SOME RECENT RESULTS ON THE EQUATION OF PRESCRIBED GAUSS CURVATURE

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In this article we discuss some recently established results concerning convex solutions $u \in C^2(\Omega)$ of the equation of prescribed Gauss curvature

$$(1) \qquad \det D^2 u = K(x)(1 + |Du|^2)^{(n+2)/2} .$$

Here Ω is a domain in $\mathbb{R}^n$, Du and $D^2 u$ denote the gradient and the Hessian of the function u, and $K(x)$ denotes the Gauss curvature of the graph of u at $(x, u(x))$, which we shall assume is positive in Ω.

We start with a necessary condition for the existence of a convex $C^2(\Omega)$ solution of (1). If u is such a solution, then the gradient mapping $Du : \Omega \to \mathbb{R}^n$ is one to one with Jacobian $\det D^2 u$, so by integrating (1) we obtain

$$\int_\Omega K = \int_\Omega \frac{\det D^2 u}{(1 + |Du|^2)^{(n+2)/2}}$$

$$= \int_{Du(\Omega)} \frac{dp}{(1 + |p|^2)^{(n+2)/2}}$$

$$\leq \int_{\mathbb{R}^n} \frac{dp}{(1 + |p|^2)^{(n+2)/2}}$$

$$= \omega_n ,$$

where ω_n is the measure of the unit ball in $\mathbb{R}^n$. Thus the condition

$$(2) \qquad \int_\Omega K \leq \omega_n$$

is necessary for the existence of a convex solution $u \in C^2(\Omega)$ of (1).

The first problem we consider is the Dirichlet problem for (1), which was recently studied by P.L. Lions [9], [10], Trudinger and Urbas [12] and Ivochkina [5]. The following theorem was proved in [12].

THEOREM 1: *Let* Ω *be a* $C^{1,1}$ *uniformly convex domain in* $\mathbb{R}^n$, $\phi \in C^{1,1}(\bar{\Omega})$ *and* $K \in C^{1,1}(\Omega)$ *a positive function such that*

(3)
$$\int_\Omega K < \omega_n$$

and

(4)
$$K(x) \le \mu \, \text{dist}(x, \partial\Omega)$$

for some positive constant μ . *Then the classical Dirichlet problem*

(5) $\quad \det D^2 u = K(x)(1 + |Du|^2)^{(n+2)/2}$ *in* Ω , $\quad u = \phi$ *on* $\partial\Omega$,

has a unique convex solution $u \in C^2(\Omega) \cap C^{0,1}(\bar{\Omega})$.

Theorem 1 can be obtained from the results of Lions [9], [10] as in [12], or directly from the results of Caffarelli, Nirenberg and Spruck [2], Krylov [6], [7], [8] and Ivochkina [5] on the existence of globally smooth solutions of the Dirichlet problem for equations of Monge-Ampère type, by using the interior second derivative estimate established in [13]. The existence of a convex solution $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ of (5) was proved under the hypotheses of Theorem 1 by Lions [9], [10], and the case $\phi \equiv 0$ was also proved by Gilbarg and Trudinger [4]. The Dirichlet problem for convex generalized solutions of (5) was studied by Bakelman [1], who proved a generalized version of Theorem 1. Additional references to this work are given in [1].

The condition (4) causes the equation (1) to become degenerate near $\partial\Omega$, which precludes us from obtaining globally smooth solutions of (5). However, a partial result on the global regularity of convex solutions of (5) is given in [12]. Specifically, if $\partial\Omega \in C^{2,1}$, $\phi \equiv 0$ and

$K^{1/n} \in C^{0,1}(\bar{\Omega}) \cap W^{2,n}(\Omega)$ (modulo convex functions), then the convex solution of (5) is in $C^2(\Omega) \cap C^{1,1}(\bar{\Omega})$.

The existence of globally smooth convex solutions of (5) was recently established by Ivochkina [5]. Her hypotheses are different to the ones of Theorem 1; in particular, K is assumed to be bounded away from zero in Ω and a restriction on the size of $|\phi|_{2;\Omega}$ is necessary.

The sharpness of the condition (4) for the classical solvability of the Dirichlet problem (5) for arbitrary smooth boundary data, at least in terms of power functions, is shown in [12] using a barrier argument. Related to this is the following global Hölder estimate which is proved in [15], and which yields nonexistence results for the Dirichlet problem (5).

THEOREM 2: *Let* Ω *be a* $C^{1,1}$ *bounded domain in* $\mathbb{R}^n$ *and* $u \in C^2(\Omega)$ *a convex solution of* (1), *where* K *satisfies*

$$(6) \qquad\qquad K(x) \geq \mu \, \text{dist}(x, \partial\Omega)^\beta$$

for some constants $\mu > 0$ *and* $\beta \in [0,1)$. *Then*

$$(7) \qquad\qquad \sup_{x,y \in \Omega} |u(x) - u(y)| \leq C|x - y|^{(1-\beta)/2n} ,$$

where C *depends only on* n *,* μ *,* β *and* Ω .

This result is an extension of the global oscillation estimate proved in [14], and is proved by a careful application of the barrier technique used there.

Although we cannot generally satisfy the boundary condition in (5) in the classical sense if (4) is not satisfied, it is possible to satisfy it in a certain optimal or generalized sense. This was proved by Bakelman [1] for generalized solutions. In [15] we have established the following result for smooth solutions.

THEOREM 3: *Let* Ω *be a* $C^{1,1}$ *uniformly convex domain in* $\mathbb{R}^n$, $\phi \in C^{1,1}(\bar{\Omega})$ *and* $K \in C^{1,1}(\Omega) \cap L^P(\Omega)$, $p > n$, *a positive function satisfying* (3). *Then there is a unique convex function* $u \in C^2(\Omega) \cap L^\infty(\Omega)$ *such that* u *solves* (1) *in* Ω,

(8) $$\limsup_{x \to y} u(x) \leq \phi(y) \qquad \text{for all} \quad y \in \partial\Omega ,$$

and if $v \in C^2(\Omega) \cap L^\infty(\Omega)$ *is another convex solution of* (1), *and* $\limsup_{x \to y} v(x) \leq \phi(y)$ *for all* $y \in \partial\Omega$, *then* $v \leq u$ *in* Ω.

The function u is therefore the supremum of the convex subsolutions of (1) which lie below ϕ on $\partial\Omega$, and the proof of the theorem shows that u is also the infimum of the convex supersolutions of (1) which lie above ϕ on $\partial\Omega$. To prove Theorem 3 we solve approximating Dirichlet problems with boundary values ϕ and obtain a sequence of $C^2(\Omega)$ convex functions converging in $C^0(\Omega)$ to a convex generalized solution u of (1), which satisfies (8) and the final conclusion of the theorem. To deduce the regularity of u we first use some measure theory to obtain information about the behaviour of u near $\partial\Omega$, and then use a standard method of Pogorelov [11] and Cheng and Yau [3]. If K satisfies (4) in $\Omega \cap B_\varepsilon(x_0)$, where $x_0 \in \partial\Omega$, then $u \in C^{0,1}(\bar{\Omega} \cap B_{\varepsilon/2}(x_0))$ and $u = \phi$ on $\partial\Omega \cap B_{\varepsilon/2}(x_0)$, while if K satisfies (6) in $\Omega \cap B_\varepsilon(x_0)$, then $u \in C^{0,(1-\beta)/2n}(\bar{\Omega} \cap B_{\varepsilon/2}(x_0))$.

The final theorem we mention summarizes the results we have proved in [14], [15] in the case

(9) $$\int_\Omega K = \omega_n .$$

THEOREM 4: *Let* Ω *be a uniformly convex domain in* $\mathbb{R}^n$ *and* $K \in C^{1,1}(\Omega) \cap L^P(\Omega)$, $p > n$, *a positive function satisfying* (9). *Then there is a convex solution* $u \in C^2(\Omega)$ *of the equation* (1), *and any two such solutions differ by a constant.*

To prove Theorem 4, we first obtain a generalized solution u of (1), which is done by solving approximating Dirichlet problems and passing to a limit with the help of an interior oscillation estimate, for example, Theorem 2 applied to smooth compactly contained subdomains of Ω . The regularity proof is similar to that in Theorem 3, and the uniqueness assertion follows from a comparison principle. If K satisfies (4) near a point $x_0 \in \partial\Omega$, then

$$\lim_{x \to x_0} u(x) = \infty ,$$

while if K satisfies (6) near $x_0 \in \partial\Omega$, and $\partial\Omega$ is $C^{1,1}$ near x_0 , then u is Hölder continuous there with exponent $(1-\beta)/2n$.

Finally, we mention that in [15], these results have been extended to Monge-Ampère equations of the form

$$\det D^2 u = f(x,u,Du) ,$$

under suitable hypotheses on f .

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