

TRACES OF ANISOTROPIC FUNCTION SPACES. APPLICATIONS

Hans Triebel

1. INTRODUCTION

Let R_2 be the two-dimensional euclidean space, the plane. Let $1 < p < \infty$ and $s = 1, 2, 3, \dots$. Then

$$(1) \quad W_p^s(R_2) = \{f \mid f \in L_p(R_2), \|f\|_{W_p^s(R_2)} = \sum_{|\alpha| \leq s} \|D^\alpha f\|_{L_p(R_2)} < \infty\}$$

are the classical Sobolev spaces, where $L_p(R_2)$ has the usual meaning (complex-valued functions). It is well-known that the trace-operator

$$(2) \quad R : f(x_1, x_2) \rightarrow f(x_1, 0)$$

is a retraction from $W_p^s(R_2)$ onto the special Besov space (\sim Lipschitz space) $B_p^{s-1/p}(R_1)$ on the real line R_1 . Here *retraction* means that there exists a linear and bounded extension operator S from $B_p^{s-1/p}(R_1)$ (the trace space) into $W_p^s(R_2)$ such that

$$(3) \quad RS = \text{id} \quad (\text{identity in } B_p^{s-1/p}(R_1)).$$

In other words, if a trace-operator is a retraction then this assertion covers both the "direct" and the "inverse" embedding theorems and indicates that R is a mapping "onto". The above-mentioned special Besov spaces $B_p^\sigma(R_1)$ with $\sigma > 0$ and $1 < p < \infty$ are defined as follows. If $t \in R_1$ and $\tau \in R_1$ then

$$(4) \quad (\Delta_\tau^1 f)(t) = f(t+\tau) - f(t), \quad \Delta_\tau^m = \Delta_\tau^1 \Delta_\tau^{m-1},$$

with $m = 2, 3, \dots$ are the usual differences. Then $B_p^\sigma(R_1)$ is the collection of all complex-valued functions such that

$$(5) \quad \|f|_{B_p^\sigma(R_1)}\|_m = \|f|_{L_p(R_1)}\| + \left(\int_0^1 \tau^{-\sigma p} \|(\Delta_\tau^m f)(\cdot)|_{L_p(R_1)}\|^p \frac{d\tau}{\tau} \right)^{1/p}$$

is finite, where m is an arbitrary (but fixed) natural number with $m > \sigma$ (different m 's yield equivalent norms).

It is well-known that the above trace assertion can be extended easily to smooth curves in the plane, e.g. to the boundary of a bounded C^∞ -domain in the plane. The situation changes drastically if one deals with anisotropic Sobolev spaces: As far as the trace-operator R from (2) is concerned one has a final answer since the early sixties, but there is no "standard way" to extend this result to traces on smooth curves, in general.

First we give some definitions and describe what is known. Let again $1 < p < \infty$ and $\bar{s} = (s_1, s_2)$ be a couple of natural numbers, where we always assume that $s_1 \leq s_2$. Then we put

$$(6a) \quad \|f|_{W_p^{\bar{s}}(R_2)}\| = \|f|_{L_p(R_2)}\| + \left\| \frac{\partial^{s_1} f}{\partial x_1^{s_1}} \Big|_{L_p(R_2)} \right\| + \left\| \frac{\partial^{s_2} f}{\partial x_2^{s_2}} \Big|_{L_p(R_2)} \right\| ,$$

and the anisotropic Sobolev space $W_p^{\bar{s}}(R_2)$ is the collection of all $f \in L_p(R_2)$ such that this norm is finite. Anisotropic Besov spaces $B_p^{\bar{s}}(R_2)$ with $1 < p < \infty$ and $\bar{s} = (s_1, s_2)$, where $0 < s_1 \leq s_2 < \infty$, are as defined via the norm

$$(6b) \quad \|f|_{B_p^{\bar{s}}(R_2)}\|_m = \|f|_{L_p(R_2)}\| + \left(\int_0^1 \tau^{-s_1 p} \left\| \left(\Delta_{\tau,1}^{m_1} f \right) (\cdot) \Big|_{L_p(R_2)} \right\|^p \frac{d\tau}{\tau} \right)^{1/p} + \left(\int_0^1 \tau^{-s_2 p} \left\| \left(\Delta_{\tau,2}^{m_2} f \right) (\cdot) \Big|_{L_p(R_2)} \right\|^p \frac{d\tau}{\tau} \right)^{1/p} ,$$

where $\bar{m} = (m_1, m_2)$ is a couple of natural numbers with $m_1 > s_1$ and $m_2 > s_2$. Furthermore, $\Delta_{\tau,1}^{m_1}$ must be understood in the sense of (4), where the differences are taken with respect to the x_1 -direction, whereas x_2 is fixed, e.g. $(\Delta_{\tau,1}^1 f)(x_1, x_2) = f(x_1 + \tau, x_2) - f(x_1, x_2)$,

similarly $\Delta_{\tau,2}^{m_2}$ is explained. For these anisotropic Sobolev-Besov spaces a final answer for the trace-operator R from (2) is well-known: Let $1 < p < \infty$, $0 < s_1 \leq s_2$ (nat. numbers in the case of the Sobolev spaces) and

$$(7) \quad \sigma = s_1 \left(1 - \frac{1}{ps_2} \right) > 0 ,$$

then R is a retraction from $W_p^{\bar{s}}(R_2)$ onto $B_p^\sigma(R_1)$ and from $B_p^{\bar{s}}(R_2)$ onto $B_p^\sigma(R_1)$, cf. [1]. The difference between the isotropic Sobolev-Besov spaces and the anisotropic ones can be explained (in our context) as follows. Let $y = \psi(x)$ be a diffeomorphic map of R_2 onto itself, say, with $\psi(x) = x$ if $|x|$ is large. Then

$$(8) \quad f(x) \rightarrow f(\psi(x))$$

is an isomorphic map of the isotropic (i.e. $s_1 = s_2$) Sobolev-Besov spaces onto itself. [This is the basis to extend trace-assertions from lines to smooth curves.] However anisotropic spaces of Sobolev-Besov type have not this property, in general. On the other hand it had been observed by Uspenskij (~ 1966) that one has at least the following substitute: Let

$$(9) \quad \psi(x) = (x_1, \psi_2(x_1, x_2)) , \quad \text{where } x = (x_1, x_2) \in R_2 ,$$

be a fibre-preserving diffeomorphic map of R_2 onto R_2 with, say, $\psi(x) = x$ if $|x|$ is large. Then (8) yields an isomorphic map from $W_p^{\bar{s}}(R_2)$ onto itself and from $B_p^{\bar{s}}(R_2)$ onto itself (recall that always $s_2 \geq s_1$). On this way, trace-assertions can be extended from the line $(x_1, 0)$ to the curve $(x_1, \psi_2(x_1, 0))$. What about traces on other curves, e.g. on $\{x \mid |x| = 1\}$ or on the model curve C_ρ defined below? The aim of this paper is to describe some results in this direction, to outline the method, and to sketch applications to

boundary value problems for semi-elliptic differential equations.

Detailed proofs may be found [2,3]. As far as basic assertions for the above anisotropic spaces are concerned we refer to [1].

2. OUTLINE OF METHODS

The above-mentioned curve C_ρ is defined as follows. Let $0 < \rho < 1$ and let $\lambda(t)$ be a monotonically increasing C^∞ -function on $(0, \infty)$ with $\lambda(t) = 2$ if $t \geq 2$ and

$$(10) \quad \lambda(t) = t^\rho \quad \text{if } 0 < t < 2.$$

Then $C_\rho = \{(t, \lambda(t)), 0 < t < \infty\}$ is our "model curve". We ask for traces of functions belonging to $W_p^{\bar{s}}(R_2)$ or $B_p^{\bar{s}}(R_2)$ on this curve. By the above-mentioned observation by Uspenskij only a neighbourhood of the origin is of interest. Let again $\bar{s} = (s_1, s_2)$ with $0 < s_1 \leq s_2 < \infty$ be the given anisotropic smoothness. We introduce the *mean smoothness* s and the *anisotropy* $\bar{a} = (a_1, a_2)$ by

$$(11) \quad \frac{1}{s} = \frac{1}{2} \left(\frac{1}{s_1} + \frac{1}{s_2} \right), \quad a_1 = \frac{s}{s_1}, \quad a_2 = \frac{s}{s_2},$$

respectively. Of course, $a_1 + a_2 = 2$. Let

$$(12) \quad |x|_{\bar{a}} = \left(|x_1|^{2/a_1} + |x_2|^{2/a_2} \right)^{1/2}, \quad x \in R_2,$$

be the *anisotropic distance* of x from the origin. Near the origin we introduce an anisotropic smooth resolution of unity $\phi = \{\phi_j(x)\}_{j=0}^\infty$ with the following properties: The ϕ_j 's are non-negative infinitely differentiable functions in R_2 ,

$$\text{supp } \phi_0 \subset \{x \mid |x|_{\bar{a}} \geq \frac{1}{2}\};$$

$$\text{supp } \phi_j \subset \{x \mid 2^{-j-1} < |x|_{\bar{a}} < 2^{-j+1}\} \quad \text{if } j = 1, 2, 3, \dots;$$

for any multi-index $\alpha = (\alpha_1, \alpha_2)$ there exists a positive number c_α

with

$$2^{-j\alpha_1 a_1 - j\alpha_2 a_2} |D^{\alpha} \phi_j(x)| \leq c_{\alpha} \quad \text{for all } j = 0, 1, 2, \dots \quad \text{and all } x \in R_2 ;$$

and

$$\sum_{j=0}^{\infty} \phi_j(x) = 1 \quad \text{for all } x \in R_2 - \{0\} .$$

The collection of all these systems ϕ is denoted by Φ_a^- . It is easy to see that Φ_a^- is not empty. Let

$$(13) \quad \|f\|_{\dot{W}_p^{\bar{s}}(R_2)}^{\phi} = \left(\sum_{j=0}^{\infty} \|\phi_j \cdot f\|_{W_p^{\bar{s}}(R_2)}^p \right)^{1/p}$$

and

$$(14) \quad \|f\|_{\dot{B}_p^{\bar{s}}(R_2)}^{\phi} = \left(\sum_{j=0}^{\infty} \|\phi_j \cdot f\|_{B_p^{\bar{s}}(R_2)}^p \right)^{1/p}$$

where $\phi \in \Phi_a^-$ and $1 < p < \infty$. (Recall that in the case of Sobolev spaces we always assume that s_1 and s_2 in $\bar{s} = (s_1, s_2)$ are natural numbers.) We ask the question whether

$$(15) \quad \|f\|_{\dot{B}_p^{\bar{s}}(R_2)}^{\phi} \sim \|f\|_{B_p^{\bar{s}}(R_2)} \quad \text{}$$

(equivalent norms). If one tries to estimate the left-hand side of (15) from above by the right-hand side then it becomes clear that an inequality of the type

$$(16) \quad \| |x|^{-s} f(x) \|_{L_p(R_2)} \leq c \|f\|_{B_p^{\bar{s}}(R_2)}$$

would be very helpful (recall that s stands for the mean smoothness).

Such an inequality cannot be expected for any $f \in B_p^{\bar{s}}(R_2)$, but it is quite clear that $f(x)$ must tend to zero sufficiently strong if

$|x| \rightarrow 0$. Recall the following well-known embedding theorem (cf. e.g.

[1]): Let $f \in B_p^{\bar{s}}(R_2)$ and let $\alpha = (\alpha_1, \alpha_2)$ be a multi-index with

$$(17) \quad \frac{1}{s_1} \left(\alpha_1 + \frac{1}{p} \right) + \frac{1}{s_2} \left(\alpha_2 + \frac{1}{p} \right) < 1 .$$

Then $(D^{\alpha} f)(x)$ is a continuous function on R_2 and there exists a

constant $c > 0$ with

$$(18) \quad \sup_{x \in \mathbb{R}_2} |(D^\alpha f)(x)| \leq c \|f\|_{B_{\bar{s}}^{\bar{s}}(\mathbb{R}_2)}$$

for all $f \in B_{\bar{s}}^{\bar{s}}(\mathbb{R}_2)$. In particular, $(D^\alpha f)(0) = 0$ makes sense for those α 's. It is easy to see that

$$(19) \quad \{f \mid f \in B_{\bar{s}}^{\bar{s}}(\mathbb{R}_2), (D^\alpha f)(0) = 0 \text{ if (17) holds}\}$$

is a finite-codimensional subspace of $B_{\bar{s}}^{\bar{s}}(\mathbb{R}_2)$. Now it comes out that the anisotropic Hardy inequality (16) holds for all functions from the space described in (19), provided that \bar{s} is "non-critical" (the precise formulation will be given in the next section). This is the crucial point of the theory. With the help of (6b) and (16) one can prove (15) (for non-critical \bar{s}). Now the problem of the trace of functions from $B_{\bar{s}}^{\bar{s}}(\mathbb{R}_2)$ (or better from the space described in (19)) on the curve C_ρ can be attacked as follows. By (15) the trace of f on C_ρ can be reduced to the traces of $\phi_j f$ on

$$(20) \quad C_\rho \cap \{x \mid 2^{-j-1} < |x|_a < 2^{-j+1}\}.$$

With the help of affine transformations one maps

$\{x \mid 2^{-j-1} < |x|_a < 2^{-j+1}\}$ onto a standard domain, hopefully that the

curves (20) are transformed in curves with uniformly bounded slopes.

This causes an additional restriction: $\frac{s_1}{s_2} \leq \rho$. However if this restriction is satisfied, then one can apply the trace-assertions from Sect. 1 to these transformed curves with embedding-constants which are independent of j . Re-transformation and a careful calculation of the dependence of all constants on j yields the desired result: The trace of functions from spaces described in (19) on C_ρ is a weighted Besov space on C_ρ . Afterwards one can extend this result to functions from $B_{\bar{s}}^{\bar{s}}(\mathbb{R}_2)$, however we shall not do this in this paper.

3. HARDY'S INEQUALITY

Let p with $1 < p < \infty$ be fixed. Then the couple $\bar{s} = (s_1, s_2)$ with $0 < s_1 \leq s_2 < \infty$ is called *critical* if there exists non-negative integers m_1 and m_2 with

$$(21) \quad \frac{1}{s_1} \left(m_1 + \frac{1}{p} \right) + \frac{1}{s_2} \left(m_2 + \frac{1}{p} \right) = 1 .$$

Otherwise \bar{s} is called *non-critical*. Of course, critical couples are exceptional couples.

Theorem 1. Let $1 < p < \infty$ and let $\bar{s} = (s_1, s_2)$ with $0 < s_1 \leq s_2 < \infty$ be non-critical.

(i) There exists a positive number c such that

$$(22) \quad \int_{R_2} |x|_{\bar{a}}^{-sp} |f(x)|^p dx \leq c \|f\|_{B_p^{\bar{s}}(R_2)}^p$$

holds for all $f \in B_p^{\bar{s}}(R_2)$ with

$$(23) \quad (D^{\alpha} f)(0) = 0 \quad \text{if} \quad \frac{1}{s_1} \left(\alpha_1 + \frac{1}{p} \right) + \frac{1}{s_2} \left(\alpha_2 + \frac{1}{p} \right) < 1 .$$

(ii) Let additionally s_1 and s_2 be natural numbers. Then there exists a positive number c such that

$$(24) \quad \int_{R_2} |x|_{\bar{a}}^{-sp} |f(x)|^p dx \leq c \|f\|_{W_p^{\bar{s}}(R_2)}^p$$

holds for all $f \in W_p^{\bar{s}}(R_2)$ with (23).

Remark 1. This is an anisotropic inequality of Hardy type. Recall that the mean smoothness s , the anisotropy \bar{a} , and $|x|_{\bar{a}}$ have been defined in (11) and (12). An extension of (22) or (24) to critical couples \bar{s} is not possible. On the other hand for non-critical couples \bar{s} , the right-hand side of (22) can be replaced by $c \|f\|_{H_p^{\bar{s}}(R_2)}$ or $c \|f\|_{F_{p,q}^{\bar{s}}(R_2)}$ with $0 < q < \infty$, where $H_p^{\bar{s}}(R_2)$ are

anisotropic Bessel-potential spaces, and $F_{p,q}^{\bar{s}}(R_2)$ are anisotropic versions of the spaces $F_{p,q}^s(R_2)$ which have been considered in [4].

4. DECOMPOSITIONS

We return to the problem (15). Let again p with $1 < p < \infty$ be fixed. Let $\bar{s} = (s_1, s_2)$ with $0 < s_1 \leq s_2 < \infty$ be an arbitrary couple. s , \bar{a} , and $|x|_{\bar{a}}$ have been defined in (11) and (12). Then

$$(25) \quad \|f|_{\dot{B}_p^{\bar{s}}(R_2)}\| = \|f|_{B_p^{\bar{s}}(R_2)}\| + \| |x|_{\bar{a}}^{-s} f(x) |_{L_p(R_2)}\|$$

and

$$(26) \quad \dot{B}_p^{\bar{s}}(R_2) = \{f | f \in L_p(R_2), \|f|_{\dot{B}_p^{\bar{s}}(R_2)}\| < \infty\}.$$

If additionally s_1 and s_2 are integers then we put

$$(27) \quad \|f|_{\dot{W}_p^{\bar{s}}(R_2)}\| = \|f|_{W_p^{\bar{s}}(R_2)}\| + \| |x|_{\bar{a}}^{-s} f(x) |_{L_p(R_2)}\|$$

and

$$(28) \quad \dot{W}_p^{\bar{s}}(R_2) = \{f | f \in L_p(R_2), \|f|_{\dot{W}_p^{\bar{s}}(R_2)}\| < \infty\}.$$

Corollary. Let \bar{s} be non-critical. Then

$$(29) \quad \dot{B}_p^{\bar{s}}(R_2) = \{f | f \in B_p^{\bar{s}}(R_2) \text{ with (23)}\}$$

and (if additionally s_1 and s_2 are natural numbers)

$$(30) \quad \dot{W}_p^{\bar{s}}(R_2) = \{f | f \in W_p^{\bar{s}}(R_2) \text{ with (23)}\}.$$

Remark 2. The corollary is a consequence of Theorem 1.

Let $\phi \in \Phi_{\bar{a}}^-$. Then $\|f|_{\dot{B}_p^{\bar{s}}(R_2)}\|^\phi$ and $\|f|_{\dot{W}_p^{\bar{s}}(R_2)}\|^\phi$ have been defined in (14) and (13), respectively.

Theorem 2. Let $\phi \in \Phi_{\bar{a}}^-$, $1 < p < \infty$, and $\bar{s} = (s_1, s_2)$ with

$0 < s_1 \leq s_2$. Then

$$(31) \quad \dot{B}_p^{\bar{s}}(\mathbb{R}_2) = \{f \mid f \in L_p(\mathbb{R}_2), \|f\|_{\dot{B}_p^{\bar{s}}(\mathbb{R}_2)}^\phi < \infty\}$$

and (if additionally s_1 and s_2 are natural numbers)

$$(32) \quad \dot{W}_p^{\bar{s}}(\mathbb{R}_2) = \{f \mid f \in L_p(\mathbb{R}_2), \|f\|_{\dot{W}_p^{\bar{s}}(\mathbb{R}_2)}^\phi < \infty\} .$$

Remark 3. This theorem and the above corollary are the precise version of the decomposition method which we mentioned in Sect. 2. In particular, for different systems $\phi \in \Phi_a^-$ the corresponding norms in (31) are equivalent. Similarly in (32).

5. TRACES ON CURVES

Let $0 < \rho < 1$ and let C_ρ be the curve described at the beginning of Sect. 2. Let τ be the arc length on C_ρ , where $\tau = 0$ corresponds to $x = 0$. Let $d(\tau)$ be a monotonically increasing C^∞ -function on $(0, \infty)$ with

$$(33) \quad d(\tau) = \tau \quad \text{if } 0 < \tau < 1 \quad \text{and} \quad d(\tau) = 2 \quad \text{if } \tau > 2 .$$

We introduce weighted Besov spaces on C_ρ , where the weights are powers of the distance-function $d(\tau)$. Let $L_p(C_\rho)$ with $1 < p < \infty$ be the usual L_p -space on C_ρ with respect to the Lebesgue measure $d\tau$, where τ stands for the arc length on C_ρ . Then $\|f\|_{B_p^\sigma(C_\rho)}^m$ is defined by (5) with C_ρ instead of R_1 and where the differences Δ_τ^m are taken with respect to the arc length. Let $1 < p < \infty$, $\sigma > 0$ and $\mu \in R_1$. Then

$$(34) \quad \|g\|_{B_p^\sigma(C_\rho, \mu)}^m = \|d^{-\sigma-\mu}g\|_{L_p(C_\rho)} + \|d^{-\mu}g\|_{B_p^\sigma(C_\rho)}^m$$

with $m > \sigma$, and $B_p^\sigma(C_\rho, \mu)$ is the collection of all complex-valued locally integrable functions g on C_ρ such that $\|g\|_{B_p^\sigma(C_\rho, \mu)}$ is

finite. Again, different m 's yield equivalent norms. This is a weighted Besov space, only the behaviour of the elements of these spaces near the origin is of interest.

Theorem 3. Let $1 < p < \infty$ and $\bar{s} = (s_1, s_2)$ with $0 < s_1 < s_2 < \infty$.

Let $\frac{s_1}{s_2} \leq \rho < 1$, $s_2 > \frac{1}{p}$ and

$$(35) \quad \sigma = s_1 \left(1 - \frac{1}{ps_2} \right), \quad \mu = \left(\frac{1}{\rho} - 1 \right) \left(\sigma - \frac{1}{p} \right).$$

Then the trace-operator R ,

$$(36) \quad R : f(x_1, x_2) \rightarrow f|_{C_\rho}$$

(restriction of f to C_ρ) is a retraction from

$$\dot{B}_p^{\bar{s}}(\mathbb{R}_2) \quad \text{onto} \quad B_p^\sigma(C_\rho, \nu).$$

If additionally s_1 and s_2 are natural numbers then R is a retraction from

$$\dot{W}_p^{\bar{s}}(\mathbb{R}_2) \quad \text{onto} \quad B_p^\sigma(C_\rho, \mu).$$

Remark 4. The formulas (35), (36) are the counterparts of (7), (2).

The method how to prove the above theorem is quite clear now: The decomposition procedure which has been mentioned in Sect. 2 as the basic ingredient has been fully established by Theorem 2 (Remark 3 and the Corollary). Now one can proceed as indicated at the end of Sect. 2.

6. BOUNDARY VALUE PROBLEMS FOR SEMI-ELLIPTIC DIFFERENTIAL EQUATIONS

Let $K = \{x \mid |x| < 1\}$ be the unit disc in the plane and let $\partial K = \{x \mid |x| = 1\}$ be its boundary. There are no problems to study spaces of the type $\dot{B}_p^{\bar{s}}(\mathbb{R}_2)$ and $\dot{W}_p^{\bar{s}}(\mathbb{R}_2)$ with $1 < p < \infty$ and

$\bar{s} = (s_1, s_2)$, $0 < s_1 \leq s_2 < \infty$ (s_1 and s_2 nat. numbers in the case of Sobolev spaces), where the "singular point" 0 is replaced by the two singular points $x^0 = (-1, 0)$ and $x^1 = (1, 0)$ on ∂K . Then $\dot{B}_p^{\bar{s}}(K)$ and $\dot{W}_p^{\bar{s}}(K)$ can be defined as the restriction of the corresponding spaces on R_2 (in the just explained modified sense) to K . In the sequel we always assume that $s_2 = 2s_1$. In this case one has both a satisfactory inner description of the spaces $\dot{B}_p^{\bar{s}}(K)$ and $\dot{W}_p^{\bar{s}}(K)$ (where only values $f(x)$ with $x \in K$ are needed) and a final answer as far as traces of functions belonging to these spaces are concerned. The latter problem is just a technical modification of the results from Sect. 5 where now $\rho = \frac{1}{2}$. It is almost clear how to define weighted Besov spaces $B_p^\sigma(\partial K, \mu)$, $\sigma > 0$, $\mu \in R_1$ and $1 < p < \infty$, where x^0 and x^1 are the singular points of ∂K (instead of 0 on C_0). We omit any details and refer to [2,3].

Let $\eta(t)$ be a C^∞ -function on $\{t \mid |t| \leq 1\}$ with $\eta(t) > 0$ if $|t| > 1$ and

$$\eta(t) \sim 1-t \quad \text{near } t=1, \quad \eta(t) \sim 1+t \quad \text{near } t=-1.$$

Then we consider the semi-elliptic differential operator

$$(37) \quad (A_\nu f)(x) = -\frac{\partial^2 f}{\partial x_1^2}(x) + \frac{\partial^4 f}{\partial x_2^4}(x) + \frac{\nu}{\eta^2(x_1)} f(x), \quad x \in K,$$

where ν is a real number. It is not really necessary that the weight $\frac{\nu}{\eta^2(x_1)}$ has this special form, but one needs in the theorem below some qualitative behaviour of a weight in front of the term with $f(x)$ in (37). Let

$$(38) \quad A_\nu f = \left(A_\nu f, f|_{\partial K}, \frac{\partial f}{\partial x_2} |_{\partial K} \right)$$

where the two latter expressions in (38) stand for the trace of f and $\frac{\partial f}{\partial x_2}$ on ∂K , respectively.

Theorem 4. Let $1 < p < \infty$. There exists a number v_0 with the following property: If $v > v_0$ and $s > 0$ then A_v yields an one-to-one mapping from $\dot{B}_p^{(s+2, 2s+4)}(K)$ onto

$$\dot{B}_p^{(s, 2s)}(K) \times_{B_p}^{s+2-\frac{1}{2p}} \left(\partial K, s+2-\frac{3}{2p} \right) \times_{B_p}^{s+\frac{3}{2}-\frac{1}{2p}} \left(\partial K, s+\frac{3}{2}-\frac{3}{2p} \right)$$

and if $s = 0, 1, 2, \dots$, from $\dot{W}_p^{(s+2, 2s+4)}(K)$ onto

$$\dot{W}_p^{(s, 2s)}(K) \times_{B_p}^{s+2-\frac{1}{2p}} \left(\partial K, s+2-\frac{3}{2p} \right) \times_{B_p}^{s+\frac{3}{2}-\frac{1}{2p}} \left(\partial K, s+\frac{3}{2}-\frac{3}{2p} \right).$$

Remark 5. Detailed formulations and proofs may be found in [3]. For the unweighted case, i.e. $v f(x)$ in (37) instead of $\frac{v}{\eta^2(x_1)} f(x)$, one has not such a final result.

REFERENCES

- [1] S.M. Nikol'skij. *Approximation of Functions of Several Variables and Imbedding Theorems*. Springer-Verlag, Berlin, Heidelberg, New York, 1975.
- [2] H. Triebel, 'Anisotropic function spaces, I, II', *Analysis Math.* 10 (1984), 53-77, 79-96.
- [3] H. Triebel, 'A priori estimates and boundary value problems for semi-elliptic differential equations: A model case', *Comm. Partial Differential Equations* 8 (1983), 1621-1664.
- [4] H. Triebel. *Theory of Function Spaces*. Birkhäuser, Boston, 1983.

Sektion Mathematik
 Universität Jena
 DDR-6900 Jena
 Universitäts-Hochhaus
 German Democratic Republic