful information that any algorithm for selecting best approximants will be unstable. On the other hand, an existence theorem for continuous selections does not necessarily give a useful algorithm. Even when M is finite-dimensional, it might not admit a continuous proximity map. For example [8, Proposition 2.6] if $E = C[-1, 1]$ and $M = \mathbb{R}f$ is the one-dimensional subspace spanned by $f(t) = t$, then there is no continuous selection for the metric projection onto M.

A variety of conditions, sufficient for the existence of continuous proximity maps, are known. They all depend on Michael's selection theorem, which requires that $P: E \to 2^M$ be lower semicontinuous. This means that, if K is any closed subset of M, then $\{a: P(a) \subseteq K\}$ is a closed subset of E.

Let $H(M)$ denote the family of all closed, bounded, convex, non-empty subsets of M. We make $H(M)$ into a metric space by equipping it with the Hausdorff metric,

$$d(A, B) = \sup(\{d(x, A): x \in B\} \cup \{d(x, B): x \in A\}) .$$

If $P: E \to H(M)$ is continuous with respect to this metric, it is easily shown to be lower semicontinuous.

Since $P(a) = M \cap B(a, d(a,M))$, it is not surprising that most conditions sufficient for M to be proximinal and P lower semicontinuous have been defined in terms of intersecting balls. The most general such property was considered by Lau [7]. He calls M a U-proximinal subspace of E if there is a function $\varepsilon: \mathbb{R}^+ \to \mathbb{R}^+$, with $\varepsilon(\rho) \to 0$ as $\rho \to 0$, such that, if $B = B(0, 1)$ is the unit ball of E, then

$$(1 + \rho)B \cap (B + M) \subseteq B + \varepsilon(\rho)(B \cap M)$$

for all $\rho > 0$.

<u>PROPOSITION 6.</u> [7, theorem 3.4]. If M is a U-proximinal subspace of E, then M is proximinal and the metric projection $P: E \to H(M)$ is continuous.

This generalizes two other results which appeared at about the same time. Following [12, p. 158] let us say that M has property (P) in E if there exist functions $\delta: \mathbb{R}^+ \to \mathbb{R}^+$ and $h: \mathbb{R}^+ \times M \to M$ such that

$$B(0, 1 + \delta(\varepsilon)) \cap B(x, 1) \subseteq B(h(\varepsilon, x), 1)$$

and $\|h(\varepsilon, x)\| \leq \varepsilon$, for all $x \in M$ and $\varepsilon > 0$. From [12, Theorems 2 and 3] it can be seen that any subspace with property (P) is proximinal, with continuous metric projection. But clearly any subspace with property (P) is U-proximinal.

Following [21] we say that M has the $1\frac{1}{2}$-ball property in E if the conditions $M \cap B(a, 1) \neq \phi$ and $\|a\| < 1 + \varepsilon$ always imply that

$$M \cap B(0, \varepsilon) \cap B(a, 1) \neq \phi .$$

As in [21] it can be shown that such a subspace is proximinal, and that $d(P(a), P(b)) \leq 2\|a - b\|$ for all $a, b \in E$. It is easy to show that the $1\frac{1}{2}$-ball property implies U-proximinality.

We remark that these latter two properties are quite different from each other. Any Banach space has the $1\frac{1}{2}$-ball property in itself, but a strictly convex space which is not uniformly convex will not have the property (P) in itself. On the other hand, if E is uniformly convex, then every subspace has property (P) [12, Proposition 1], but no proper subspace has the $1\frac{1}{2}$-ball property [22, p. 301].

Anyway, we can now give a few examples of subspaces which admit continuous proximity maps. Most of these examples can be found in [7], [12] or [21].

PROPOSITION 7. In each of the following cases, M is a U-proximinal subspace of E.

(i) $E = C(X)$, where X is compact, Hausdorff, and M is any closed (self-adjoint) subalgebra

(ii) E is uniformly convex, and M is any subspace

(iii) E is the operator space $B(\ell_p, \ell_q)$, where $p < \infty$, and M is the subspace of compact operators

(iv) $E = B(L_1(\mu), \ell_1)$ where μ is any measure, and $M = K(L_1(\mu), \ell_1)$

(v) $E = B(F, c_0)$ where F is any Banach space, and $M = K(F, c_0)$.

If M is U-proximinal in E, it is not hard to show that, for $a_i \in E \setminus M$,

$$d(a_1 - P(a_1), a_2 - P(a_2)) \le d(a_1 - a_2, M) + \max_{i=1}^{2} d(a_i, M)\,\varepsilon(2d(a_1 - a_2, M)/d(a_i, M)) .$$

It then follows from the methods of [21, section 1] that the proximity map can be chosen to be continuous, homogeneous and quasi-additive. The latter term means that $\pi(x+m) = \pi(x) + m$ whenever $m \in M$. This leads us to consider the promised non-linear generalization of projections.

A retract $\pi\colon E \to M$ is called a semiprojection if it is quasi additive and homogeneous. If there is also a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ for which the identity

$$\|x\| = f(\|\pi(x)\|, \|x - \pi(x)\|)$$

holds, then M is said to be a semisummand in E. If the annihilator M^0 is a semisummand in E^*, then M is said to be a semiideal in E. If M^{00} is a semisummand in E^{**} then M is called a semiidealoid in E. There is no need to extend this series of definitions, since M^0 is a semisummand in E^* whenever M^{000} is a semisummand in E^{***}. These concepts, and the fundamental results concerning them, are due to Mena-Jurado, Payá-Albert and Rodríguez-Palacios [13].

If M is a semisummand whose semiprojection is linear, then M is said to be a summand in E. If M^0 is a summand in E^*, then M is said to be an ideal in E. These concepts were first considered by Alfsen and Effros [1] for the L and M norms on $\mathbb{R}^2$. They called M an L-summand of E if it was a summand, with $f(\alpha, \beta) = \alpha + \beta$. If M^0 was an L-summand of E^*, they called M an M-ideal in E. Their characterization of M-ideals in terms of intersecting balls [1, Theorems 5.8 and 5.9] shows that every M-ideal has the $1\frac{1}{2}$-ball property. Mena-Jurado et. al. [13] showed that if M^{00} is a summand in E^{**}, then M^0 is already a summand in E^*.

If M is a complemented subspace, it is obviously possible to renorm E in such a way that M becomes a semisummand. A typical example of a nonlinear semiprojection is the proximity map from $C_{\mathbb{R}}(X)$ onto the Chebyshev subspace $\mathbb{R}1$. In this case we have $f(\alpha, \beta) = \alpha + \beta$. We do not know of any semisummand which is not a complemented subspace.

It seems plausible that the following conditions could be equivalent.

(A) M^0 is a complemented subspace of E^*

(B) M has the $1\frac{1}{2}$-ball property in E (possibly after renorming E)

(C) M is a semiideal(oid) in E (possibly after renorming E) .

The arguments from [21, section 1] show that (C) implies (A). According to R. Payá-Albert [private communication], (C) implies (B). Concerning the other possible implications, not much seems to be known.

If $P: E \to H(M)$ is continuous, Michael's theorem guarantees the existence of a continuous selection. Suppose P is Lipschitz continuous; must it admit a Lipschitz continuous selection? This is a significant unsolved problem. A positive solution would show that (B) implies (A). For, as we remarked earlier, P is Lipschitz continuous when M has the $1\frac{1}{2}$-ball property. A result of Lindenstrauss [10, Theorem 3 (a)] ensures that M^0 is complemented whenever M is the range of a Lipschitz continuous retract on E . Likewise, an example satisfying (B) but not (A) would show that there can be no Lipschitz version of Michael's selection theorem.

For the reverse implication (does (A) imply (B)?) we have the following partial result. First recall that if $y, y_\alpha \in \ell_1$ and $y_\alpha(n) \to y(n)$ for all n , then $\|y_\alpha\| - \|y - y_\alpha\| \to \|y\|$.

<u>THEOREM 8</u>. If M is isomorphic to c_0 , then E can be renormed so

that M becomes an M-ideal in E.

PROOF. Since E^*/M^0 is isomorphic to ℓ_1, the open mapping theorem permits us to write $E^* = Y \oplus M^0$, where Y is isomorphic to ℓ_1. Let $|\cdot|$ denote the ℓ_1 norm on Y, and $\|\cdot\|$ the original norm on E^*. Since $\{y; \|y\| \le 1\}$ is bounded, its weak* closure is contained in $B(0, r)$, for some $r > 0$. Define a new norm on E^* by

$$|||y + z||| = r|y| + \|z\|, \qquad \text{for } y \in Y \text{ and } z \in M^0.$$

This clearly makes M^0 an L-summand in E^*. It remains to show that $|||\cdot|||$ is induced by some equivalent norm for E.

So let $y_\alpha + z_\alpha \to y + z$, weak*. Since M^0 is weak* closed, we may suppose that z_α converges weak* to some $z_2 \in M^0$. Put $z_1 = z - z_2$. Then $y_\alpha - y$ converges weak* to z_1 and so

$$\|z_1\| \le r \liminf |y_\alpha - y| .$$

Clearly $\|z_2\| \le \liminf \|z_\alpha\|$ and $|y_\alpha| - |y_\alpha - y| \to |y|$. Then

$$\begin{aligned} |||y + z||| &\le r|y| + \|z_1\| + \|z_2\| \\ &\le r(|y| + \liminf |y_\alpha - y|) + \liminf \|z_\alpha\| \\ &= r \liminf |y_\alpha| + \liminf \|z_\alpha\| \\ &\le \liminf |||y_\alpha + z_\alpha||| . \end{aligned}$$

Thus $|||\cdot|||$ is weak* lower semicontinuous, as required. //

The first paragraph of this proof shows that M^0 is complemented in E^*, whenever M is isomorphic to c_0. But every M-ideal has the $1\frac{1}{2}$-ball property. So this theorem establishes a stronger conclusion than we are interested in, under a very strong hypothesis. We show now that the stronger conclusion does not follow from the weaker hypothesis (A).

EXAMPLE 4. If $E = \ell_1$, there exists a subspace M such that

(i) M^0 is complemented in E^*

(ii) no renorming of E can make M into an ideal.

PROOF. Let M be the subspace exhibited by Lindenstrauss [9]. He showed that M^0 is complemented in E^*, but that M is not complemented in any dual space - in particular E.

If some renorming of E makes M into an ideal, then, by [13, Theorem 2.6] we can suppose that M is already an M-ideal. Being uncomplemented, it cannot be an M-summand. But any proper M-ideal contains an isomorphic copy of c_0, according to [2, Corollary 4.5]. This is clearly impossible for a subspace of ℓ_1. //

A similar argument shows that the same conclusions hold when M is the predual of the James-tree space [11] and $E = M^{**}$. It would be interesting to know whether, in these two examples, E can be renormed so as to give M the $1\frac{1}{2}$-ball property.

Finally, a word of warning concerning the implication (A) $\Rightarrow$ (C). If (A) holds, then obviously E^* can be renormed so that M^0 becomes a summand. We cannot conclude (C) from this, since the new norm on E^* might not be a dual norm. Example 4 illustrates this.

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