ASYMPTOTICALLY STABLE SOLUTIONS OF THE NAVIER-STOKES EQUATIONS AND ITS GALERKIN APPROXIMATIONS

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In many numerical and theoretical studies in fluid dynamics, especially in meteorology and oceanography, simpler truncated systems called Galerkin approximations or spectral systems, are studied instead of the full system of partial differential equations. These are finite dimensional systems of ordinary differential equations, usually with only linear and quadratic terms, which are obtained by truncating infinite dimensional systems involving the time-dependent coefficients of Fourier-like series expansions of the solutions of the partial differential equations. An implicit assumption here is that the qualitative behaviour of the solutions of the truncated system closely resemble that of the solutions of the full system of partial differential equations. This is known not to be true, in the Lorenz equations for example, when the truncation is too severe or the type of behaviour under consideration too complicated. It is, however known from the work of Foias, Prodi and Temam [2,3,4] that a compact attracting set for the Navier-Stokes equations is essentially finite-dimensional. In addition Constantin, Foias and Temam [1] have recently shown for the Navier-Stokes equations that the presence of an asymptotically stable steady solution in a Galerkin approximation, defined in terms of the eigenfunctions of the Stokes operator, of sufficiently high order implies the existence of a nearby asymptotically stable solution in the full Navier-Stokes equations. Their proof makes considerable use of the spectral properties of the linear operators in the Galerkin approximations and the Navier-Stokes equations linearized about steady solutions.

Such a simple spectral theory is not available for more complicated attracting sets such as periodic or almost periodic solutions, let alone strange attractors. There is however an extensive theory, see for example Yoshizawa [6], which characterizes the stability of an attracting set in terms of Lyapunov functions. I have been looking at the problem of whether the Navier-Stokes equations has a stable attracting set of a certain kind whenever a Galerkin approximation of sufficiently high order has a stable attracting set of the same kind from the view point of Lyapunov stability theory. I had completed my proof for an asymptotically stable steady solution, the simplest case, when I received a preprint of Constantin, Foias and Temam's paper. Their results are stronger than what I obtained using Lyapunov theory. Nevertheless I shall outline my results here as they give an indication of how Lyapunov functions can be used. To facilitate the exposition I shall restrict attention to two-dimensional spatially periodic domains, although the results also hold for more general domains and boundary conditions. At the end of this paper I shall briefly discuss extensions to attracting sets such as periodic and almost periodic solutions, the details of which I shall present elsewhere.

MATHEMATICAL PRELIMINARIES

Consider the Navier-Stokes equations

$$\frac{\partial \underset{\sim}{u}}{\partial t} - \nu\Delta \underset{\sim}{u} + \underset{\sim}{u}\cdot\nabla \underset{\sim}{u} = -\nabla p + \underset{\sim}{f} \tag{1}$$

$$\nabla\cdot \underset{\sim}{u} = 0 \tag{2}$$

on a unit square domain Ω in $\mathbb{R}^2$ and suppose that $\underset{\sim}{u}$ is spatially periodic in Ω . Following Temam [5], define

$$H^m_{per} = \left\{ f:\Omega \to \mathbb{R} \; ; \; f(\underset{\sim}{x}) = \sum_{\underset{\sim}{k} \in \mathbb{Z}} c_{\underset{\sim}{k}} e^{2i\pi \underset{\sim}{k}\cdot\underset{\sim}{x}} , \quad \bar{c}_{\underset{\sim}{k}} = c_{-\underset{\sim}{k}} \text{ and } \sum_{\underset{\sim}{k} \in \mathbb{Z}} |\underset{\sim}{k}|^{2m} |c_{\underset{\sim}{k}}|^2 < \infty \right\}$$

and

$$V_j = \{\underset{\sim}{u} \in (H^j_{per})^2 \; ; \; \nabla\cdot\underset{\sim}{u} = 0\}$$

for $j = 0,1$ and 2. (V_0 is written H in [5]). Consider the inner products and norms

$$(\underset{\sim}{f},\underset{\sim}{g}) = \int_\Omega \underset{\sim}{f}(\underset{\sim}{x})\cdot\underset{\sim}{g}(\underset{\sim}{x})\,d\underset{\sim}{x} \; , \quad |\underset{\sim}{f}| = \sqrt{(\underset{\sim}{f},\underset{\sim}{f})} \quad \text{for } \underset{\sim}{f},\underset{\sim}{g} \in V_0$$

and

$$((\underset{\sim}{f},\underset{\sim}{g})) = \int_\Omega \nabla\underset{\sim}{f}(\underset{\sim}{x}):\nabla\underset{\sim}{g}(\underset{\sim}{x})\,d\underset{\sim}{x} \; , \quad \|\underset{\sim}{f}\| = \sqrt{((\underset{\sim}{f},\underset{\sim}{f}))} \quad \text{for } \underset{\sim}{f},\underset{\sim}{g} \in V_1$$

with $|\Delta\underset{\sim}{f}|$ on V_2 .

Let P be the orthogonal projection of $(H^0_{per})^2$ onto H and write the Navier-Stokes equations (1)-(2) as

(3) $$\frac{\partial\underset{\sim}{u}}{\partial t} - \nu\Delta\underset{\sim}{u} + P(\underset{\sim}{u}\cdot\nabla\underset{\sim}{u}) = P\underset{\sim}{f}$$

on V_2 . Then there is a unique solution $\underset{\sim}{u}$ for each initial $\underset{\sim}{u}(0) \in V_2$ which exists for all $t \geq 0$. Moreover there is a constant $K = K(|\underset{\sim}{f}|, 1/\nu)$ such that

(4) $$|\underset{\sim}{u}(t)| \ , \ \|\underset{\sim}{u}(t)\| \text{ and } |\Delta\underset{\sim}{u}(t)| \leq K$$

provided $\underset{\sim}{u}(0)$ also satisfied these bounds. Clearly any steady solution $\bar{u}$ of (4), that is ,

(5) $$-\mu\Delta\bar{u} + P(\bar{u}\cdot\nabla\bar{u}) = Pf$$

also satisfies the bounds (4).

Let $\underset{\sim}{\phi}_1, \underset{\sim}{\phi}_2, \underset{\sim}{\phi}_3, \ldots$ be an orthonormal basis of V_0 consisting of the eigenfunctions of the Stokes equation

(6) $$-P\Delta\underset{\sim}{\phi}_k = \lambda_k \underset{\sim}{\phi}_k$$

with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_k \to \infty$. The $\underset{\sim}{\phi}_1, \underset{\sim}{\phi}_2, \underset{\sim}{\phi}_3, \ldots$ are also orthogonal in V with norm $\underset{\sim}{\phi}_k = \lambda_k^{1/2}$. Let P_m be the orthogonal projection of $(H^0_{per})^2$ onto the linear span of $\{\underset{\sim}{\phi}_1, \underset{\sim}{\phi}_2, \ldots, \underset{\sim}{\phi}_m\}$ in V_0 for $m \geq 1$. The m^{th}*-order Galerkin approximation* of the Navier-Stokes equations (3) is the system of differential equations in $P_m V_0$

(7) $$\frac{\partial \underset{\sim}{u}_m}{\partial t} - \nu\Delta\underset{\sim}{u}_m + P_m(\underset{\sim}{u}_m\cdot\nabla\underset{\sim}{u}_m) = P_m\underset{\sim}{f}$$

with initial data $\underset{\sim}{u}_m(0) \in P_m H$. Existence and uniqueness results analogous to the full Navier-Stokes equations also hold for a Galerkin approximation (7) of any order, and the same bounds (4) hold provided the initial data $\underset{\sim}{u}_m(0)$ also satisfies (4), uniformly in $m = 1,2,3,\ldots$ Similarly, any steady solution $\bar{u}_m$ of (7), that is

$$(8) \qquad -\nu\Delta\underset{\sim}{u}_m + P_m(\underset{\sim}{u}_m\cdot\nabla\underset{\sim}{u}_m) = P_m\underset{\sim}{f}$$

also satisfies the bounds (4), uniformly in $m = 1,2,3,\ldots$

The perturbation $\bar{\underset{\sim}{u}}_m = \underset{\sim}{u}_m - \bar{\underset{\sim}{u}}_m$ of a solution $\underset{\sim}{u}_m$ from a steady solution $\bar{\underset{\sim}{u}}_m$ of an m^{th}-order Galerkin approximation satisfies the differential equation

$$(9) \qquad \frac{\partial\underset{\sim}{\sigma}_m}{\partial t} - \nu\Delta\underset{\sim}{\sigma}_m + P_m(\underset{\sim}{\sigma}_m\cdot\nabla\bar{\underset{\sim}{u}}_m + \bar{\underset{\sim}{u}}_m\cdot\nabla\underset{\sim}{\sigma}_m) + P_m(\underset{\sim}{\sigma}_m\cdot\nabla\underset{\sim}{\sigma}_m) = 0$$

in P_mV_0. With $\underset{\sim}{X}(t) \in \mathbb{R}^m$ defined by $\underset{\sim}{\sigma}_m(t,\underset{\sim}{x}) = \sum_{j=1}^m X_j(t)\underset{\sim}{\phi}_j(\underset{\sim}{x})$, this can be written as a linear-quadratic differential equation in $\mathbb{R}^m$

$$(10) \qquad \frac{d\underset{\sim}{X}}{dt} = A\underset{\sim}{X} + \underset{\sim}{b}(\underset{\sim}{X},\underset{\sim}{X})$$

where H is an $m\times m$ real matrix and $\underset{\sim}{b} : \mathbb{R}^{2m} \to \mathbb{R}^m$ bilinear.

Following Yoshizawa [6], the zero solution of (10), or equivalently (9), is (exponentially) *asymptotically stable* if there exists a $\lambda > 0$ such that for each $\varepsilon > 0$ there can be found a $\delta = \delta(\varepsilon) > 0$ such that $|\underset{\sim}{X}(t)| \leqq \varepsilon e^{-t}$ for all $t \geqq 0$ whenever $|\underset{\sim}{X}(0)| \leqq \delta$, that is $|\underset{\sim}{u}_m(t) - \bar{\underset{\sim}{u}}_m| = |\underset{\sim}{\sigma}_m(t)| \leqq \varepsilon e^{-\lambda t}$ for all $t \geqq 0$ whenever $|\underset{\sim}{u}_m(0) - \bar{\underset{\sim}{u}}_m| = |\underset{\sim}{\sigma}_m(0)| \leqq \delta$. An analogous definition will be used for the asymptotic stability of a steady solution $\bar{u}$ of the Navier-Stokes equations (3), namely with the subscript m deleted above. Necessary and sufficient conditions for the asymptotic stability of the zero solution of (10) have been given by

Yoshizawa [6; theorems 11.6, 19.1] in terms of the existence of a Lyapunov function, that is a continuous function $W : \mathbb{R}^+ \times \mathbb{R}^m \to \mathbb{R}^+$ such that for all $|\underset{\sim}{X}|, |\underset{\sim}{X}'| \leqq R$ and $t \geqq 0$

$$(11.1) \qquad |\underset{\sim}{X}| \leqq W(t,\underset{\sim}{X}) \leqq L|\underset{\sim}{X}|$$

$$(11.2) \qquad |W(t,\underset{\sim}{X}) - W(t,\underset{\sim}{X}')| \leqq L|\underset{\sim}{X}-\underset{\sim}{X}'|$$

and

$$(11.3) \qquad D^+_{(10)} W(t,\underset{\sim}{X}) \leqq -\lambda W(t,\underset{\sim}{X}) \ ,$$

for some positive constants L, R and λ, where $D^+_{(10)}W$ is the upper right hand derivative of W along solutions of the differential equation (10), that is

$$(12) \qquad D^+_{(10)} W(t,\underset{\sim}{X}) = \overline{\lim_{h\to 0^+}}\, h^{-1}\{W(t,\underset{\sim}{X}+h(A\underset{\sim}{X}+\underset{\sim}{b}(\underset{\sim}{X},\underset{\sim}{X}))) - W(t,\underset{\sim}{X})\}$$

Finally bounds of the form

$$|\underset{\sim}{u}\cdot\nabla\underset{\sim}{\sigma}| \leqq M|\underset{\sim}{u}|^{1/2}|\Delta\underset{\sim}{u}|^{1/2}\|\underset{\sim}{v}\|$$

and

$$|\underset{\sim}{u}\cdot\nabla\underset{\sim}{v}| \leqq M|\underset{\sim}{u}|^{1/2}\|\underset{\sim}{u}\|^{1/2}|\underset{\sim}{v}|^{1/2}|\Delta\underset{\sim}{v}|^{1/2} \ ,$$

here $\underset{\sim}{u},\underset{\sim}{v} \in V_2$ and M is a positive constant depending only on Ω and the type of boundary conditions, are needed in the sequel.

MAIN RESULT

The main result here is to show that when a Galerkin approximation (7) has an asymptotically stable steady solution, then the Navier-Stokes equations (3) has a nearby steady solution which is also asymptotically stable, provided the order of the Galerkin approximation is sufficiently high and its steady solution sufficiently strongly asymptotically stable. This result is thus weaker than that of Constantin, Foias and Temam [1], which makes no restriction on how strongly asymptotically stable the steady solution is. This is a consequence of the simpler techniques used here.

THEOREM

Let $\underset{\sim}{u}_m$ *be an asymptotically stable steady solution of an* m^{th}*-order Galerkin approximation* (7) *with Lyapunov function* W *and corresponding constants* L,R *and* λ . *Then, the Navier-Stokes equations* (3) *have a steady solution* $\bar{\underset{\sim}{u}}$ *with*

$$\|\bar{\underset{\sim}{u}}-\bar{\underset{\sim}{u}}_m\| \leq \lambda_{m+1}^{-1/4}$$

provided m *is sufficiently large. Moreover,* $\bar{\underset{\sim}{u}}$ *is asymptotically stable when* λ *is sufficiently large, depending on the constants* K,L,M *and* ν .

I shall only give a sketch of the proof, which has three main parts. The first part is to show that for any solution $\underset{\sim}{u}$ of the Navier-Stokes equations (3) starting sufficiently close to $\bar{\underset{\sim}{u}}_m$, $\underset{\sim}{P}_m u(t)$ remains close enough to $\bar{u}_m$ to allow the Lyapunov function W to be used for $\sigma_m = P_m u - \bar{u}_m$, provided m is sufficiently large. From Yoshizawa [6; page 118], the zero solution of (10) is stable under persistent perturbations g, that is, a solution of

$$\text{(13)} \qquad \frac{dX}{dt} = AX + b(X,X) + g$$

satisfies $|X(t)| \leqq R$ provided $|X(0)| \leqq \eta R$ and $|g| \leqq \eta\lambda R/L$ for some $0 < \eta < 1$. This is applied to a solution u of the Navier-Stokes equations (3) satisfying bounds (4), for which $\sigma_m - P_m u - \bar{u}_m$ satisfies

$$\frac{\partial\sigma_m}{\partial t} - \nu\Delta\sigma_m + P_m(\sigma_m \cdot \nabla\bar{u}_m + \bar{u}_m \cdot \nabla\sigma_m) + P_m(\sigma_m \cdot \nabla\sigma_m)$$

$$= P_m\left((P_m u)\cdot\nabla(P_m u) - u\cdot\nabla u\right) = g \;.$$

Since $|u-P_m u| \leqq \|u\|/\lambda_{m+1}^{1/2}$ and $\|u-P_m u\| \leqq |\Delta u|/\lambda_{m+1}^{1/2}$, it follows that $|g| \leqq 2MK^2/\lambda_{m+1}^{1/2}$. Hence for λ_{m+1} sufficiently large,

$$|P_m u(t) - \bar{u}_m| \leqq R$$

provided $|P_m u(0) - \bar{u}_m| \leqq \eta R$.

The second part of the proof is to show that the Navier-Stokes equations (3) have a steady solution $\bar{u}$ which is close to $\bar{u}_m$, provided m is sufficiently large. This will be done by showing that each Galerkin approximation of order $\ell \geqq m$ has a steady solution

$\bar{u}_\ell$ near to $\bar{u}_m$, and then taking a convergent subsequence to obtain $\bar{u}$. The ℓ^{th}-order steady Galerkin approximation for $\ell \geqq m$ can be written as

$$
\begin{aligned}
&-\nu\Delta\sigma_m + P_m(\sigma_m \cdot \nabla\bar{u}_m + \bar{u}_m \cdot \nabla\sigma_m) = -P_m(\sigma_m \cdot \nabla\sigma_m) \\
(14.1)\qquad &-P_m(g_m \cdot \nabla g_m + g_m \cdot \nabla\bar{u}_m + \bar{u}_m \cdot \nabla g_m + g_m \cdot \nabla\sigma_m + \sigma_m \cdot \nabla g_m)
\end{aligned}
$$

in $P_m V_2$ and

$$
\begin{aligned}
&-\nu\Delta g_m + (P_\ell - P_m)(g_m \cdot \nabla\bar{u}_m + \bar{u}_m \cdot \nabla g_m) + (P_\ell - P_m)(g_m \cdot \nabla g_m) \\
(14.2)\qquad &= (P_\ell - P_m)(f - \bar{u}_m \cdot \nabla\bar{u}_m - g_m \cdot \nabla\sigma_m - \sigma_m \cdot \nabla g_m)
\end{aligned}
$$

in $(P_\ell - P_m)V_2$, where $\sigma_m = P_m\bar{u}_\ell - \bar{u}_m$ and $g_m = (P_\ell - P_m)\bar{u}_\ell$. This can in turn be written as a fixed point

$$\sigma_m = F_1(\sigma_m, g_m) \ , \quad g_m = F_2(\sigma_m, g_m)$$

for a continuous mapping (F_1, F_2) of the finite-dimensional space $\ell_m V_2 \otimes (P_\ell - P_m)V_1$ into itself. In view of the asymptotic stability of $\bar{u}_m$ the linear operator on the left hand side of (14.1) has a bounded inverse from $P_m V_2$ into $P_m V_0$, with norm bounded by K and $1/\lambda$, uniformly in m. (see for example, proposition 2.5 in Constantin, Foias and Temam [1]). Also for m sufficiently large

(15) $$\nu\lambda_{m+1}^{1/2}|\underset{\sim}{g}_m|\|\underset{\sim}{g}_m\| \leqq \int_\Omega \text{LHS}(14.2)\ g_m$$

(see for example proposition 4.1 in Foias, Manley, Temam and Treve [2]).
A fixed point $\bar{\underset{\sim}{u}}_\ell$ is then obtained with

(16) $$|\Delta(P_m\bar{\underset{\sim}{u}}_\ell-\bar{\underset{\sim}{u}}_m)| \leqq \lambda_{m+1}^{-3/4} \quad \text{and} \quad \|(P_\ell-P_m)\bar{\underset{\sim}{u}}_\ell\| \leqq \lambda_{m+1}^{-1/4}$$

uniformly in $\ell \geqq m$, provided m is sufficiently large that λ_{m+1} is greater than an expression involving the constants M,K,ν .
As these steady solutions $\bar{\underset{\sim}{u}}_\ell$ all satisfy the bounds (4) and (16), a compactness argument can be used to obtain a steady solution $\bar{\underset{\sim}{u}}$ of the Navier-Stokes equations (3) which also satisfies the bounds (4) and (16).

The final part of the proof is to show that $\bar{\underset{\sim}{u}}$ is asymptotically stable provided the constant λ is sufficiently large. Let $\underset{\sim}{\sigma}_m = P_m(\underset{\sim}{u}-\bar{\underset{\sim}{u}})$ and $\underset{\sim}{g}_m = \underset{\sim}{u}-\bar{\underset{\sim}{u}}_m - \underset{\sim}{\sigma}_m$ for any solution $\underset{\sim}{u}$ of the Navier-Stokes equations (3) satisfying bounds (4). Then σ_m and $\underset{\sim}{g}_m$ satisfy the perturbation equations

$$\frac{\partial\underset{\sim}{\sigma}_m}{\partial t} - \nu\Delta\underset{\sim}{\sigma}_m + P_m(\underset{\sim}{\sigma}_m\cdot\nabla\bar{\underset{\sim}{u}}_m+\bar{\underset{\sim}{u}}_m\cdot\nabla\underset{\sim}{\sigma}_m) + P_m(\underset{\sim}{\sigma}_m\cdot\nabla\underset{\sim}{\sigma}_m)$$

(17.1) $$= P_m(\underset{\sim}{\sigma}_m\cdot\nabla\bar{\underset{\sim}{u}}_m+\bar{\underset{\sim}{u}}_m\cdot\nabla\underset{\sim}{\sigma}_m - (\underset{\sim}{u}-\bar{\underset{\sim}{u}})\cdot\nabla\bar{\underset{\sim}{u}}\cdot\nabla(\underset{\sim}{u}-\bar{\underset{\sim}{u}}))$$

$$+ P_m(\underset{\sim}{\sigma}_m\nabla\underset{\sim}{\sigma}_m - (\underset{\sim}{u}-\bar{\underset{\sim}{u}})\cdot\nabla(\underset{\sim}{u}-\bar{\underset{\sim}{u}}))$$

and

$$\frac{\partial g_m}{\partial t} - \nu\Delta\underset{\sim}{g}_m + (P-P_m)(\underset{\sim}{g}_m\cdot\nabla\bar{\underset{\sim}{u}}_m+\bar{\underset{\sim}{u}}_m\cdot\nabla\underset{\sim}{g}_m)+(P-P_m)(\underset{\sim}{g}_m\cdot\nabla\underset{\sim}{g}_m)$$

(17.2) $$= -(P-P_m)(\underset{\sim}{\sigma}_m\cdot\nabla\bar{\underset{\sim}{u}}+\bar{\underset{\sim}{u}}\cdot\nabla\underset{\sim}{\sigma}_m+\underset{\sim}{\sigma}_m\nabla\underset{\sim}{g}_m+\underset{\sim}{g}_m\cdot\nabla\underset{\sim}{\sigma}_m+\underset{\sim}{\sigma}_m\cdot\nabla\underset{\sim}{\sigma}_m)$$

Considering (17.1) as an equation of the form (13) and using various bounds such as (4) and (16), and the first part of the proof, gives

$$
\begin{aligned}
(18.1)\qquad D^{+}_{(13)}W^2 + 2\lambda W^2 &\leqq L\,|RHS(17.1)|\,W \\
&\leqq C_1\lambda_{m+1}^{-1/8}W^2 + C_2\|\underset{\sim}{g}_m\|\cdot W \\
&\leqq C_1\lambda_{m+1}^{-1/8}W^2 + \frac{\nu}{2}\|\underset{\sim}{g}_m\|^2 + \frac{2}{\nu}C_2^2W^2
\end{aligned}
$$

Multiplying (17.2) by $\underset{\sim}{g}_m$, integrating over Ω and using the inequality (15), gives

$$
\begin{aligned}
(18.2)\qquad \frac{1}{2}\frac{d}{dt}|\underset{\sim}{g}_m|^2 + \frac{\nu}{2}\|\underset{\sim}{g}_m\|^2 + \frac{\nu}{4}\lambda_{m+1}|\underset{\sim}{g}_m|^2 &\leqq |RHS(17.2)|\;|\underset{\sim}{g}_m| \\
&\leqq C_3\lambda_{m+1}^{-1/2}W^2 + \frac{\nu}{8}\lambda_{m+1}|\underset{\sim}{g}_m|^2
\end{aligned}
$$

Here C_1, C_2 and C_3 depend on the constants K, L and M. Combining (18.1) and (18.2) gives

$$
(19)\qquad D^{+}_{(13)}(W^2+|\underset{\sim}{g}_m|^2) + \gamma(W^2+|\underset{\sim}{g}_m|^2) \leqq 0 \quad ,
$$

where $\gamma > 0$ provided m is sufficiently large and $\lambda \geqq 8\, C_2^2/\nu$, say. From (11.1) and (19)

$$|\underset{\sim}{\sigma}_m(t)|^2+|\underset{\sim}{g}_m(t)|^2 \leqq W^2(t,\underset{\sim}{X}(t))+|\underset{\sim}{g}_m(t)|^2$$

$$\leqq (W^2(0,\underset{\sim}{X}(0))+|\underset{\sim}{g}_m(0)|^2 e^{-\gamma t}$$

$$\leqq L|\underset{\sim}{u}(0)-\bar{u}|^2 \; e^{-\gamma t}$$

$$\rightarrow 0 \text{ as } t \rightarrow \infty \quad ,$$

provided $|\underset{\sim}{u}(0)-\bar{u}|$ is sufficiently small. This shows that $\underset{\sim}{\bar{u}}$ is (locally) asymptotically stable, provided λ is sufficiently large. This completes the outline of the proof of the theorem.

CONCLUDING REMARKS

The proof of Constantin, Foias and Temam [1] makes considerable use of the spectral properties of the linearized operators

$$A_\ell(\underset{\sim}{\bar{u}}_\ell)\underset{\sim}{\sigma}_\ell = -\nu\Delta\underset{\sim}{\sigma}_\ell + P_\ell(\underset{\sim}{\sigma}_\ell\cdot\nabla\underset{\sim}{\bar{u}}_\ell+\underset{\sim}{\bar{u}}_\ell\cdot\nabla\underset{\sim}{\sigma}_\ell)$$

of $P_\ell V_2$ into $P_\ell V_0$ for $\ell \geqq m$, in particular the continuity of the operators A_ℓ in the vicinity of $\underset{\sim}{\bar{u}}_m$. They can thus establish the asymptotic stability of the steady solution $\underset{\sim}{\bar{u}}$ of the Navier-Stokes equations (3), without any explicit restriction, such as λ being sufficiently large as was required above. Such a simple spectral theory is not available for more complicated attracting sets, such as periodic and almost periodic solutions or strange attractors. Lyapunov functions do however exist for more general asymptotically stable attractors. A simple extension of the above proof carries over to perturbations about a time-dependent solution $\underset{\sim}{\bar{u}}_m(t)$, but this is unrealistic (in fact impossible for time-independent forcing) as it requires the perturbations and the limiting trajectory to have identical phase. More realistic is the orbital asymptotic stability of the attracting set $\Gamma_m = V\{\bar{u}_m(t); t \geqq 0\}$. The Lyapunov functions $W(t,X)$ here then satisfy constraints such as

$$\text{dist}\{X(t),\Gamma_m\} \leqq W(t,X(t)) \leqq L.\text{dist}\{X(t),\Gamma_m\} ,$$

involving the distance of (unperturbed) solution $X(t)$ from the limiting set Γ_m . See, for example, Yoshizawa [6] for further details.

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