APPROXIMATION BY COMPACT OPERATORS BETWEEN CLASSICAL FUNCTION SPACES

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Interest in approximating a bounded linear operator T on a Hilbert space H originated with Gohberg and Krein [7, Section II.7]. They showed, constructively, that there is always a compact operator C which minimizes $\|T - C\|$. In contemporary terminology, the compact operators $K(H)$ form a proximinal subspace of $B(H)$. Another constructive proof of this fact was later given by Holmes and Kripke [9], and a comparison of the two constructions was made by Bouldin [4]. An abstract proof has also been given by Alfsen and Effros [1, Corollary 5.6].

More recently, various authors [2,3,11,12, 13, 16] have considered this problem for operators between general Banach spaces E and F. For which E and F is $K(E,F)$ a proximinal subspace of $B(E,F)$? In this expository talk, we will summarize what is known when E and F are classical function spaces - that is, $C(X)$, where X is compact and Hausdorff, $L_p(\mu)$ where $1 \le p < \infty$, or the sequence space c_0. There is no need to consider $L_\infty(\mu)$ since every such space is isometric to some $C(X)$. It will, of course, be necessary to distinguish the cases $p = 1$ and $p > 1$. Our first result establishes proximinality in the case $F = c_0$. We remark that this is nontrivial, since $K(E,c_0)$ is always a proper subspace of $B(E,c_0)$, when E is infinite dimensional, by [10] or [14].

THEOREM 1. [2] *For any Banach space* E , $K(E,c_0)$ *is proximinal in* $B(E,c_0)$.

PROOF. Given $T \in B(E,c_0)$ we have $T^*e_n \to 0$ (weak*) where $T^* : \ell_1 \to E^*$ and (e_n) is the usual basis for ℓ_1 . Let $d = \limsup \|T^*e_n\|$ and, for $n \in \mathbb{N}$, let $r_n = \max\{0, 1 - d/\|T^*e_n\|\}$. If C is any compact operator, then $\|C^*e_n\| \to 0$ and so $\|T - C\| \geq d$. Let $D \in B(c_0)$ be the diagonal operator determined by the sequence (r_n) . Since $r_n \to 0$, D is compact and so is $C = DT$. Finally $\|T - C\| = \sup \|T^*e_n - C^*e_n\| \leq d$. So C is a compact approximant to T .

Establishing proximinality of the compact operators in the remaining cases is more difficult. The most general condition sufficient for proximinality was defined by Lau [11]. He calls $K(E,F)$ a U-proximinal subspace if for all $\varepsilon > 0$, there exists a $\delta > 0$ (depending only on ε) such that, for all $T \in B(E,F)$ and $C \in K(E,F)$ with $\|T\| \leq 1$ and $\|T+C\| \leq 1 + \delta$, there exist $\tilde{T} \in B(E,F)$ and $\tilde{C} \in K(E,F)$ with $T+C = \tilde{T}+\tilde{C}$, $\|\tilde{T}\| \leq 1$ and $\|\tilde{C}\| \leq \varepsilon$. This generalizes several sufficient conditions considered by other authors. In most cases, proximinality of the compact operators is established by first proving U-proximinality. The proof of Theorem 1 can easily be modified to show that $K(E,c_0)$ is U-proximinal in $B(E,c_0)$. The other positive results are summarized as follows.

THEOREM 2. $K(E,F)$ *is proximinal in* $B(E,F)$ *in each of the following cases.*

(1A) $E = c_0$, $F = C(X)$

(1B) $E = c_0$, $F = L_1(\mu)$

(1C) $E = c_0$, $F = L_p(\mu)$, *where* $1 < p < \infty$

(2A) $E = C(X)$, $F = C(Y)$, *where* X *is dispersed and* Y *is Stonean or* $X = Y$ *is the one-point compactification of a discrete set, and the scalars are real*

(3A) $E = L_1(\mu)$, $F = C(X)$

(3B) $E = L_1(\mu)$, $F = \ell_1(\Gamma)$, *where* Γ *is discrete*

(3C) $E = L_1(\mu)$, $F = L_p(\nu)$, *where* $1 < p < \infty$

(4A) $E = L_p(\mu)$, *where* $1 < p < \infty$, $F = C(X)$

(4C) $E = \ell_p(\Gamma)$, $F = \ell_q(\Delta)$ *where* $1 < p,q < \infty$ *and* Γ , Δ *are discrete.*

PROOF. (1A). See Mach [12].

(1B) All operators are compact, by [15].

(1C) Since F is reflexive, an application of Schur's lemma shows that all operators are compact.

(2A) See [16].

(3A) This follows from the representation theorem for operators taking values in $C(X)$ [5, Theorem $\overline{\text{IV}}$.7.1] and the well known fact every subalgebra of a $C(X)$ space is proximinal.

(3B) U-proximinality was established first in [11], then by a different method in [16].

(3C) This follows by duality from case (4A).

(4A) This is a special case of [13, Corollary 6].

(4C) For $p \leq q$, this follows from the methods of [8]. For $p > q$, every operator is compact [15].

The first negative result was due to Feder [6] who showed, amongst other things, that $K(\ell_\infty)$ is not proximinal in $B(\ell_\infty)$. His results depend on the observation that $K(\ell_1,E)$ is proximinal in $B(\ell_1,E)$ if and only if {compact subsets of E} is a proximinal subset of {closed, bounded subsets of E} , with respect to the Hausdorff metric. He then constructs, with some difficulty, a subset of $L_1(0,1)$ with no best compact approximant. Further negative results follow by duality. We summarize them.

THEOREM 3. *In each of the following cases,* $K(E,F)$ *is not proximinal in* $B(E,F)$.

(2A) $E = C(X)$, $F = C(Y)$ *where* X *contains a perfect subset and* Y *is Stonean or* X *and* Y *both contain copies of the Cantor set or* X *contains the Cantor set and* Y *contains* ω^2 .

(3B) $E = L_1(\mu)$, $F = L_1(\nu)$ *where* ν *is not a discrete measure.*

PROOF. (2A) See [3], [6] or [16].

(3B) This follows from [6, Theorem 3].

Looking over these results, we see that nothing at all is known about the cases which would be numbered (2B), (2C) and (4B), and that the classification is incomplete in several other cases. Of the various questions left open, the following seem to be the most interesting.

PROBLEM 1. *Is* $K(C(X), L_p(\mu))$ *proximinal in* $B(C(X), L_p(\mu))$ *?*

PROBLEM 2. *Is* $K(L_r(\nu), L_p(\mu))$ *proximinal in* $B(L_r(\nu), L_p(\mu))$ *when* $r > 1$ *and* μ, ν *are not necessarily discrete?*

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