AN APPROXIMATION THEOREM FOR ORDER BOUNDED OPERATORS

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The object of this paper is to outline some recent work with P.G. Dodds, B. de Pagter and A.R. Schep [1]. In the following E and F will be Riesz spaces and T will be a positive operator from E to F. For proofs which are not given the reader is referred to a forthcoming paper [1]. Our aim is to approximate in a purely order theoretic way any operator in the order interval $[0,T]$ of the space of all regular operators between E and F with operators of a particularly simple kind with respect to T. For the sake of convenience we will assume that $E = C(K)$ (except in corollary 7), that the normal integrals on F, denoted $F_n^\sim$, separate the points of F and that F is Dedekind complete. The latter has as a consequence that the space of all order bounded (= regular) operators from E to F, denoted by $L_b(E,F)$ is itself a Dedekind complete Riesz space.

Every element $f \in C(K)$ determines a multiplication operator $g \to gf$ on $C(K)$, which is called a multiplier. Abstractly such operators $\sigma : C(K) \to C(K)$ are defined by the conditions that $|\sigma(g)| \wedge |h| = 0$ whenever $|g| \wedge |h| = 0$ and that σ is order bounded.

We are interested in the set of all operators R in $[0,T]$ for which there exist $n \in \mathbb{N}$, multipliers $\sigma_1,\ldots\sigma_n$ on $C(K)$ and order projections $\pi_1,\ldots\pi_n$ on F such that $R = \sum_{i=1}^{n} \pi_i T\sigma_i$. The set of all those operators will be labelled $\ell(T)$. The elements of $\ell(T)$ serve as approximating operators in $[0,T]$.

The following terminology is needed. If L is a Riesz space and

φ is an order bounded functional on L, then $\rho_{\varphi}(f) = |\varphi|(|f|)(f \in L)$ defines a seminorm on L. For a set of order bounded functionals M on L, we define $|\sigma|(L,M)$ to be the locally convex topology generated by the set of all seminorms ρ_{φ} with $\varphi \in M$. Every $0 \leq \varphi \in F_n^{\sim}$ and every $0 \leq x \in E$ determines an element $\Phi_{\varphi,x}$ in the space of normal integrals on $L_b(E,F)$ by $\Phi_{\varphi,x}(S) = \langle Sx,\varphi \rangle$ for all $S \in L_b(E,F)$. Taking $\mathcal{F} = \{\Phi_{\varphi,x} | 0 \leq \varphi \in F_n^{\sim}, 0 \leq x \in E\}$, we have all the notation to state lemma 1.

<u>Lemma 1</u>. $\ell(T)$ is $|\sigma|(L_b(E,F),\mathcal{F})$-dense in $[0,T]$.

The proof of lemma 1 is largely based on a convenient formula for the infimum of two positive operators from E to F. Indeed, for every $0 \leq S,R \in L_b(E,F)$, $0 \leq x \in E$ and $0 \leq \varphi \in F_n^{\sim}$ we have $\langle (R \wedge S)(x),\varphi \rangle =$ $\inf \sum_{i,j} \langle \pi_j Sx_i,\varphi \rangle \wedge \langle \pi_j Rx_i,\varphi \rangle$, where the infimum is taken over all finite subsets $\{x_1,\ldots x_n\} \subset E^+$ with $\sum_i x_i = x$ and all finite subsets of mutually disjoint band projections $\{\pi_1,\ldots,\pi_m\}$ on F with $\sum_j \pi_j = Id_F$.

However, we have in mind a more intrinsic way of characterizing $[0,T]$ in terms of $\ell(T)$. For this purpose we need more structural information about $\ell(T)$. By considering the tensor product of the band projections on F with the multipliers on E we obtain the following result.

<u>Lemma 2</u>. $\ell(T)$ is a sublattice of $[0,T]$.

If L is a Riesz space and K is a subset of L, $\mathcal{I}K$ is defined to be the set of all $f \in L$ for which there exists a subset $\{f_\tau\} \subset K$ with $f_\tau \uparrow_\tau f$. $\mathcal{D}K$ is defined by replacing $\uparrow$ in the preceding sentence by $\downarrow$ (and $\mathcal{I}K$ by $\mathcal{D}K$). The following up-down theorem by D.H. Fremlin suits the situation (see [3]).

Theorem 3. If L is a Dedekind complete Riesz space, if M is a solid subspace of the normal integrals on L which separates the points of L and if K is a sublattice of L, then the closure of K for $|\sigma|(L,M)$ is $\mathcal{DIDI}K$.

We employ theorem 3 by taking $L = L_b(E,F)$, $M = F$, $K = \ell(T)$. Lemma 1, Lemma 2 and some routine inspections of the situation, together with theorem 3 now yield:

Theorem 4. $\mathcal{DIDI}\ \ell(T) = [0,T]$

Because the characterization in theorem 4 is intrinsic, we can now derive a much stronger approximation theorem, (due to Kalton and Saab [4]).

Theorem 5. If ρ is an order continuous Riesz seminorm on the principal ideal generated by T in $L_b(E,F)$, if $S \in [0,T]$ and $\varepsilon > 0$, then there exists $S' \in \ell(T)$ with $\rho(S - S') < \varepsilon$.

Apart from being interesting in their own right, these theorems have nice consequences. The main reason for this is the preservation of certain properties of T in $\ell(T)$. For instance, every element of $\ell(T)$ is compact if T is compact. A straightforward application is the following majorization result by Dodds and Fremlin (see [2]).

Corollary 6. If F is an AL-space and $E = C(K)$, if $0 \leq S \leq T$ are operators from E to F, and T is a compact operator, then S is a compact operator.

To discuss another application we have to abandon the assumption $E = C(K)$. Instead, we assume that E is a Banach lattice with quasi-interior point, i.e. with an element $u \in E$ such that E is norm dense in the principal ideal generated by u. We borrow the abstract definition for the multipliers from the $C(K)$ situation, i.e. the multipliers are the order bounded operators $\sigma : E \to E$ with $|\sigma(g)| \wedge |h| = 0$ as soon as $|g| \wedge |h| = 0$. The multipliers form a Riesz space under pointwise operations and, in fact, this Riesz space is Riesz isomorphic to a $C(K)$-space. Using the same techniques, the statements in theorem 4 and theorem 5 remain valid. The latter will be used in the proof of our next corollary. (Again due to Kalton and Saab [4]).

Corollary 7. If E and F are Banach lattices and F has order continuous norm (so no restrictions on E at all), if $0 \leq S \leq T$ are operators from E to F and T is a Dunford-Pettis operator, then S is a Dunford-Pettis operator.

We sketch a proof of this corollary. To prove that S is a Dunford-Pettis operator we have to show that for every sequence $(a_n)_{n \in \mathbb{N}}$ of elements of E, which converges weakly to zero, $\|S(a_n)\| \to 0$. Suppose $a_n \to 0$ weakly. By taking $y = \sum_{n=0}^{\infty} 2^{-n}|a_n|$ we may assume that E has a quasi-interior point, namely y. Let A be the solid hull of $\{a_n \mid n \in \mathbb{N}\}$ and $B = \{\varphi \in F^* \mid \|\varphi\| \leq 1\}$. Define for every R in the principal ideal generated by T in $L_b(E,F)$, $\rho(R) = \sup\{|\langle Ra,\varphi\rangle| \mid a \in A, \varphi \in B\}$. It can be shown that ρ is an order continuous Riesz seminorm on the principal ideal generated by T in $L_b(E,F)$. Therefore, there exists by the remarks preceding corollary 7 an element S' in $\ell(T)$ with $\rho(S - S') \leq \varepsilon$. As S' is a Dunford-Pettis operator it easily follows that $\|Sa_n\| \to 0$.

References

[1] G.J.H.M. Buskes, P.G. Dodds, B. de Pagter, A.R. Schep, *Up-down*

theorems in the centre of $L_b(E,F)$.

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[4] N.J. Kalton and Paulette Saab, *Ideal properties of regular operators between Banach lattices* (to appear, Illinois J. of Math.).

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