

MORSE INEQUALITIES AND ESTIMATES FOR THE
NUMBER OF SOLUTIONS OF NONLINEAR EQUATIONS

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In this lecture, I want to discuss how Morse inequalities apply in more general situations where mappings are not bounded below, where there are degenerate critical points, and where we have some information of how solutions of the corresponding differential equations join critical points. We then discuss an application to elliptic partial differential equation and some counterexamples which show that our estimates for the number of solutions of the elliptic problem are, in a sense, best possible.

The proof of most of the results given depend upon the homotopy index of Conley [3]. However, nearly all the results can be understood without knowing the homotopy index.

1. MORSE INEQUALITIES

We discuss various generalizations of the Morse inequalities. There are two points I should make clear at the outset. Firstly, our approach is based on Conley's homotopy index [3]. This is a much more general invariant for bounded solutions of autonomous ordinary differential equations which should be much better known and used. Secondly, we will restrict ourselves to problems on a finite-dimensional space though one could construct a theory for some infinite-dimensional problems by using Rybakowski's generalization of the homotopy index to infinite-dimensional semi-flows ([13], [14]).

We start by reviewing the classical Morse inequalities. Assume that F is a C^2 function on a compact manifold M of dimension n . The

critical points of f are the points where $\nabla f(x) = 0$. A critical point x is said to be non-degenerate if $D^2f(x)$ is invertible. (Note that we define $D^2f(x)$ by using a suitable chart round x .) Let $i(x)$ denote the number of negative eigenvalues of $D^2f(x)$. This is called the index of x . Similarly, we can define $i(L)$ for any self-adjoint linear map L . (This makes sense even if x is degenerate.) Assume now that all the critical points of f on M are non-degenerate. It follows easily that there are only a finite number of critical point of f on M . Let $\{x_i\}_{i=1}^r$ denote them and let c_i denote the Betti numbers of M . (These are non-negative integers which depend only on M and are determined from the homology of M .) Finally, let M_k denote the number of critical points of index k . Then the classical Morse inequalities are that

$$(1) \quad \sum_{j=0}^s (-1)^{s-j} M_j \geq \sum_{j=0}^s (-1)^{s-j} c_j$$

for $0 \leq s \leq n$, with equality when $s = n$. Note that, for $s = n$, this is simply the Poincaré-Hopf theorem relating the Euler characteristic of M to the degrees of critical points on M . The inequalities (1) are known as the Morse inequalities and have been used a great deal.

We want to consider some generalizations. Firstly, we rewrite the classical Morse inequalities in a form which is more convenient for generalizations. If x_m is a non-degenerate critical point of f , let $B_j(f, x_m) = 1$ if $j = i(x_m)$ and to be zero otherwise (where $0 \leq j \leq n$). These numbers are called the Betti numbers of the critical point x_m . The Morse inequalities can now be written as

$$(2) \quad \sum_{j=0}^s (-1)^{s-j} \sum_{i=1}^r B_j(f, x_i) \geq \sum_{j=0}^s (-1)^{s-j} c_j$$

(It is easily checked that these are the same inequalities as before.) In this form, it turns out that the Morse inequalities are still true even if f has degenerate critical points provided that f has only a finite

number of critical points. In this case, the Betti numbers $B_j(f, x_i)$, $0 \leq j \leq n$, of a degenerate critical point x_i are defined to be the Betti numbers of $h(-\nabla f, T)$, where h denotes the homotopy index in the sense of [3] and T is a small neighbourhood of x_i . (It can be shown that the definition is independent of T .) *In addition, the Morse inequalities (2) hold for functions f on \mathbb{R}^n provided that f has isolated critical points, that the correct c_j are used (where c_j now depends on f) and provided that (i) the critical points of f lie in a compact set and (ii) there is a compact set W such that every heteroclinic orbit of the differential equation*

$$(3) \quad x'(t) = \nabla f(x(t))$$

lies in W . (By a heteroclinic orbit, we mean a solution $W(t)$ of (3) such that $W(t) \rightarrow a$ as $t \rightarrow -\infty$ and $W(t) \rightarrow b$ as $t \rightarrow \infty$, where a and b are critical points of f . (The reason for the occurrence of heteroclinic orbits is that they are the only non-constant bounded solutions of (3) defined on \mathbb{R} . Here, we are using that the critical points of f are isolated.) Assumption (ii) above holds in many cases. For example, it holds if there is a $K > 0$ such that $\|\nabla f(x)\| \geq K$ if $\|x\|$ is large. Hence we have found a version of the Morse inequalities which apply to functions on \mathbb{R}^n with degenerate critical points. Note that our functions need not be bounded above or below. (There is nothing special about \mathbb{R}^n : we could consider any n -dimensional manifold.) Note that our approach above follows Conley [3] and Dancer [5].

For the above Morse inequalities to be useful in analysis, we have to be able to calculate the c_i and to understand the Betti numbers of degenerate critical points. We discuss the first of these problems and then return to the second. Proofs can be found in [5].

We assume as before that f is a C^2 function on \mathbb{R}^n satisfying

conditions (i) and (ii). The numbers c_i are defined to be the Betti numbers of $h(-\nabla f, B_r)$, where B_r is a ball with centre zero and large radius r . However, this formula is often not convenient for computations. More importantly from this formula and homotopy invariance properties of the homotopy index, we see that if $f_t : \mathbb{R}^n \rightarrow \mathbb{R}$ are C^2 functions depending on $t \in [0,1]$ such that the map $(x,t) \rightarrow \nabla_x f_t(x)$ is continuous and such that

$$(4) \quad \|\nabla_x f_t(x)\| \geq K > 0$$

for $\|x\| \geq R_1$ and $t \in [0,1]$, then the c_i 's are the same for each f_t . This means that we can often calculate the c_i 's by deforming f to a simpler map. For example, if we can deform f (as above) to the map $x \rightarrow \langle Lx, x \rangle$, where L is self-adjoint and invertible, one easily sees that $c_j = \delta_{j,i(L)}$. (These conditions certainly hold if $\nabla f(x) - Lx = o(\|x\|)$ as $\|x\| \rightarrow \infty$.) As a second example, if one can deform f to a map with no critical points, one easily sees that $c_i = 0$ for all i . Thus one can calculate the c_i in many cases.

Now, we need to consider the calculation of the Betti numbers of degenerate critical points. Without loss of generality, one can assume that the critical point is at the origin. My basic point here is that it is possible to understand quite well the Betti numbers of a degenerate critical point. Indeed, the problems with the calculation of the Betti numbers are similar to the problems of the calculation of the degree of a degenerate critical point. I should point out that there were two earlier definitions of the Betti numbers of a critical point (Morse [12] and Gromoll and Meyer [7].) Our definition is equivalent to theirs but seems easier to work with. For example, a question in [7] can be easily resolved from our approach.

Let $k = \dim N(D^2 f(0))$, where N denotes the kernel. Then it can be

proved that $B_j(f,0) = 0$ if $j < i(0)$ or if $j > i(0) + k$. Moreover if $B_{i(0)}(f,0) \neq 0$, then $B_j(f,0) = \delta_{i(0),j}$ while if $B_{i(0)+k}(f,0) \neq 0$ then $B_j(f,0) = \delta_{t,j}$ where $t = i(0) + k$. (Hence, in all other cases, $B_j(f,0) = 0$ for $j \leq i(0)$ and $j \geq i(0) + k$.) In particular, we see that $B_j(f,0)$ can only be non-zero for at most $k-1$ values of j (at most 1 if $k = 1$). Note that $B_j(f,0)$ are all non-negative integers. (They can be all zero.) In general, several of the $B_j(f,0)$ can be non-zero and they may each be larger than 1. These possibilities make the possible presence of degenerate critical points a major impediment to estimating numbers of solutions. The above restrictions on the Betti numbers of a degenerate critical point are best possible in the sense that given a linear operator L , and non-negative integers $a_j, j = i(L)+1, \dots, i(L)+k-1$, then there is a C^∞ function f such that $\nabla f(0) = 0$, $D^2f(0) = L$, f has an isolated critical point at 0 and $B_j(f,0) = a_j$ for $j = i(L)+1, \dots, i(L)+k-1$. Here $k = \dim N(L)$. (We do not know if f can be chosen real analytic.)

I discuss briefly some of the steps in the proofs of the above results. The idea is to use the implicit function theorem to show that the equation

$$(5) \quad P\nabla f(r+v) = 0$$

has a unique small solution $v(r) \in N(D^2f(0))^\perp$ for each small $r \in N(D^2f(0))$. Here $I-P$ is the orthogonal projection onto $N(D^2f(0))$.

One can then prove that $B_j(f,0) = B_{j-i(L)}(g,0)$ where $L = D^2f(0)$, $g(r) = f(r+v(r))$ and g is considered as a function on $N(D^2f(0))$.

(This is essentially the Liapounov-Schmidt reduction.) It turns out that

$B_j(f,0) = \delta_{j,i(L)}$ exactly when 0 is a local minimum of g and

$B_j(f,0) = \delta_{j,i(L)+k}$ exactly when 0 is a local maximum of g .

If we know more about g , one can obtain more information. If f is

C^∞ , then, generically, g is of the form $g_1 + g_2$ where $\nabla g_1(x) \neq 0$ for $x \in N(D^2 f(0)) \setminus \{0\}$, $g_1(\alpha x) = \alpha^s g_1(x)$ and $g_2(x) = o(\|x\|^s)$ as $x \rightarrow 0$. In this case, $B_j(g, 0)$ is simply the $(j-1)$ th Betti number of $\{r \in N(D^2 f(0)) : \|r\| = 1, g_1(r) \geq 0\}$. Note that, if $s = 3$ (or if f is odd and $s = 4$) one can calculate g_1 without having to solve (5). This makes the computations much easier. It can be shown that $B_j(f, 0)$ can only be large if $k + s$ is large.

It can be proved that $\sum_{j=1}^n (-1)^j B_j(f, 0) = \text{index}(f, 0)$, where $\text{index}(f, 0)$ denotes the degree of the isolated zero of ∇f at the origin. (This is to be expected since the last Morse inequality is an inequality.) It follows easily from this formula and our comments above that, if $k = 1$, the Betti numbers of f at zero are determined from $i(0)$ and $\text{index}(f, 0)$ while, if $k = 2$, the same is true except that we now also need to distinguish between when g has a maximum and a minimum at 0.

There is a rather different way one can look at these questions. Suppose that one has a critical point constructed by some minimax argument. Then one tries to see what one can discover about its Betti numbers. Assume $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is C^2 , f has an isolated local minimum at z and there is a $v \in \mathbb{R}^n$ such that $f(v) \leq f(z)$. Finally assume that $\|(\nabla f)(y_n)\|$ has a positive lower bound whenever $f(y_n)$ is bounded and $\|y_n\|$ is large. Then it is well known that f has a critical point on $f^{-1}(c)$, where c is defined as follows. (This is known as the mountain pass theorem.) We let ℓ denote the set of continuous paths joining z to v and $c = \inf_{\ell} \sup_{q \in \ell} f(q)$. We then have the following result.

THEOREM: Hofer ([8], [9]), Dancer [5], Tian [16]. Assume that (i) f satisfies the above assumptions, that (ii) each critical point in $f^{-1}(c)$ is isolated in \mathbb{R}^n and that (iii) the smallest real eigenvalue of $D^2 f(x)$ is simple for x in \mathbb{R}^n . Then there is a critical point b in $f^{-1}(c)$ such that $B_j(f, b) = \delta_{1, j}$.

The interest in this theorem is that it gives complete information on the Betti numbers of b even though b may be quite degenerate. The theorem is very useful because a great many critical points in analysis are constructed by the above procedure. We will consider applications later. Note that assumption (iii) tends to be automatically satisfied in applications to second-order elliptic equations. If assumption (iii) is deleted, one can only prove that $B_1(f,b) \neq 0$. (The result above is not true without some such assumption.) More recently Hofer [10] has proved a version of the theorem where the isolatedness assumption is deleted. He and Ekelund have used this result to obtain some very nice theorems on the minimal period problem in Hamiltonian Mechanics. It would be nice to know something about the Betti numbers of critical points established by other minimax arguments.

As in Bott [1], the Morse inequalities (2) still hold if there are manifolds of critical points rather than just isolated critical points. (We now have to talk about the Betti numbers of the manifold of critical points.)

One advantage of the present approach to Morse inequalities is that one can sometimes get extra information if one knows enough on which critical points are joined by heteroclinic orbits (where the direction of flow is important). Assume that f satisfies our basic conditions earlier for a map on \mathbb{R}^n (or a compact manifold). Assume that the critical points of f can be written as a disjoint union $\bigcup_{i=1}^d M_i$ in such a way that if W is a heteroclinic orbit of $x'(t) = -\nabla f(x(t))$ with $W(-\infty) \in M_i$ and $W(\infty) \in M_j$, then $j \geq i$. (This last condition is the important condition.) Such a decomposition is called a *Morse decomposition*. With each M_i , one can associate Betti numbers $B_j(f, M_i)$. (To do this, one chooses a neighbourhood T_i of M_i union the heteroclinic orbits joining points of M_i such that $\bar{T}_i \cap M_j = \emptyset$ for $j \neq i$. Then $B_j(f, M_i)$ are defined to be

the Betti numbers of $h(-\nabla f, T_1)$.) It can then be proved that the Morse inequalities hold for the M_j , that is,

$$(6) \quad \sum_{j=0}^s (-1)^{s-j} \sum_{i=1}^d B_j(f, M_i) \geq \sum_{j=0}^s (-1)^{s-j} c_j$$

(with c_j as before) for $0 \leq s \leq n$ with equality if $s = n$. Note that, in general, we only expect results of the above type to hold if there is some restriction on the heteroclinic orbits. I should clarify our assumptions on heteroclinic orbits. We can always renumber the M_i so that our condition holds if and only if there is no cycle among the M_i 's (where, by a cycle, we mean heteroclinic orbits $\{P_r\}_{r=1}^p$, such that $P_r(-\infty) \in M_{j(r-1)}$, $P_r(\infty) \in M_{j(r)}$ and $j(1) = j(p)$). To evaluate the $B_j(f, M_i)$, we try to deform f to a map where M_i contains a single critical point or is empty. We will give an example later. Note that the above result still holds even if M_i contains an infinite number of critical points but is closed provided we change the definition of heteroclinic orbit to require that $w(t) \rightarrow M_i$ as $t \rightarrow -\infty$ and $w(t) \rightarrow M_j$ as $t \rightarrow \infty$ (rather than approaching single critical points). These results are special cases of results in [3].

Lastly, it is possible to use the Morse type approach to critical point theory to give a different proof of the well-known result that any C^2 map on a compact manifold M has at least category M critical points. The idea here is to use the "box-like" neighbourhood of a critical point constructed in [5] to show that $\{x : f(x) \leq a\}$ changes category by at most 1 if a crosses a critical value containing a single critical point. In addition, one can use similar ideas to show that if $w \in H^r(M)$ has the property that $w^S \neq 0$ (where we mean the cup product), then there exist distinct critical points $\{x_i\}_{i=1}^S$ in M such that $B_{ri}(f, x_i) \neq 0$. This shows how the product structure of the cohomology ring gives some information on the type of critical points we obtain. Note that the above

results answer some questions left open in [5] and that the "box-like" neighbourhood we mentioned above is used to obtain further properties of $B_j(f, x_i)$ in [5].

2. APPLICATIONS

In this section, we consider rather briefly two applications of the ideas in Section 1 and mention some related counter-examples.

Firstly, we consider the equation

$$(7) \quad \begin{aligned} -\Delta u &= f(x, u) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Here, it is assumed that Ω is a bounded domain in \mathbb{R}^n , f is C^1 on $\bar{\Omega} \times \mathbb{R}$, $f(x, 0) = 0$ for $x \in \Omega$, $f_2'(x, 0) = a(x) > 0$ in Ω and $f_2'(x, t) \rightarrow b(x)$ as $|t| \rightarrow \infty$ uniformly in x (though this could be weakened to the corresponding assumption on f). Finally, assume that $\int_{\Omega} \frac{1}{2} |\nabla u|^2 - \frac{1}{2} b u^2 dx > 0$ if $u \in W^{2,2}(\Omega) \cap W^{1,2}(\Omega)$ and $u \neq 0$, that 1 is not an eigenvalue of

$$(8) \quad -\Delta h = \lambda a h$$

and that the second eigenvalue of this problem is less than 1 (for Dirichlet boundary conditions). Then (cf. Hofer [8]) (7) has at least 4 non-trivial solutions. We sketch a proof. Clearly, we can assume that (7) has only a finite number of solutions. We first use positive operator theory to construct a non-trivial non-negative solution u_+ of (7) such that u_+ has degree 1 and every eigenvalue of $-\Delta - f_2'(x, u_+)I$ has no negative eigenvalues (cf. [6]). It is not difficult to deduce that u_+ is a local minimum of the functional $\int_{\Omega} \frac{1}{2} |\nabla u|^2 - F(x, u) dx$ on $W^{1,2}(\Omega)$, where $F(x, t) = \int_0^t f(x, s) ds$. We can also construct a non-positive solution u_- with similar properties. We can now use a Liapounov-Schmidt

reduction to reduce our problem to an equivalent finite-dimensional problem and then use Theorem 1 (essentially with $u_+ = v$, $u_- = z$ or vice versa) to construct a critical point b with Betti numbers $\delta_{1,j}$. Since the linearization at 0 has at least 2 negative eigenvalues, it follows from our earlier comments that $b \neq 0$. By our earlier formula for the degree in terms of the Betti numbers, b has degree -1 . Thus u_+ , u_- each have degree 1 while b has degree -1 and 0 has degree ± 1 (because it is non-degenerate). Since the total degree is easily seen to be 1, it follows that there must be another solution, as required. (Note that, in the sketch of the proof above, we have not been careful to distinguish between (7) and the equivalent finite-dimensional problem.) As in [5], the condition that $-\Delta - aI$ is invertible (for Dirichlet boundary conditions) can be weakened considerably and indeed, if in the last step of the proof we use Morse inequalities rather than degree, we see that the above argument only fails if the Betti numbers of the critical point 0 have a rather particular form.

If $f(x,t)$ is odd in t , a theorem of Clark [2] implies that (7) has at least $2k$ non-trivial solutions if the k th eigenvalue of (8) (counting multiplicity) is less than 1. One might hope for a similar result without the oddness assumption. Unfortunately this is false. Recently [6], the author has constructed an example for Ω the open unit ball in \mathbb{R}^n for which the $(n+1)$ th eigenvalue is less than 1 but there are only 4 non-trivial solutions (and all 5 solutions are non-degenerate). The example can be constructed so that f is "nearly independent of x " and "nearly odd in t ". The essential idea is to make a small perturbation from the case where f is independent of x . Note that for the above counter-example, the branches of solutions bifurcating from 0 of

$$\begin{aligned} \Delta u &= \lambda f(x,u) && \text{in } B \\ u &= 0 && \text{on } \partial B \end{aligned}$$

for $\lambda \leq 1$ must have a complicated structure in the case where the first $(n+1)$ eigenvalues of (8) are simple. (The counter-example can be chosen so this holds.)

The above methods can also be used to obtain partial results on Lazer and McKenna's conjecture [11] on the number of solutions of asymptotically homogeneous problems (as in [5] or [8]) and to construct a counter-example to a slight variant of their conjecture.

Lastly, we illustrate very briefly the use of Morse decompositions. Consider (7) again but with different assumptions on a and b (where, for simplicity, we now assume f is independent of x). We assume that $b > a > 0$, that neither a nor b are eigenvalues of $-\Delta$ (for Dirichlet boundary conditions) that a is greater than the first eigenvalue of $-\Delta$, that (a,b) contains an eigenvalue of $-\Delta$ and finally that $tf(t) > 0$ for $t \neq 0$. Then (7) has a solution which changes sign in Ω . We sketch a proof avoiding numerous technical details. Assume the result is false. We then have a Morse decomposition of the solutions $M_1 \cup M_2 \cup M_3$, where M_1 denotes the non-trivial positive solutions, M_2 is the non-trivial negative solutions and $M_3 = \{0\}$. (To see that this is indeed a Morse decomposition, one uses the maximum principle to see that, if a solution $u(t,x)$ of the natural parabolic equation corresponding to (7) is positive in Ω for some $\tau > 0$, then the same is true for $t > \tau$. If one replaces f by λf and choose λ large, we see that M_1 is empty. Since one can use homotopy invariance arguments to prove that $B_j(\lambda f, M_1)$ is independent of λ , it follows that $B_j(f, M_1) = 0$ for all j . Similarly, $B_j(f, M_2) = 0$ for all j . One can then easily get a contradiction to the Morse inequalities (6) (since (a,b) intersects the eigenvalue of $-\Delta$). The above result admits a number of variants.

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