HAMILTONIAN SYSTEMS WITH MONOTONE TRAJECTORIES

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1. INTRODUCTION

Recently Helmut Hofer and I [1] studied a class of ordinary differential equations on $\mathbb{R}^{2n}$ which possess a Hamiltonian structure of a non-standard type. We considered the class of equations which can be written in the form

$$\dot{q}(t) = Sp(t) \ , \quad \dot{p}(t) = -V'(q(t)) \ , \quad (q(t),p(t)) \in \mathbb{R}^n \times \mathbb{R}^n \ , \ t \in \mathbb{R} \ ,$$

where S is a non-singular invertible self-adjoint operator on $\mathbb{R}^n$ with one negative and $n-1$ positive eigenvalues, and V' denotes the gradient of a smooth potential function V. The total energy, or Hamiltonian, which is conserved along trajectories is

$$H(q(t),p(t)) = \frac{1}{2} (Sp(t),p(t)) + V(q(t)) \ .$$

Because S is not positive-definite the quadratic form $(Sp(t),p(t))$ may be negative, and since the corresponding quadratic form in classical Hamiltonian dynamics is the positive kinetic energy functional, our theory does not include classical particle dynamics as a special case. However, systems such as ours do arise in applied mathematics, the most familiar example being the nonlinear Sturm Liouville problems for equations and systems.

EXAMPLE 1 NONLINEAR STURM LIOUVILLE EQUATIONS

The familiar equation

$$-u''(t) = f(u(t)) \ , \quad t \in \mathbb{R} \ ,$$

may be written as a Hamiltonian system

$$\dot{q}(t) = -p(t) \ , \quad \dot{p}(t) = F'(q(t)), \ t \in \mathbb{R} \ ,$$

where F denotes a primitive of f. In this case $n = 1$, $S : \mathbb{R} \to \mathbb{R} = -\mathrm{Id}$, and $V = -F$.

REMARK In some ways this example is the prototype of the theory to be developed shortly, since for $n > 1$ the thrust of the hypothesis on S is to make the problem essentially one-dimensional.

EXAMPLE 2 BONA-SMITH EQUATIONS [2]

The equations

$$c(u-\frac{1}{3}u'') = \eta-\frac{1}{3}\eta'' + \frac{1}{2}u^2 + d_1 \ ,$$

$$c(\eta-\frac{1}{3}\eta'') = u+u\eta + d_2 \ ,$$

have solutions representing travelling wave solutions of Boussinesq equations modelling long waves in a nonlinear dispersive medium. Here c is the phase speed of the wave and d_1 and d_2 are constants. If we put $\underset{\sim}{q} = (u,\eta)$

$$V(u,\eta) = \frac{1}{2}(u^2+\eta^2+u^2\eta) - cu\eta+d_1\eta+d_2u ,$$

and

$$S = \begin{pmatrix} 3/c^2 & 3/c \\ 3/c & 0 \end{pmatrix} ,$$

then taking $p = \dot{q}$, we obtain a system of the class about to be considered.

REMARK A rather complete picture of the travelling wave phenomena associated with this and similar Boussinesq systems may be obtained as special cases of the theory below. The same is true of the next example.

EXAMPLE 3 FITZ-HUGH NAGUMO EQUATION [3]

An extension of the Fitz-Hugh Nagumo equations

$$y_t = D_1\Delta u+f(u)-w , \quad w_t = D_2\Delta w+\varepsilon(u-\gamma w) ,$$

has associated with it the stationary problem which in one dimension takes the form

$$D_1u'' = w-f(u) , \quad D_2w'' = \varepsilon(\gamma w-u), \; D_1,D_2 > 0 ,$$

which can be written in Hamiltonian form, where the Hamiltonian is $\frac{1}{2}(D_1p_1^2-(D_2/\varepsilon)p_2^2) + F(q_1) - q_1q_2+\frac{1}{2}\gamma q_2^2$, and $(q_1,q_2) = (u,w),(p_1,p_2)=(\dot{u},\dot{w})$. Clearly the quadratic form is indefinite, and the problem is in the class we defined earlier.

The salient feature of the whole class of systems under consideration is that conservation of total energy imposes a natural ordering on the trajectories of solutions. Before considering more general situations it is instructive to examine EXAMPLE 1 where the observation is trivial. In this example $\frac{1}{2}\dot{q}(t)^2 + F(q(t)) = c$, a constant, along trajectories, and between the zeros of $c-F(q(t))$ the function $\dot{q}(t)$ does not change sign ; in other words q is a monotonic function of t between the zeros of $c-F(q(t))$. Of course this observation is central when one comes to a phase plane analysis of solutions of such problems. The higher dimensional analogue is the following.

Let e denote a normalised eigenvector corresponding to the unique negative eigenvalue of S , and let

$$P = \{q \in \mathbb{R}^n : (S^{-1}q,q) \leqq 0 \ , \ (q,e) \geqq 0\} \quad .$$

Then P is a closed convex cone in $\mathbb{R}^n$, and in the usual way it induces the natural ordering on $\mathbb{R}^n$:

$$q_1 \leqq q_2 \quad \text{if and only if} \quad q_2 - q_1 \in P \ .$$

Now suppose that $(q(t),p(t))$ is a solution of the Hamiltonian system $\dot{q} = Sp$, $\dot{p} = -V'(q)$, so that

$$(Sp(t),p(t)) + V(q(t)) = c \ , \quad t \in \mathbb{R} \ .$$

Then between the zeros of $c-V(q(t))$, the function (Sp,p) does not change sign. In order words $(S^{-1}\dot{q}(t),\dot{q}(t)) = (Sp(t),p(t))$ does not change sign between the zeros of $c-V(q(t))$. Now if the latter is positive between two of its zeros (if any), then $\dot{q}(t) \in P$ or $-P$, and so the trajectory $q(t)$ is monotonic in this interval. (Note than $\dot{q}$ cannot pass from P to $-P$, or vice versa, except at a zero of $c-V(q(t))$.) In the case when $n=2$, the situation is even more straightforward, for then the complement of $P^{\circ} \cup (-P^{\circ})$ (where $^{\circ}$ denotes interior) is the union of two disjoint convex cones Q and $-Q$. In this case q is monotonic increasing with respect to $\pm P$ or to $\pm Q$ when $c-V(q(t)) \neq 0$, and the type monotonicity involved changes only at the zeros of $c-V(q(t))$.

Returning to the general case there is one further observation which is central in this theory. If a trajectory is monotonic with respect to P, say, and $V(q(t))-c > 0$ on (a,b) where $V(q(b)) = c$ then $(S^{-1}\dot{q}(b),\dot{q}(b)) = (Sp(b),p(b)) = c-V(q(b)) = 0$. In other words as the spatial component q of a solution passes through a point of potential energy c, then the velocity of the trajectory is constrained to lie in a very small subset of $\mathbb{R}^n$, namely on the conical surface $\{q : (S^{-1}q,q) = 0\} \subset \mathbb{R}^n$.

These features of this class of systems are very potent in an analysis of their trajectories. Essentially what distinguishes them from classical Hamiltonian systems is that here the "kintic energy" term (Sp,p) can get very large and negative to compensate for the potential energy V becoming very large and positive. Of course this would be true if S were simple indefinite: the fact that it is indefinite with precisely one negative eigenvalue gives the monotonicity of trajactories which is so essential to the subsequent development. At this stage it is not clear what analogous theory might be available when S has more than one negative eigenvalue.

We make the following assumptions about V in order to obtain the main results of [1].

Suppose that there exists a bounded, open convex set $C \subset \mathbb{R}^n$ *such that* $V > 0$ *in* $C, V = 0$ *on* ∂C, *and if* $q \in \partial C$ with $V'(q) = 0$, *then* $C \subset q+P$. *(Note we do not exclude the possibility that* $V'(q) \neq 0$, $q \in \partial C$. *However, there is at most one zero of* V' *on* ∂C, *and* C *lies completely in the cone* $q+P$ *in this case.) We also require that at certain points* ∂C *is strictly convex ; namely if* $(SV'(q), V'(q)) = 0$, *then* $(V''(q)SV'(q), SV'(q)) < 0$.

<u>REMARK</u> If the boundary ∂C is strongly convex everywhere then the last condition is automatically satisfied. Even when we do not have a strongly convex boundary, the result below may still be true, being proved by an approximation argument.

Under the above hypotheses we obtain a result about the existence of bounded orbits whose projection onto q-space connect two points of the boundary of ∂. These orbits are of a very special type among the class of periodic orbits for dynamical systems.

<u>DEFINITION</u> *We say an orbit is of type* s *(or an* s*-orbit) if* $q(t+T) = q(T-t)$ *for all* $t \in \mathbb{R}$ *and* $p(T) = 0$ *for every* $T \in \mathbb{R}$ *such that* $p_i(T) = 0$ *for some* $i = 1,2,\ldots,n$.

This means that all the components of $p = (p_1,\ldots,p_n)$ are synchronised so that if one vanishes, then they all vanish, and the orbit in q space is symmetric about $t = T$.

The hypotheses above led to the following observation.

<u>THEOREM</u> *There exists an* s-*orbit such that* $q(0) \in \partial C$, $p(0) = 0$, *and* $q(t) \in \bar{C}$ *for all* $t \in \mathbb{R}$. *Moreover, if* $V'(q) \neq 0$, $q \in \partial C$, *there is at least one periodic* s-*orbit, of period* T, *joining two points of* ∂C *by monotone trajectories in* C *every half period, and* $q(0) = q(T/2) \in \partial C$, $p(0) = p(T/2) = 0$.
If $V'(q^*) = 0$, $q^* \in \partial C$, *then there are no periodic* s-*orbits with* $q(0) \in \partial C$ *and* $p(0) = 0$, *but there exists a homoclinic* s-*orbit such that* $q(0) \in \partial C, p(0) = 0$, *and* $q(t) \to q^*$ *as* $t \to \infty$, $q(t) \in C$, $t > 0$.

The proof of this is obtained using a shooting argument in $\mathbb{R}^n$, and the main tool to be employed is the Brouwer degree of a continuous mapping on an open subset of $\mathbb{R}^n$. To define precisely what map is involved, we need to examine the geometrical set-up. The details of these proofs are to be found in [1], so here I will confine attention to a heuristic justification of the claims made in the next section. In the next section also the result is established using an argument slightly different from that given in [1].

2. AN OUTLINE OF THE PROOF

We take as coordinate axes orthogonal lines spanned by the eigenvectors of S , and we may assume, without loss of geneality, that the n^{th} eigenvector corresponds to the unique negative eigenvalue which is -1. Thus $Se_n = -e_n$ and $Se_i = \lambda_i e_i$, $\lambda_i > 0$, $i=1,2\ldots,n-1, \|e_i\| =1, i=1,2,\ldots,n$. For the sake of a convenient notation, abbreviate by putting $e = e_n$. Recall that $P = \{q \in \mathbb{R}^n : (S^{-1}q,q) \leq 0, (q,e) \geq 0\}$. So now let $\Gamma \subset \partial C$ be the set

$$\{q \in \partial C \; : \; (SV'(q),V'(q)) < 0 \;, \; (V'(q),e) < 0\} \; .$$

<u>REMARK</u> Γ is the set of all points $q \in \partial C$ whose outward normal makes an acute angle with e (that is, $(V'(q),e) < 0$) , and whose tangent plane to ∂C is parallel to a tangent plane to the level set $(S^{-1}q,q) = a$ for some $a < 0$. In particular, the boundary $\partial\Gamma$ of Γ in ∂C , is the set

$$\partial\Gamma = \{q \in \partial C \; : \; (SV'(q),V'(q)) = 0 \;, \; (V'(q),e) < 0\}$$

and consists of those points of ∂C whose outward normal makes an acute angle with e , and whose tangent plane is parallel to one of the tangent planes to the conical surface $(S^{-1}q,q) = 0$.

If $V'(q) \neq 0$, $q \in \partial C$, then there is another set $\tilde{\Gamma}$ defined analogously using $(V'(q),e) > 0$, instead of $(V'(q),e) < 0$. If $V'(q^*) = 0$, $q^* \in \partial C$, then $C \subset q^* + P$, and as a consequence $\Gamma = \{q \in \partial C \; : \; (SV'(q),V'(q)) < 0\}$.
This remark is very easily observed by drawing a diagram in $\mathbb{R}^2$, and is proved in [1].

Let P denote the orthogonal projection defined in $\mathbb{R}^n$ by

$$Pq = q - (q,e)e \; ,$$

The following result is geometrically clear, and may be proved from the Implicit Function Theorem.

<u>LEMMA</u> *The projection* $P : \bar{\Gamma} \to P(\bar{\Gamma})$ *is a homeomorphism and* $P(\partial\Gamma) = \partial(P\Gamma) \subset \mathbb{R}^{n-1}$.

The next result is also geometrically obvious, but its proof is slightly more subtle .

<u>LEMMA</u> *The sets* $\bar{\Gamma} \subset \partial C$ *and* $P(\bar{\Gamma}) \subset \mathbb{R}^{n-1}$ *are contractible in themselves to a point.*

The proof of this is based on the observation that the evolution equation

$$\dot{q}(t) = SV'(q(t)) - ((SV'(q(t)),V'(q(t))V'(q(t))/\|V'(q(t))\|^2)$$

leaves $\bar{\Gamma}$ invariant, and contracts it to a neighbourhood of the convex set $\{q \in \partial C : V'(q)/\|V'(q)\| = -e\}$. Hence $\bar{\Gamma}$ is contractible to a point, and consequently so is $\overline{P(\Gamma)}$.

Now for any open bounded set $\Omega \subset \mathbb{R}^{n-1}$ and any continuous function $f : \bar{\Omega} \to \mathbb{R}^{n-1}$ with $f(q) \neq 0$, $q \in \partial\Omega$, let $\deg(\Omega,f,0)$ denote the Brouwer degree of f with respect to Ω and 0 . Then the following result is well-known.

<u>LEMMA</u> *If* Ω *is contractible in itself to a point, and if* $(f(q),\nu(q)) > 0$ *for all* $q \in \partial\Omega$ *, where* $\nu(q)$ *is the outward unit normal to* $\partial\Omega$ *at* q *(which is supposed to be well-defined) then*

$$\deg(\Omega,f,0) = 1 \ .$$

We will use this in the next section, after we have defined a shooting map. The significance of the set $\bar{\Gamma}$ is revealed by the following.

If $(q(t),p(t))$, $t > 0$, is a solution of the initial-value problem

$$\dot{q}(t) = Sp(t), \; \dot{p}(t) = -V(q(t)), \; t > 0 ,$$

$$p(0) = 0, \quad q(0) \in \partial C ,$$

then from elementary calculus we find that

$$\tau(q(0)) = \sup \{t > 0 : q(t) \in C\} > 0 \quad \text{if} \quad q(0) \in \Gamma$$

and

$$\{t > 0 : q(t') \in C, t' \in (0,t)\} = \phi \text{ if } \quad q(0) \in \partial\Gamma .$$

Moreover, if $q(0) \in \partial C$, $p(0) = 0$, we know by conservation of energy that $\frac{1}{2}(Sp(t),p(t)) + V(q(t)) = 0$ for all t, whence $\dot{q}(t) \in -P$ for all $t \in (0,\tau)$. It is an immediate consequence of our hypotheses that the mapping $q(0) \to \tau(q(0))$ is continuous from $\bar{\Gamma}$ into $\mathbb{R}$, and hence by standard continuous dependence theory for initial value problems that $q(0) \to \dot{q}(\tau(q(0))) = Sp(\tau(q(0)))$ is a continuous mapping from $\bar{\Gamma}$ to $\mathbb{R}^n$. There is no guarantee that $\tau(q(0) < \infty$, and indeed, in the case when homoclinic orbits are sought, it is important that the possibility $\tau(q(0)) = +\infty$, $q(0) \in \Gamma$, should not be excluded. Note however that in any case $\{\|\dot{q}(t)\| : t \in (0,\tau(q(0)))\}$ is bounded because of the differential equation, and hence the mapping $\Theta(q(0)) = \{\dot{q}(\tau(q(0))/\tau(q(0))\}$ defines a continuous mapping Θ on Γ. Now from the differential equation and the fact that $\tau(q(0))=0, q(0) \in \partial\Gamma$ we find that

$$\Theta(q_n(0)) = \left\{ -\int_0^{\tau(q_n(0))} SV'(q_n(w))\,dw \right\} / \tau(q_n(0))$$

$$\to \; -SV'(q(0)) \quad ,$$

where $q_n(0) \in \Gamma$ and $q_n(0) \to q(0) \in \partial\Gamma$ as $n \to \infty$. Hence Θ can be extended continuously to $\bar{\Gamma}$ by putting $\Theta(q(0)) = -SV'(q(0)), q(0) \in \partial\Gamma$. Now recall from the introduction that there is a natural constraint on $\dot{q}(\tau)$ since $V(q(\tau)) = 0$; namely $(S^{-1}\dot{q}(t), \dot{q}(\tau)) = 0$. Now we define a mapping $\theta : P(\bar{\Gamma}) \to \mathbb{R}^{n-1}$ as follows : for $x \in P(\bar{\Gamma})$,

$\theta(x) = P \circ \Theta \circ P_\Gamma^{-1}(x)$, where P_Γ denotes the restriction of P to $\bar{\Gamma}$.

Now on $P(\partial\Gamma) = \partial(P\Gamma)$ we know that

$$\theta(x) = -PSV'(P_\Gamma^{-1}(x)) \quad ,$$

and a straightforward calculation of the normal $\nu(x)$ to $\partial(P\Gamma)$ at x ensures that $(\theta(x), \nu(x)) > 0$ for all $x \in \partial(P\ )$. Hence $\deg(P\Gamma, \theta, 0) = 1$, and there exists a point $x \in P\Gamma$ such that $\theta(x) = 0$. This means that there exists a point $q(0) \in \Gamma$ such that $\Theta(q(0)) = 0$.

There are only two circumstances under which this can happen. One is when $\dot{q}(\tau(q(0))) = 0$, and the other is when $\tau(q,(0)) = \infty$. In the first case the fact that $\dot{q}(\tau(q(0))) = 0$ means that the orbit is symmetric about $t = \tau(q(0))$, by the uniqueness theorem for initial value problems. Hence we have established the existence of a periodic s-orbit. In the second case $\tau(q(0)) = \infty$, and so the trajectory $q(t)$ lies in C and $\dot{q}(t) \in -P$ for all $t > 0$. Hence for all $t_1 > t_2 > 0$,

$$\left|(q(t_1)-q(t_2),e)\right| = \left|\int_{t_2}^{t_1} (\dot{q}(t),e)\right| \geq (\text{const.}) \int_{t_1}^{t_2} \|\dot{q}(t)\| dt ,$$

and since $q(t) \in C$, a bounded set, for all $t > 0$, we find that $\|\dot{q}\| \in L_1(0,\infty)$. Now the fact that $\|\dot{q}\| = \|V'(q)\|$ is bounded for all $t > 0$ implies that $\dot{q}(t) \to 0$ as $t \to \infty$. We then infer from the monotonicity of trajectories in q-space that $q(t) \to q^* \in \partial C$ where $V'(q^*) = 0$. This completes the outline of the proof.

REMARK It is clear from this sketch that homoclinic s-orbits exist only if there is a point where $V'(q^*) = 0$, $q^* \in \partial C$. It is also clear that if there is a point $q^* \in \partial C$ where $V'(q^*) = 0$, then there cannot be a periodic s-orbit in C of the type described. To see this it is sufficient to observe that if $q(0) \in \Gamma$, then $q(\tau(q(0))) \notin \bar{\Gamma}$, because no point of Γ is ordered relative to any other point of $\bar{\Gamma}$ and $q(\tau(q(0)) < q(0)$ for all $q(0) \in \Gamma$. However, since the hypothesis $C \subset P+q^*$ implies that $\bar{\Gamma} = \{q \in \partial C \backslash \{0\} : (SV'(q),V'(q)) \leq 0\}$, the only possibility is that $(SV'(q(\tau(q(0)))),V'(q(\tau(q(0)))) > 0$. Clearly if $q(\tau(q(0))) = q^*$, then $\tau(q(0)) < +\infty$. It is a matter of elementary calculus then to verify that $V(q(t)) < 0$, $t > \tau(q(0))$, and so $q(t)$ does not lie in $\bar{C}$ for all $t > 0$.

FURTHER EXTENSIONS

The proof whose outline has just been given will be given in an article by Helmut Hofer and myself, to appear shortly. It has obviously the potential to treat situations a great deal more general than those described in the hypotheses above. Additionally, because of the stability of the Brouwer degree methods, we can obtain continuous dependence results for parameter dependent problems at no extra cost. Further, using the Brouwer degree to define a local index of solutions, we can obtain an algebraic count of their multiplicity, and consider questions of their bifurcation. It might be amusing to finish by explaining how the methods above lead to existence of solutions in examples where uniqueness is certainly false. Consider two bounded strongly convex regions C_1 and C_2 where two functions V_1 and V_2 are positive, and where $V_i'(q) \neq 0$, $q \in \partial C_i$, i=1,2. Then we know that in each of these sets there is an s-periodic solution of the corresponding Hamiltonian system.

Now take a tubular neighbourhood T_i of each of these orbits, and define a new function V such that $V_i = V$ on T_i, i=1,2, and $V > 0$ in C , $V = 0$ on ∂C where C is a strongly convex set containing T_i, i=1,2. Then the Hamiltonian system corresponding to V has at least one periodic orbit, according to our theory. But we know it has at least two by construction. The Brouwer degree probably suggests it has three!

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