THE REGULARITY OF WEAK SOLUTIONS

TO PARABOLIC SYSTEMS

Yan Ziqian

There has been a lot of work devoted to the regularity of weak
solutions to elliptic systems since De Giorgi [3] had shown in 1968 that
his celebrated regularity result for elliptic equations cannot be
extended to systems. Various methods such as the direct approach, the
indirect approach and the hole-filling technique were developed to
study the partial and everywhere regularity for quasilinear and
nonlinear elliptic systems (see e.g. [4], [8] and the references cited
there). It is reasonable to ask the following questions. How about the
problem of regularity for parabolic systems? Are the results for
elliptic systems still true for them? And do the methods mentioned
above work in the parabolic case? Basically, the answers to these
questions are positive.

Let Ω be an open set in $\mathbb{R}^n$, $T > 0$ and $Q = \Omega \times (0,T)$.
Denote $z = (x,t)$, where $x \in \mathbb{R}^n$, $t \in \mathbb{R}$. For $z_i = (x_i, t_i) \in \mathbb{R}^{n+1}$,
$i = 1,2$, introduce the parabolic metric

$$d(z_1, z_2) = |x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}} .$$

Throughout this paper we use the convention that repeated indices are to
be summed for α and β from 1 to n , for i and j from 1 to
N , but not for k .

In 1973, M. Giaquinta and E. Giusti [5] first studied the regularity of weak solutions for uniformly parabolic systems

$$u_t^i - D_\alpha\left[A_{ij}^{\alpha\beta}(z,u)D_\beta u^j\right] = 0 \ , \quad i = 1,\ldots,N \ , \quad z \in Q$$

with bounded and continuous coefficients $A_{ij}^{\alpha\beta}(z,u)$, using the indirect approach. The direct approach, for which the L^p-estimate for the spatial derivatives of weak solutions is crucial, has been carried on in S . Campanato [1,2], M. Giaquinta and M. Struwe [7], under *strictly controlled* and *quadratic* growth conditions respectively. This work generalized the partial regularity result of [5]. By means of the hole-filling technique, an everywhere regularity result for diagonal parabolic systems with quadratic growth was obtained by M. Struwe [9] and improved by M. Giaquinta and M. Struwe [6].

S. Campanato considered first (in [1]) the quasilinear system

$$(1) \qquad u_t^i - D_\alpha[A_{ij}^{\alpha\beta}(z,u)D_\beta u^j] + a_i^\alpha(z,u) = B_i(z,u,Du) \ , \quad i = 1,\ldots,N$$

and then (in [2]) the nonlinear system

$$(2) \qquad u_t^i - D_\alpha A_i^\alpha(z,u,Du) = B_i(z,u,Du) \ , \quad i = 1,\ldots,N \ .$$

He assumed, among other conditions, that A_i^α and B_i have the following growth, which he called strictly controlled:

$$|A_i^\alpha(z,u,p)| \leq C(|p|+|u|^{\frac{\gamma}{2}}+f_{i\alpha}) \ ,$$

$$|B_i(z,u,p)| \leq C(|p|^{2(1-\frac{1}{\gamma})}+|u|^{\gamma-1}+f_i)$$

with $f_{i\alpha}$ and f_i belonging to some L^p spaces and

$$1 \leq \gamma < \begin{cases} 2(n+2)/n & \text{if} \quad n \geq 2 \ , \\ \\ 4 & \text{if} \quad n = 1 \ . \end{cases}$$

He established the L^p-estimate for the spatial derivatives of weak solutions of system (1) and (2), similar to those in elliptic case, and showed partial Hölder continuity of every solution of system (1) or (2), with a singular set Q_0 , closed in Q , such that

$$H^n(Q_0) = 0 \quad \text{or} \quad \text{meas } Q_0 = 0 \ ,$$

where H^n stands for the Hausdorff measure (relative to the parabolic metric d) of dimension n , and meas the Lebesgue measure of dimension $n+1$.

For systems (1) with quadratic growth, M. Giaquinta and M. Struwe [7] proved the L^p-estimate and partial Hölder continuity, and provided the singular set Q_0 with a refined estimate:

$$H^{n-\epsilon}(Q_0) = 0 \quad \text{for some} \quad \epsilon > 0 \ .$$

A few questions about the results mentioned above can be raised.

(i)   Can we get L^p-estimate for system (2) with controllable

growth for the critical case $\gamma = 2(n+2)/n$ and without

exception of $n=1$? Does there also exist an L^p-estimate for

nonlinear systems (2) (not only quasilinear systems (1) !)

under the natural growth conditions?

(ii) Is every weak solution of system (1) or (2) partially Hölder-

continuous under both natural and controllable growth

conditions including the case $\gamma = 2(n+2)/n$? How about the

Hausdorff measure of its singular set?

(iii) For what sort of parabolic systems does any weak solution have

everywhere regularity?

(iv) Under what conditions can further regularity be obtained?

    Here I wish to briefly describe some results by myself [10], which

have partly answered the above questions.

    Consider parabolic systems

$$(3) \quad u^i_t - D_\alpha A^\alpha_i(z,u,Du) = B_i(z,u,Du) \ , \quad i = 1,\ldots,N \ , \quad z \in Q \ .$$

Assume that the controllable growth conditions hold, i.e.

$$(4) \quad A^\alpha_i(z,u,p)p^i_\alpha \geq \lambda |p|^2 - C|u|^{\frac{2(n+2)}{n}} - f^2 \ , \quad \lambda > 0 \ , \quad 0 \leq f \in L^\sigma(Q) \ ,$$

$$(5) \quad |A^\alpha_i(z,u,p)| \leq C(|p|+|u|^{\frac{n+2}{n}} + f_{i\alpha}) \ , \quad 0 \leq f_{i\alpha} \in L^\sigma(Q) \ ,$$

(6) $|B_i(z,u,p)| \leq C(|p|^{\frac{n+4}{n+2}} + |u|^{\frac{n+4}{n}} + f_i)$, $0 \leq f_i \in L^\tau(Q)$.

For $z_0 = (x_0,t_0) \in Q$,   and $R > 0$,   we denote

$$B_R = B(x_0,R) = \{x \in \mathbb{R}^n : |x-x_0| < R\} ,$$

$$I_R = I(t_0,R) = \{t : t_0 - R^2 < t < t_0\} ,$$

$$Q_R = Q(z_0,R) = B(x_0,R) \times I(t_0,R)$$

and

$$\fint_{Q_R} \sim = \frac{1}{\operatorname{meas} Q_R} \int_{Q_R} \sim .$$

THEOREM 1. *Suppose that* (4), (5) *and* (6) *hold with* $\sigma > 2$,
$\tau > \frac{2(n+2)}{n+4}$. *If* $u \in L^2(0,T;H^1(\Omega,\mathbb{R}^N)) \cap L^\infty(0,T;L^2(\Omega,\mathbb{R}^N))$ *is a weak
solution of system* (3), *then there exists an exponent* $p > 2$ *such
that* $|Du| \in L^p_{loc}(Q)$. *Moreover, for* $Q(z_0,R) \subset Q(z_0,4R) \subset Q$,   *we have*

$$\left[\fint_{Q_R} \left[|Du|^2 + |u|^{\frac{2(n+2)}{n}}\right]^{p/2} dz\right]^{1/p} \leq C\left[\left[\fint_{Q_{4R}} \left[|Du|^2 + |u|^{\frac{2(n+2)}{n}}\right] dz\right]^{1/2}\right.$$

$$\left. + \left[\fint_{Q_{4R}} F^p dz\right]^{1/p}\right] ,$$

provided $R < R_0$, *where* C *and* R_0 *are constants depending on* u ,
and

$$F = f + \sum_{i,\alpha} f_{i\alpha} + \sum_{i} f_{i}^{\frac{n+2}{n+4}} \; .$$

Next we consider system (3) under natural growth conditions:

(7) $\quad A_i^\alpha(z,u,p)p_\alpha^i \geq \lambda|p|^2 - f^2 \; , \quad 0 \leq f \in L^\sigma(Q) \; ,$

(8) $\quad |A_i^\alpha(z,u,p)| \leq C(|p|+f_{i\alpha}) \; , \quad 0 \leq f_{i\alpha} \in L^\sigma(Q) \; ,$

(9) $\quad |B_i(z,u,p)| \leq C(|p|^{2-\delta}+f_i) \; , \; 0 < \delta < \dfrac{n}{n+2} \; , \quad 0 \leq f_i \in L^\tau(Q)$

or

(9)' $\quad |B(z,u,p)| \leq a|p|^2 + b \; , \quad 0 \leq b \in L^\tau(Q) \quad \text{and} \quad 2aM < \lambda \; , \quad \text{where}$

$M = \sup_{Q}|u| \; .$

THEOREM 2. *Suppose that* (7), (8) *and* (9) *or* (9)' *hold with* $\sigma > 2$ *and* $\tau > 1$ *, and that* $u \in L^2(0,T;H^1(\Omega,\mathbb{R}^N)) \cap L^\infty(Q,\mathbb{R}^N)$ *is a weak solution of system* (3), *with* $\sup_{Q}|u| = M$ *. Then there exists* $p > 2$ *such that* $|Du| \in L^p_{loc}(Q)$ *. Moreover, for* $Q(z_0,R) \subset Q(z_0,4R) \subset Q$ *, we have*

$$\left[\fint_{Q_R} |Du|^p dz \right]^{1/p} \leq C \left[\left[\fint_{Q_{4R}} |Du|^2 dz \right]^{1/2} + \left[\fint_{Q_{4R}} F^p \, dz \right]^{1/p} \right] \; ,$$

provided $R < R_0$ *, where* C *and* R_0 *are constants depending on* u *, and*

$$F = f + \sum_{i,\alpha} f_{i\alpha} + \sum_{i} f_i^{1/2} \quad (\text{or} \quad b^{1/2}) \ .$$

Applying the L^p-estimates to quasilinear systems

(10) $u_t^i - D_\alpha[A_{ij}^{\alpha\beta}(z,u)D_\beta u^j + a_i^\alpha(z,u)] = B_i(z,u,Du)$, $i = 1,\ldots,N$, $z \in Q$

with assumptions:

(11) $\lambda|\xi|^2 \le A_{ij}^{\alpha\beta}(z,u)\xi_\alpha^i \xi_\beta^j \le \varLambda|\xi|^2$, $\varLambda \ge \lambda > 0$,

(12) There exists a continuous, bounded, nondecreasing and concave function ω such that $\omega(0) = 0$ and

$$\sum |A_{ij}^{\alpha\beta}(z,u) - A_{ij}^{\alpha\beta}(z_0,u_0)|^2 \le \omega(|z-z_0|^2 + |u-u_0|^2)$$

for any $z,z_0 \in Q$ and any $u,u_0 \in \mathbb{R}^N$,

(13) $|a_i^\alpha(z,u)| \le C(|u|^{\frac{n+2}{n}} + f_{i\alpha})$, $f_{i\alpha} \in L^\sigma(Q)$,

(14) $|B_i(z,u,p)| \le C(|p|^{\frac{n+4}{n+2}} + |u|^{\frac{n+4}{n}} + f_i)$, $f_i \in L^\tau(Q)$,

we can get the partial Hölder continuity for their solutions. Precisely, we have

THEOREM 3. *Suppose that* $u \in L^2(0,T;H^1(\Omega,\mathbb{R}^N)) \cap L^\infty(0,T;L^2(\Omega,\mathbb{R}^N))$ *is a weak solution of system* (10) *and that* (11)-(14) *hold with* $\sigma > n+2$ *and* $\tau > \dfrac{(n+2)^2}{n+4}$. *Then* u *is Hölder continuous in an open set* $Q' \subset Q$

and $H^{n+2-p}(Q\backslash Q') = 0$, where $p > 2$ is the number appearing in Theorem 1.

As mentioned previously, M. Giaquinta and M. Struwe [7] have already proved the partial Hölder continuity for system (10) with quadratic growth. Theorem 3 is not an immediate consequence of that result, although controllable growth is much weaker than quadratic growth. This is because in the former case, weak solutions are not required to be bounded.

As far as everywhere regularity is concerned, we have the following Theorems 4 and 5.

Consider the parabolic system of triangular form, i.e., system

(15) $u_t^i - D_\alpha[A_{ij}^{\alpha\beta}(z,u)D_\beta u^j + a_i^\alpha(z,u)] = B_i(z,u,Du)$, $i = 1,\ldots,N$, $z \in Q$

with $A_{ij}^{\alpha\beta}$ satisfying

(16) $A_{ij}^{\alpha\beta}(z,u) = 0$ when $j > i$, $\lambda|\xi|^2 \leq A_{kk}^{\alpha\beta}(z,u)\xi_\alpha\xi_\beta \leq \Lambda|\xi|^2$

for all $i,j,k = 1,\ldots,N$, all $z \in Q$, $u \in \mathbb{R}^N$, $\xi \in \mathbb{R}^n$ and some $\Lambda \geq \lambda > 0$.

THEOREM 4. *Under controllable conditions* (13) *and* (14) *with* $\sigma > n+2$, $\tau > \frac{n}{2} + 1$, *every solution* $u \in L^2(0,T;H^1(\Omega,\mathbb{R}^N)) \cap L^\infty(0,T;L^2(\Omega,\mathbb{R}^N))$ *of triangular parabolic system* (15), (16) *is (locally) Hölder continuous in* Q .

THEOREM 5. *Under the natural growth conditions*

$$|a_i^\alpha(z,u)| \leq f_{i\alpha} \in L^\sigma(Q) \ , \ \sigma > n+2 \ ,$$

$$|B_i(z,u,p)| \leq C|p|^{2-\delta} + f_i \ ,$$

$$0 < \delta < \frac{n}{n+2} \ , \ f_i \in L^\tau(Q) \ , \ \tau > \frac{n}{2} + 1 \ ,$$

every solution $u \in L^2(0,T;H^1(\Omega,\mathbb{R}^N)) \cap L^\infty(Q,\mathbb{R}^N)$ *of a triangular parabolic system* (15), (16) *is (locally) Hölder continuous in* Q .

Unfortunately, we have found no way out yet to show the everywhere regularity for the triangular parabolic system (15), (16) with quadratic growth (9)'. As for the system of diagonal form

$$(17) \quad u_t^i - D_\alpha[a^{\alpha\beta}(z)D_\beta u^i] = B_i(z,u,Du) \ , \ i = 1,\ldots,N \ , \ z \in Q \ .$$

M. Giaquinta and M. Struwe [6] proved the following result:

Let $u \in L^2(0,T;H^1(\Omega,\mathbb{R}^N)) \cap L^\infty(Q,\mathbb{R}^N)$ with $\sup_Q |u| = M$ be a weak solution of system (17) which satisfies the conditions:

$$\lambda|\xi|^2 \leq a^{\alpha\beta}\xi_\alpha\xi_\beta \leq \Lambda|\xi|^2 \ , \ \Lambda \geq \lambda > 0 \ ,$$

$$|B(z,u,p)| \leq a|p|^2 + b \ ,$$

$$aM < \lambda \ , \ b \in L^\tau(Q) \ , \ \tau > \frac{n}{2} + 1 \ .$$

Then u is (locally) Hölder continuous in Q .

Let us come back to the general quasilinear system (1) and state a further regularity result.

THEOREM 6. *Suppose* $u \in C_{loc}^{\mu,\mu/2}(Q,\mathbb{R}^N)$, $0 < \mu \leq 1$, *is a weak solution of system* (1). *Assume that*

$$A_{ij}^{\alpha\beta}(z,u)\xi_\alpha^i\xi_\beta^j \geq \lambda|\xi|^2 \quad for\ some \quad \lambda > 0\ ,$$

$A_{ij}^{\alpha\beta}(z,u)$ *and* $a_i^\alpha(z,u)$ *are Hölder continuous with exponent* ν *in* x,u *and* $\nu/2$ *in* t , *and*

$$|B(z,u,p)| \leq C(1+|p|^2)\ .$$

Then the derivatives Du *of* u *are (locally) Hölder continuous in* Q *with the same exponents* ν *and* $\nu/2$ *in* x *and* t *respectively.*

Starting with Theorem 6, higher regularity can be obtained from the linear theory.

In conclusion, I wish to point out that M. Struwe [9] has given a global regularity result. But, in general, the regularity up to the boundary has not been studied extensively.

REFERENCES

[1]   S. Campanato, L^p-regularity and partial Hölder continuity for
       solutions of second order parabolic systems with strictly
       controlled growth, Ann. Mat. Pura Appl., 128(1980), 287-316.

[2]   S. Campanato, On the nonlinear parabolic systems in divergence
       form, Hölder continuity and partial Hölder continuity of the
       solutions, Ann. Mat. Pura Appl., 144(1984), 83-112.

[3]   E. De Giorgi, Un esempio di estremali discontinue per un
       problema variazionale di tipo ellittico, Boll. UMI, 4(1968),
       135-137.

[4]   M. Giaquinta, Multiple integrals in the calculus of variations
       and nonlinear elliptic systems. Princeton University Press,
       Princeton, New Jersey, 1983.

[5]   M. Giaquinta and E. Giusti, Partial regularity for the solutions
       to nonlinear parabolic systems, Ann. Mat. Pura Appl., 97(1973),
       253-266.

[6]   M. Giaquinta and M. Struwe, An optimal regularity result for a
       class of quasilinear parabolic systems, manuscripta math.,
       36(1981), 223-239.

[7]   M. Giaquinta and M. Struwe, On the partial regularity of weak
       solutions of nonlinear parabolic systems, Math. Z., 179(1982),
       437-451.

[8]   S. Hildebrandt, Nonlinear elliptic systems and harmonic
       mappings, Proc. Beijing Symp. Diff. Geo. and Diff. Eq., Science
       Press, Beijing, 1981.

[9]   M. Struwe, On the Hölder continuity of bounded weak solutions of
       quasilinear parabolic systems, manuscripta math., 35(1981),
       125-145.

[10]   Yan Ziqian, L^p-regularity and Hölder continuity for solutions of
       second order parabolic systems, Research Report CMA-R37-85,
       Australian National University.

Jilin University
Changchun
CHINA