ERROR ESTIMATES FOR A FIRST KIND INTEGRAL EQUATION AND AN ASSOCIATED BOUNDARY VALUE PROBLEM

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1. INTRODUCTION

This paper deals with an integral equation method for obtaining numerical solutions to the two-dimensional interior and exterior Dirichlet problem

(1.1a) $$\Delta U = 0 \quad \text{on} \quad \mathbb{R}^2 \backslash \Gamma$$

(1.1b) $$U = g \quad \text{on} \quad \Gamma$$

(1.1c) $$U(X) = O(1) \quad \text{as} \quad |X| \to \infty .$$

Here

$$\Delta = \left(\frac{\partial}{\partial X_1}\right)^2 + \left(\frac{\partial}{\partial X_2}\right)^2 , \qquad X = (X_1, X_2) \in \mathbb{R}^2$$

is the Laplacian, Γ is a simple closed C^∞ curve in the plane, and g is a given function on Γ. It will sometimes be necessary to refer to the bounded and unbounded components of $\mathbb{R}^2 \backslash \Gamma$, and these will be denoted by Ω_+ and Ω_- respectively. Also, $\nu(Y)$ denotes the unit normal at $Y \in \Gamma$ pointing into Ω_+, and σ denotes the arc length measure on Γ.

The boundary value problem (1.1) can be reduced to an integral equation on Γ by seeking a representation of the solution in the form of a single layer potential

(1.2) $$U(X) = \frac{1}{\pi} \int_\Gamma \log\left(\frac{1}{|X - Y|}\right) v(Y)\, d\sigma(Y) + \omega , \qquad X \in \mathbb{R}^2 ,$$

with v an unknown function on Γ and ω an unknown constant. If $v \in L_p(\Gamma)$ for some $p > 1$ then the formula (1.2) defines a function

U which is harmonic on $\mathbb{R}^2\backslash\Gamma$ and continuous on $\mathbb{R}^2$, hence (1.1a) and (1.1b) are satisfied if the pair (v,ω) is a solution of the first kind integral equation

$$\frac{1}{\pi}\int_\Gamma \log\left(\frac{1}{|X-Y|}\right) v(Y)\, d\sigma(Y) + \omega = g(X)\,, \qquad X \in \Gamma\,. \tag{1.3a}$$

Also, U is bounded at infinity (1.1c) iff

$$\int_\Gamma v\, d\sigma = 0\,, \tag{1.3b}$$

in which case

$$U(\infty) = \omega\,. \tag{1.4}$$

Methods for the numerical solution of (1.3) have been investigated by various authors including [1], [3], [4], [5], [6] and [7].

An alternative and more traditional approach is to represent the solution to (1.1) using double layer potentials

$$\frac{1}{\pi}\int_\Gamma \frac{\nu(Y)\cdot(X-Y)}{|X-Y|^2} v_\pm(Y)\, d\sigma(Y)\,, \qquad X \in \Omega_\pm\,.$$

This has the advantage of leading to integral equations of the second kind which are easier to handle numerically than (1.3), but on the other hand the single layer potential (1.2) yields a much simpler representation for the gradient of U, namely

$$\nabla U(X) = \frac{1}{\pi}\int_\Gamma \frac{Y-X}{|Y-X|^2} v(Y)\, d\sigma(Y)\,, \qquad X \in \mathbb{R}^2\backslash\Gamma\,.$$

Moreover, if v is Hölder continuous then the boundary values of the gradient

$$(\nabla U)_\pm(X) = \lim_{Z\to X,\ Z\in\Omega_\pm} \nabla U(Z)\,, \qquad X \in \Gamma$$

exist (classically) and are given by the formula

$$(\nabla U)_\pm(X) = \left(\mp v(X)\ \nu(X) + \frac{1}{\pi}\int_\Gamma \frac{Y-X}{|Y-X|} v(Y)\, d\sigma(Y)\right., \qquad x \in \Gamma\,,$$

where $⨍$ indicates a Cauchy principal value integral. In particular the normal derivatives of U are

$$(1.5)\quad \left(\frac{\partial U}{\partial \nu}\right)_{\pm}(X) \qquad \mp v(X) + \frac{1}{\pi}\int_{\Gamma} \frac{\nu(X)\cdot(Y-X)}{|Y-X|^2}\, v(Y)\, d\sigma(Y) \quad , \qquad X \in \Gamma \ ,$$

and here there is no need for a Cauchy principal value because the function

$$(X,Y) \mapsto \frac{\nu(X)\cdot(Y-X)}{|Y-X|^2}$$

is in $C^{\infty}(\Gamma^2)$.

2. THE INTEGRAL EQUATION ON THE TORUS

The one dimensional torus

$$\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$$

is the group of real numbers under addition modulo 2π , and we shall usually think of the elements of $\mathbb{T}$ as points in the interval between 0 and 2π . Parametrize Γ using a C^{∞} diffeomorphism

$$\gamma = (\gamma_1, \gamma_2) : \mathbb{T} \to \Gamma$$

and note that the derivative $\dot{\gamma}$ satisfies

$$|\dot{\gamma}(t)| \neq 0 \quad \text{for all } t \ .$$

Let

$$(2.1)\qquad u(t) = v[\gamma(t)]\ |\dot{\gamma}(t)| \qquad f(t) = g[\gamma(t)]$$

then (1.3) can be rewritten as

$$(2.2a)\quad \frac{1}{\pi}\int_0^{2\pi} \log\left(\frac{1}{|\gamma(x) - \gamma(t)|}\right) u(t)\ dt + \omega = f(x) \quad , \qquad 0 \le x \le 2\pi$$

$$(2.2b)\qquad \int_0^{2\pi} u(t)\ dt = 0 \ .$$

In the special case where Γ is the unit circle one can take

$$\gamma(t) = e^{it} \ , \qquad 0 \le t \le 2\pi \ ,$$

and then

$$|\gamma(x) - \gamma(t)| = 2\left|\sin\left(\frac{x-t}{2}\right)\right| .$$

In the general case we use the decomposition

$$\frac{1}{\pi}\log\left(\frac{1}{|\gamma(x)-\gamma(t)|}\right) = \Lambda(x-t) + k(x,t)$$

where

$$\Lambda(t) = \frac{1}{\pi}\log\left(\frac{1}{2|\sin(t/2)|}\right) \tag{2.3}$$

is singular but

$$k(x,t) = \begin{cases} \frac{1}{\pi}\log\left(\frac{2\left|\sin\left(\frac{x-t}{2}\right)\right|}{|\gamma(x)-\gamma(t)|}\right) & \text{if } x \neq t \pmod{2\pi} \\ \frac{1}{\pi}\log\left(\frac{1}{|\dot{\gamma}(t)|}\right) & \text{if } x = t \pmod{2\pi} \end{cases}$$

satisfies

$$k \in C^{\infty}(\mathbb{T}^2) . \tag{2.4}$$

Define the integral operators

$$Au(x) = \Lambda * u(x) = \int_0^{2\pi} \Lambda(x-t)\ u(t)\ dt$$

$$Ku(x) = \int_0^{2\pi} k(x,t)\ u(t)\ dt$$

then (2.2) can be written

$$(A+K)u + \omega = f \quad \text{on } \mathbb{T} \tag{2.5a}$$

$$\int_0^{2\pi} u\ dt = 0 . \tag{2.5b}$$

In section 5 we shall present a numerical method for obtaining approximate solutions to these equations, and will then carry out an error analysis. The latter relies on existence, uniqueness and regularity results which are proved in section 4 and which involve the function spaces to be introduced below.

3. BESOV SPACES

The books [2] and [10] are suitable background references for the material of this section, and our notation is consistent with them.

Since $\mathbb{T}$ is a one dimensional C^∞ manifold the Sobolev spaces

$$W_p^s = W_p^s(\mathbb{T}) \ , \quad s \in \mathbb{Z} \ , \quad 1 < p < \infty$$

and the Besov spaces

$$B_{pq}^s = B_{pq}^s(\mathbb{T}) \ , \quad s \in \mathbb{R} , \quad 1 < p < \infty \ , \quad 1 \le q \le \infty$$

can be defined in terms of the corresponding spaces on the real line, but they can also be described directly as follows.

Denote the norm on $L_p(\mathbb{T})$ by

$$\|u\|_p = \left(\int_0^{2\pi} |u(t)|^p \, dt\right)^{1/p}$$

and let D be the operator of differentiation in the sense of distributions on $\mathbb{T}$. For integers $s \ge 0$ the space $W_p^s(\mathbb{T})$ consists of those distributions u on $\mathbb{T}$ for which the norm

$$\|u\|_{W_p^s} = \sum_{\ell=0}^{s} \|D^\ell u\|_p$$

is finite, whereas for integers $s < 0$

$$\|u\|_{W_p^s} = \sup\{|\langle u,\varphi\rangle| \ : \ \varphi \in C^\infty(\mathbb{T}) \quad \text{and} \quad \|\varphi\|_{W_{p'}^{-s}} \le 1\} \ .$$

With respect to these norms W_p^s is a reflexive Banach space (note $p \ne 1,\infty$) , and satisfies the duality relation

$$(3.1) \qquad (W_p^s)' = W_{p'}^{-s} \ , \qquad \frac{1}{p} + \frac{1}{p'} = 1 \ .$$

The Besov spaces arise naturally by real interpolation of the Sobolev spaces, with

$$(3.2) \qquad B_{pq}^s = (W_p^{s_0}, W_p^{s_1})_{\theta,q}$$

for integers $s_0 \neq s_1$, and

$$s = (1-\theta)s_0 + \theta s_1 , \qquad 0 < \theta < 1 .$$

In order to write down a convenient norm for $B^s_{pq}(\mathbb{T})$ we need some more notation. Let

$$\hat{u}_m = \int_0^{2\pi} e^{-imt}\, u(t)\, dt , \qquad m \in \mathbb{Z}$$

and define the function e_m by

$$e_m(t) = e^{imt} ,$$

then the Fourier series for u is

$$u = \frac{1}{2\pi} \sum_{m\in\mathbb{Z}} \hat{u}_m\, e_m .$$

For any distribution u on $\mathbb{T}$, the sum converges in the sense of distributions and can be decomposed into

(3.3)
$$u = \frac{1}{2\pi} \sum_{j=0}^{\infty} \Sigma^{(j)} u$$

where

$$\Sigma^{(j)} u = \begin{cases} \hat{u}_0\, e_0 & \text{if } j = 0 \\ \displaystyle\sum_{2^{j-1} \le |m| < 2^j} \hat{u}_m\, e_m & \text{if } j \neq 0 \end{cases} .$$

One can now put

$$\|u\|_{B^s_{pq}} = \begin{cases} \left(\displaystyle\sum_{j=0}^{\infty} \left(2^{sj}\, \|\Sigma^{(j)} u\|_p \right)^q \right)^{1/q} & \text{if } 1 \le q < \infty \\ \displaystyle\sup_{j\ge 0} 2^{sj}\, \|\Sigma^{(j)} u\|_p & \text{if } q = \infty . \end{cases}$$

If E is any of the above spaces of distributions on $\mathbb{T}$, then we shall write $\mathring{E}$ for the closed subspace of E consisting of those distributions u having zero mean value on $\mathbb{T}$, in other words

$$\mathring{E} = \{u \in E : \hat{u}_0 = 0\} .$$

Also, the notation

$$J : E \rightleftharpoons F$$

will indicate that J is an isomorphism of E onto F in the category of Banach spaces.

4. SOLVABILITY OF THE INTEGRAL EQUATIONS

In this section we first study A as an operator on the Besov spaces introduced above, and then deduce the required results for the integral equation (2.5).

Observe that the derivative of the function (2.3) is

$$D\Lambda(t) = -\frac{1}{2\pi}\cot\mathrm{an}(t/2) ,$$

hence the operator A is related to the Hilbert transform

$$Hu(x) = \frac{1}{2\pi i}\int_0^{2\pi} \mathrm{cotan}\left(\frac{t-x}{2}\right) u(t)\, dt$$

by

(4.1) $$DA = iH ,$$

a fact which makes it easy to prove the following.

4.1 Theorem. (i) *For* $s \in \mathbb{R}$, $1 < p < \infty$ *and* $1 \le q \le \infty$,

$$A : \overset{\circ}{B}{}^{s}_{pq}(\mathbb{T}) \rightleftharpoons \overset{\circ}{B}{}^{s+1}_{pq}(\mathbb{T}) .$$

(ii) *For* $m \in \mathbb{Z}$ *and* u *a distribution on* $\mathbb{T}$,

$$(Au)^\wedge_m = \lambda_m \hat{u}_m$$

where

$$\lambda_m = \hat{\Lambda}_m = \begin{cases} 0 & \text{if } m = 0 \\ \dfrac{1}{|m|} & \text{if } m \ne 0 . \end{cases}$$

Proof. (i) It suffices to show

$$A : \overset{\circ}{W}{}^{s}_{p}(\mathbb{T}) \rightleftharpoons \overset{\circ}{W}{}^{s+1}_{p}(\mathbb{T}) \ , \qquad s \in \mathbb{Z} \ , \quad 1 < p < \infty$$

since the result then follows by interpolation (3.2). Moreover, A has a symmetric kernel so by duality (3.1) it suffices to consider integers $s \geq 0$. Finally, because D comutes with A and satisfies

$$D : \overset{\circ}{W}{}^{s+1}_{p}(\mathbb{T}) \rightleftharpoons \overset{\circ}{W}{}^{s}_{p}(\mathbb{T}) \ ,$$

one has only to use (4.1) and the well known result [11, p.159]

$$H : \overset{\circ}{L}_{p}(\mathbb{T}) \rightleftharpoons \overset{\circ}{L}_{p}(\mathbb{T}) \ , \qquad 1 < p < \infty \ .$$

(ii) By a familiar property of convolutions,

$$(Au)^{\wedge}_{m} = (\Lambda * u)^{\wedge}_{m} = \hat{\Lambda}_{m} \, \hat{u}_{m}$$

and hence $\lambda_m = \hat{\Lambda}_m$. Using the change of variable $z = e^{it}$,

$$\lambda_0 = \int_0^{2\pi} \Lambda(t) \, dt = \mathrm{Re}\left(\frac{1}{2\pi i} \int_{|z|=1} \frac{2}{z} \log\left(\frac{1}{z-1}\right) dz\right) = 0 \ ,$$

while for $m > 0$ the change of variable $z = e^{-it}$ gives

$$\mathrm{i}m \, \lambda_m = (D\Lambda)^{\wedge}_{m} = i \, \frac{1}{2\pi i} \int_{|z|=1} z^{m-1} \, \frac{z+1}{z-1} \, dz$$

$$= i \, \frac{1}{2} \operatorname*{res}_{z=1} z^{m-1} \, \frac{z+1}{z-1} = i \ ,$$

so $\lambda_m = 1/m$. Finally, since Λ is an even function $\lambda_m = \lambda_{-m}$ and therefore $\lambda_m = 1/(-m)$ for $m < 0$. □

Introduce the operator

$$\mathcal{A}(u,\omega) = (A + K)u + \omega$$

so that the integral equation (2.5) can be written

$$\mathcal{A}(u,\omega) = f \tag{4.2a}$$

(4.2b) $$\hat{u}_0 = 0 \ .$$

Let $\mathbb{K}$ denote either the real or complex number field, then the main result for this section can be stated as follows.

4.2 Theorem. *For* $s \in \mathbb{R}$, $1 < p < \infty$ *and* $1 \le q \le \infty$

$$A : \overset{\circ}{B}{}^{s}_{pq}(\mathbb{T}) \times \mathbb{K} \rightleftharpoons B^{s+1}_{pq}(\mathbb{T}) \ .$$

Proof. Decompose A into $A = A_1 + K$ where $A_1(u,\omega) = Au + \omega$ and $K(u,\alpha) = Ku$, then A_1 is an isomorphism by 4.1 and K is compact because K has a smooth kernel (2.4). Thus, A is a Fredholm operator with zero index, and it suffices to show A is one-one. Suppose $A(u,\alpha) = 0$ with $\hat{u}_0 = 0$, and let v be the corresponding function on Γ satisfying (2.1). It follows from 4.1 that v is smooth, so the function U defined by (1.2) solves the homogeneous Dirichlet problem, i.e. (1.1) holds with $g = 0$. Since U is bounded at infinity the maximum principle applies on the exterior region Ω_- as well as on the interior region Ω_+, hence $U = 0$ on $\mathbb{R}^2$. By (1.4), this means $\omega = 0$, and finally by (1.5),

$$2v = \left(\frac{\partial U}{\partial \nu}\right)_- - \left(\frac{\partial U}{\partial \nu}\right)_+ = 0 \quad \text{on } \Gamma \ . \qquad \square$$

5. A NUMERICAL METHOD

We shall now analyse a Galerkin method for obtaining numerical solutions to the integral equation (4.2).

Denote the space of trigonometric polynomials of degree $\le n$ by

$$T_n = \operatorname{span}\,\{e_m : |m| \le n\} \ ,$$

and denote the normalized L_2 inner product by

$$(\varphi|u) = \frac{1}{2\pi}\int_0^{2\pi} \overline{\varphi(t)}\, u(t)\, dt\ ,$$

then look for $(u_n,\omega_n) \in \mathring{\mathcal{T}}_n \times \mathbb{K}$ satisfying the Galerkin equations

(5.1) $$(\varphi \mid (A+K)u_n + \omega_n) = (\varphi|f) \quad \text{for all} \quad \varphi \in \mathcal{T}_n\ .$$

Since 4.1(ii) implies

$$(e_\ell \mid Ae_m) = \lambda_m\, \delta_{\ell m}\ ,$$

if we write

(5.2) $$u_n = \sum_{1\le|m|\le n} z_m \frac{e_m}{\lambda_m}$$

then the unknowns $z_{\pm1},\ldots,z_{\pm n}$ and ω_n must satisfy the $(2n+1)\times(2n+1)$ linear algebraic system

$$\sum_{1\le|m|\le n} \left(\delta_{\ell m} + \frac{1}{\lambda_m}(e_\ell|Ke_m)\right) z_m + \delta_{\ell 0}\omega_n = (e_\ell|f)\ , \qquad |\ell| \le n\ .$$

The inclusion of the factor $1/\lambda_m$ in the mth term of the expansion (5.2) guarantees that the ℓ_2 condition number of the coefficient matrix is bounded as $n \to \infty$.

If the operator

$$P_n : L_2(\mathbb{T}) \to \mathcal{T}_n$$

is the orthogonal projection onto the subspace $\mathcal{T}_n$, then the Galerkin equations (5.1) are equivalent to

$$P_n[(A+K)u_n + \omega_n] = P_n f\ .$$

Noting that

$$P_n u = \frac{1}{2\pi} \sum_{|m|\le n} \hat{u}_m\, e_m$$

is just a truncation of the Fourier series for u , it is clear P_n commutes with A and satisfies $P_n 1 = 1$. Therefore, if the operator $\mathcal{A}_n$ is defined by

$$\mathcal{A}_n(u,\omega) = (A + P_nK)u + \omega\ ,$$

then the Galerkin solution $(u_n, \omega_n) \in \mathring{T}_n \times \mathbb{K}$ must satisfy

(5.3) $$A_n(u_n, \omega_n) = P_n f \ .$$

In the error analysis which follows, the generic constant c is independent of n and (u, ω) .

5.1 Theorem. *For* $s \in \mathbb{R}$, $1 < p < \infty$ *and* $1 \leq q \leq \infty$, *the solutions of* (4.2) *and* (5.3) *satisfy the asymptotic error estimate*

$$\|u_n - u\|_{\mathring{B}^s_{pq}(\mathbb{T})} + |\omega_n - \omega| \leq c\|(I - P_n)u\|_{B^s_{pq}(\mathbb{T})} \ .$$

Proof. First note

$$(A_n - A)(u, \omega) = (P_n - I)Ku \ ,$$

then observe that since K has a smooth kernel (2.4),

$$\lim_{n\to\infty} \|(I - P_n)K : \mathring{B}^s_{pq}(\mathbb{T}) \to B^{s+1}_{pq}(\mathbb{T})\| = 0 \ .$$

(This can be proved easily using 5.2 below.) Hence, A_n converges to A in the operator norm and the stability estimate

$$\|A_n^{-1} : B^{s+1}_{pq}(\mathbb{T}) \to \mathring{B}^s_{pq}(\mathbb{T}) \times \mathbb{K}\| \leq c$$

follows immediately from 4.2. Now use the identity

$$A_n(u_n - u,\ \omega_n - \omega) = (P_n - I)Au = A(P_n - I)u$$

and the inequality

$$\|A(P_n - I)u\|_{B^{s+1}_{pq}} \leq c\|(I - P_n)u\|_{B^s_{pq}} \ . \qquad \square$$

In order to deduce the rate of convergence of the Galerkin solution (u_n, ω_n) , we shall now estimate

$$(I - P_n)u = \frac{1}{2\pi} \sum_{|m|>n} \hat{u}_m e_m \ ,$$

which is just the tail of the Fourier series of u .

5.2 Theorem. *For* $-\infty < s < r < \infty$, $1 < p < \infty$ *and* $1 \le q \le \infty$,

$$\|(I - P_n)u\|_{B^s_{pq}(\mathbb{T})} \le c\, n^{s-r} \|u\|_{B^r_{p\infty}(\mathbb{T})} .$$

Proof. The imbedding $B^s_{p1} \subset B^s_{pq}$ means that it suffices to consider $q = 1$. Given $n \ge 1$, let ℓ be the unique integer satisfying $2^{\ell-1} \le n < 2^{\ell}$, then

$$\|\Sigma^{(j)}(I - P_n)u\|_p = \begin{cases} 0 & \text{if } 0 \le j \le \ell-1 \\ \|\sum\limits_{n<|m|<2^{\ell}} \hat{u}_m e_m\|_p & \text{if } j = \ell \\ \|\Sigma^{(j)} u\|_p & \text{if } j \ge \ell+1 . \end{cases}$$

The Marcinkiewicz multiplier theorem [9] implies

$$\|\sum_{n<|m|<2^{\ell}} \hat{u}_m e_m\|_p \le c \|\Sigma^{(\ell)} u\|_p$$

with c independent of n , and therefore

$$\begin{aligned} \|(I - P_n)u\|_{B^s_{p1}} &\le c \sum_{j=\ell}^{\infty} 2^{sj} \|\Sigma^{(j)} u\|_p \\ &\le c\left(\sum_{j=\ell}^{\infty} 2^{(s-r)j}\right) \sup_{j\ge\ell} 2^{rj} \|\Sigma^{(j)} u\|_p \\ &\le c\, n^{s-r} \|u\|_{B^r_{p\infty}} . \end{aligned}$$

□

6. CONCLUSION

Let v_n be the function on Γ satisfying

$$u_n(t) = v_n[\gamma(t)]\, |\dot{\gamma}(t)| ,$$

and denote the corresponding single layer potential by

$$U_n(X) = \frac{1}{\pi} \int_{\Gamma} \log\left(\frac{1}{|X - Y|}\right) v_n(Y)\, d\sigma(Y) + \omega_n , \qquad X \in \mathbb{R}^2 .$$

The function U_n furnishes an approximation to the solution U of the Dirichlet problem (1.1), and the results of the last section will now be used to estimate the error $U_n - U$. We write

$$\partial^{\alpha} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_2}\right)^{\alpha_2}, \qquad |\alpha| = \alpha_1 + \alpha_2$$

where $\alpha = (\alpha_1, \alpha_2)$ is an ordered pair of nonnegative integers.

6.1 Lemma. *If* U *is the solution of the Dirichlet problem* (1.1), *then the boundary values of its partial derivatives satisfy*

$$\|(\partial^{\alpha}U)_{\pm}\|_{B^{s}_{pq}(\Gamma)} \leq c\, \|g\|_{B^{s+|\alpha|}_{pq}(\Gamma)}$$

for $s > 0$, $1 < p < \infty$ *and* $1 \leq q \leq \infty$.

Proof. Since U is bounded at infinity the two dimensional Kelvin transform reduces the problem to estimating $(\partial^{\alpha}U)_{+}$. This is done using the trace theorem [10, 4.7.1]

$$\|(\partial^{\alpha}U)_{+}\|_{B^{s}_{pq}(\Gamma)} \leq c\, \|\partial^{\alpha}U\|_{B^{s+1/p}_{pq}(\Omega_+)}, \qquad s > 0$$

$$\leq c\, \|U\|_{B^{s+|\alpha|+1/p}_{pq}(\Omega_+)}$$

and the regularity theorem [10, 5.5.2 (including Remark 1*)]

$$\|U\|_{B^{s+|\alpha|+1/p}_{pq}(\Omega_+)} \leq c\, \|g\|_{B^{s+|\alpha|}_{pq}(\Gamma)}. \qquad \square$$

6.2 Theorem. *For* $1 < p < \infty$ *and* $r > |\alpha| + \frac{1}{p}$,

$$\|\partial^{\alpha}(U_n - U)\|_{L_{\infty}(\mathbb{R}^2\setminus\Gamma)} \leq c\, n^{|\alpha| + 1/p - r}\, \|g\|_{B^{r}_{p\infty}(\Gamma)}.$$

Proof. Since $\partial^{\alpha}(U_n - U)$ is harmonic on $\mathbb{R}^2\setminus\Gamma = \Omega_+ \cup \Omega_-$ and is bounded at infinity, the maximum principle implies

$$\|\partial^{\alpha}(U_n - U)\|_{L_{\infty}(\Omega_{\pm})} \leq \|[\partial^{\alpha}(U_n - U)]_{\pm}\|_{L_{\infty}(\Gamma)}.$$

In the appendix we give an elementary proof of the imbedding

$$B^{1/p}_{p1}(\Gamma) \subset C(\Gamma) \,, \tag{6.1}$$

which means

$$\| [\partial^\alpha(U_n - U)]_\pm \|_{L_\infty(\Gamma)} \le c \| [\partial^\alpha(U_n - U)]_\pm \|_{B^{1/p}_{p1}(\Gamma)} \,.$$

Now put $g_n = U_n|\Gamma$ and apply 6.1 to the function $U_n - U$ to obtain

$$\| [\partial^\alpha(U_n - U)]_\pm \|_{B^{1/p}_{p1}(\Gamma)} \le c \, \| g_n - g \|_{B^{|\alpha|+1/p}_{p1}(\Gamma)} \,.$$

Finally let $s = |\alpha| + 1/p$, then by 4.2, 5.1 and 5.2

$$\begin{aligned}
\| g_n - g \|_{B^s_{p1}(\Gamma)} &\le c \| A(u_n - u, \omega_n - \omega) \|_{B^s_{p1}(\mathbb{T})} \\
&\le c (\| u_n - u \|_{B^{s-1}_{p1}(\mathbb{T})} + |\omega_n - \omega|) \\
&\le c \, n^{(s-1)-(r-1)} \| u \|_{B^{r-1}_{p\infty}(\mathbb{T})} \\
&\le c \, n^{s-r} \| g \|_{B^r_{p\infty}(\Gamma)} \,.
\end{aligned}$$

□

Thus, the uniform rate of convergence of $\partial^\alpha U_n$ is limited only by the smoothness of the data g . Moreover, the next result shows that away from Γ the convergence is faster than any power of n^{-1} , even when the data is far from smooth.

6.3 Theorem. *Fix* $\delta > 0$ *and let*

$$\Omega_\delta = \{ X \in \mathbb{R}^2 : \mathrm{dist}(X, \Gamma) > \delta \}$$

then for $-\infty < s < r < \infty$,

$$\| \partial^\alpha (U_n - U) \|_{L_\infty(\Omega_\delta)} \le c \, n^{s-r} \| g \|_{B^r_{p\infty}(\Gamma)} \,,$$

where c *depends on* δ , α , s , r *and* p .

Proof. The function

$$(X,Y) \mapsto \log\left(\frac{1}{|X-Y|}\right)$$

is C^∞ for $|X-Y| > \delta\cdot$, and consequently

$$\|\partial^\alpha(U_n - U)\|_{L_\infty(\Omega_\delta)} \le c(\|u_n - u\|_{B^{s-1}_{pq}(\mathbb{T})} + |\omega_n - \omega|)$$

$$\le c\, n^{s-r} \|g\|_{B^r_{p\infty}(\Gamma)} . \qquad \square$$

We have been content to present here these basic error estimates, and refer to [8] for a discussion of some practical aspects of implementing the numerical method such as quadrature errors and fast Fourier transforms. In conclusion, I wish to express my thanks to Professor H. Triebel for a number of conversations which were helpful in the preparation of this paper.

APPENDIX

Here is a proof of the imbedding (6.1).

Lemma. *If* $1 \le p \le q \le \infty$ *then there is a constant* c *such that*

$$\|u\|_q \le c\, n^{1/p - 1/q} \|u\|_p \quad \text{for all} \quad u \in T_n .$$

Proof. Denote the Dirichlet kernel by

$$d_n(t) = \frac{1}{2\pi} \sum_{|m|\le n} e^{imt} = \frac{1}{2\pi} \frac{\sin[(n+\frac{1}{2})t]}{\sin(t/2)}$$

then for $u \in T_n$,

$$u = d_n * u$$

and hence by Young's inequality

$$\|u\|_q \le \|d_n\|_\rho \|u\|_p , \qquad 1 - \frac{1}{\rho} = \frac{1}{p} - \frac{1}{q} .$$

When $p = q$ the result is trivial, thus it suffices to show

$$\|d_n\|_\rho \le c\, n^{1-1/\rho} \qquad \text{for } \rho > 1 .$$

If $\rho = \infty$ then

$$\|d_n\|_\infty = d_n(0) = \frac{1}{2\pi}(2n+1) \le c\, n ,$$

and since

$$|d_n(t)| \le \frac{1}{2t}, \qquad 0 < t < \pi ,$$

if $1 < \rho < \infty$ then

$$\int_0^{2\pi} |d_n(t)|^\rho \, dt = 2 \int_0^{\pi} |d_n(t)|^\rho \, dt$$

$$\le c \left(\int_0^{\pi/n} n^\rho \, dt + \int_{\pi/n}^{\pi} t^{-\rho} \, dt \right)$$

$$\le c\, n^{\rho - 1} . \qquad \square$$

Theorem. *For* $1 < p < \infty$ *there are imbeddings*

$$B_{p1}^{1/p - 1/q}(\mathbb{T}) \subset L_q(\mathbb{T}) , \qquad p \le q \le \infty$$

$$B_{p1}^{1/p}(\mathbb{T}) \subset C(\mathbb{T}) .$$

Proof. Since $\Sigma^{(j)} u \in T_{2^j}$ the lemma implies

$$\|\Sigma^{(j)} u\|_q \le c\, 2^{(1/p - 1/q)j} \|\Sigma^{(j)} u\|_p$$

and therefore

$$\|u\|_q \le \frac{1}{2\pi} \sum_{j=0}^{\infty} \|\Sigma^{(j)} u\|_q$$

$$\le c \sum_{j=0}^{\infty} 2^{(1/p - 1/q)j} \|\Sigma^{(j)} u\|_p$$

$$= c\, \|u\|_{B_{p1}^{1/p - 1/q}(\mathbb{T})} .$$

In particular, if $u \in B_{p1}^{1/p}(\mathbb{T})$ then the sum (3.3) converges uniformly and since each term $\Sigma^{(j)} u$ is continuous it follows that u is continuous. □

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