SOME FULLY NON-LINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

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In this paper I wish to discuss the classical solvability of the first boundary value problem for a class of non-linear parabolic equations of second order. The equations to be considered arise from symmetric functions in a natural way analagous to the equations considered by Caffarelli, Nirenberg and Spruck [C.N.S] in the elliptic case. They are also motivated by the proposed analogue of the Monge-Ampère equation of Krylov [K], which is considered here as a first special case. I do not present the proofs of the results described, but only rough indications of the methods involved. The work constitutes the central results of the latter half of my doctoral dissertation [R.1].

In [K.1] Krylov considered the problem of defining an appropriate evolution equation corresponding to the Monge-Ampère equation

1) $\det D^2u = f(x)$

The equation he proposed took the form

1') $-D_tu.\det D^2u = f(x,t)$

In [K.1], Krylov studies this equation on cylindrical domains with zero boundary values and with the right hand side depending only on x and t. There he proved results relating to the existence of "generalized solutions", while in subsequent work ([K.2], [K.3]) he proved further results for classical solvability. A related problem, that of the evolution of convex surfaces with velocity proportional to Gauss curvature, has been studied by Tso [Ts]. By mimicking the methods of the elliptic theory (for example in the dissertation of John Urbas [U.1]) these results may be extended to the classical solvability of the equation

$$1'') \quad -D_t u . \det D^2 u = f(x,t,u,Du)$$

for smooth boundary values on a somewhat different class of domains to those considered by Krylov. It is mainly in the fact that the elliptic methods (in particular the calculation of Pogorelov [P.1]) may be naturally extended to the equation 1") that its claim to be the 'correct' analogue lies.

The elliptic Monge-Ampère equation is known to be classically solvable for general boundary values only on convex domains. It is easy to see that for a corresponding result for the Krylov equation 1"), the class of domains to be considered should not be cylindrical but rather 'parabola shaped'. More precisely, the natural class of domains for this problem consists of domains Ω satisfying

$$2) \quad \Omega = \{(x,t) \in \mathbb{R}^{n+1} \mid \rho(x,t) < 0\}$$

where ρ is some smooth function uniformly convex with respect to x and uniformly decreasing with respect to t. For such domains the following theorem holds :

<u>THEOREM 1</u> *Suppose* Ω *is a natural domain,* $\varphi \in C^{2,1}(\overline{\Omega})$, *and* f *is a positive function in* $C^2(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ *satisfying* $f_z \geqslant 0$ *and the structure condition*

$$0 \leqslant f(x,t,z,p) \leqslant \mu(|z|)d^\beta|p|^\alpha$$

for all $(x,t) \in \tilde{N}$ *a neighbourhood of* $\partial\Omega$, $z \in \mathbb{R}$, $|p| \geqslant \mu(|z|)$, μ *a nondecreasing function,* $\beta = \alpha - n - 2 \geqslant 0$. *Then the classical first boundary value problem has a unique convex decreasing solution in* $C^{2,1}(\Omega) \cap C^{0,1;0,1/n+1}(\overline{\Omega})$.

As in the elliptic case, this theorem is proved using the method of continuity. Thus it suffices to establish a priori estimates for u and its derivatives. These estimates are proved by methods essentially identical to the elliptic case (see [G.T.] for the global estimates, [T.U.] or [U.] for the local second derivative estimates, and [U.] for the boundary gradient estimate). The necessary Hölder estimates for second space and first time derivatives may be proved using the weak Harnack inequality of Gruber, a non-probalistic proof of which may be found in [R.1] or [R.2].

The above theorem suggests that the Krylov equation is indeed a good parabolic version of the Monge-Ampère equation. However, it has the following disagreeable feature; it has no stationary solutions (indeed the domains considered cannot be stationary). It is normal to expect the elliptic solutions of an equation to be stationary solutions of the corresponding parabolic equations. To obtain an equation with this property, which is also solvable under reasonable conditions, I consider the more general situation of an equation arising from a symmetric function of $n+1$ variables. To see how such a function gives rise to a parabolic equation, for a function F defined on the space of $(n+1)\times(n+1)$

real matrices, consider equations of the form

3) $F(\overline{H}[u]) = \psi(x,t)$

where for convenience of notation, I denote by $\overline{H}[u]$ the matrix

$$\overline{H}[u] = \begin{pmatrix} -D_t u & 0 \\ 0 & D^2 u \end{pmatrix}$$

Given a symmetric function f on $n+1$ variables, one may define a function F by evaluating f at the eigenvalue vector

$$\overline{\lambda}[u] \equiv (\lambda_1[u], \ldots, \lambda_n[u], -D_t u)$$

where $\lambda_i[u]$ is defined to be the i-th eigenvalue of D^2u.

I now define a class of functions f so that the equation 3) is parabolic on appropriate u. Let f be a real valued function on $\mathbb{R}^{n+1}$, $f = f(\lambda_1, \ldots, \lambda_n, \lambda_t)$ satisfying

4) f is symmetric and smooth in all its arguments

Also suppose that a positive cone $\Upsilon \subset \mathbb{R}^{n+1}$ is given with vertex at the origin and containing the cone $\{\lambda_i > 0 \; ; \; i = 1, \ldots, n+1\}$ and symmetric in the λ_i. This cone is to determine the equations on which our operator is to be parabolic. As extra conditions on f I require

5) $\partial f/\partial\lambda_i > 0$ on Υ

6) f is concave on Υ

Note that 5) is simply the requirement that

$$F[u] \equiv f(\lambda_1[u], \ldots, \lambda_n[u], -D_t u)$$

is parabolic on those u such that $\bar{\lambda}[u] \in \Upsilon$. Property 6) is the essential condition used by Evans [E.] and subsequent authors for the proof of classical solvability of uniformly elliptic fully non-linear equations.

It is also required for application of the method of continuity, as well as for the estimates, that F be locally uniformly parabolic; i.e. I require that

$$7) \quad \begin{cases} \text{For every } C > 0 \text{ and compact } K \subset \Upsilon \text{ there exists } R = R(C,K) \\ \text{such that } f(\lambda_1, \ldots, \lambda_n, \lambda_{n+1} + R) \geqslant C \qquad \forall \lambda \in K \end{cases}$$

and

$$8) \quad f(R\lambda) \geqslant C \qquad \forall \lambda \in K$$

The next condition controls the behaviour of f near the extremal points of the cone Υ.

$$9) \quad \exists\ \bar{\psi}_0 \text{ such that } \overline{\lim_{\lambda \to \lambda_0}} f(\lambda) \leqslant \bar{\psi}_0 \qquad \forall \lambda_0 \in \partial\Upsilon$$

For the inhomogeneous term ψ I require that

$$10) \quad \psi \in C^\infty(\bar{\Omega});\ 0 < \psi_0 = \min \psi \leqslant \max \psi = \psi_1;\ \psi_0 > \bar{\psi}_0.$$

This condition ensures (indirectly) that the prospective solution actually remains in a compact subset of the parabolic functions.

Now I turn to the conditions to be satisfied by the domain Ω. To obtain the most general solvability theorem, it is again necessary to consider domains which are variable in t. The theorem to be formulated will establish the solvability for arbitrary smooth boundary values on a such a class of domains

The domains to be treated will be assumed to be reasonably smooth; i.e. I do not consider domains with corners, and so cylindrical domains are not included. Note that this means that the usual compatibility conditions at the corners do not arise. My first assumption is that sufficiently close to $(0,0) \in \partial\Omega$, the surface $\partial\Omega$ may be represented as

$$11) \quad x_n = \rho(x',t) = \tfrac{1}{2}\sum_{\alpha<n} \kappa_\alpha x_\alpha^2 - \tau.t + o(|x'|^3+t^2) \qquad \ldots \; t \leqslant 0$$

where the positive x_n-axis is the interior normal to $\partial\Omega$ with respect to x, and κ_α are the principle curvatures of $\partial\Omega$ at $(0,0)$ with respect to x, and τ is the "time derivative" of $\partial\Omega$. The condition tying in the nature of F now becomes

$$12) \quad \forall(x,t) \in \partial\Omega \;,\; \exists R \text{ such that } (\kappa_1, \ldots, \kappa_{n-1}, \tau, R) \in \Upsilon$$

Note that this condition does not make sense where κ_i or τ are not well defined. I therefore assume that for t sufficiently small, $\partial\Omega$ may be represented as

11') $t = \vartheta(x)$

and insist that

12') $\vartheta - t$ is an admissible function

where the term "admissible" is defined by the following.

DEFINITION *A function* $u \in C^{2,1}(\overline{\Omega})$ *is called admissible if at every* $(x,t) \in \overline{\Omega}$, $\overline{\lambda}[u](x,t) \in \Upsilon$

Note that with this definition, condition 12') is equivalent to the requirement that

$$\overline{\lambda}[\vartheta - t](0,x) = (\vartheta_{11}(x), \ldots, \vartheta_{nn}(x), 1) \in \Upsilon$$

Thus the restrictions on $\partial\Omega$ take on a purely pointwise form. It is also clear that the class of domains thus defined is, apart possibly from the smoothness requirement, the maximal class for the following theorem.

The main result of this article is the obvious extension of Theorem 2 of [C.N.S.].

THEOREM 2 *Suppose conditions* 4) - 12') *all hold, and that* $\varphi \in C^\infty(\partial\Omega)$. *Then there exists a unique admissible solution* $u \in C^\infty(\Omega)$ *to the boundary value problem*

13) $F[u] = \psi(x,t)$ *in* Ω, $u = \varphi$ *on* $\partial\Omega$

The method of proof is as before; apriori estimates are derived for solutions, and then the method of continuity is invoked. The usual bootstrapping methods give the higher regularity in the interior. The estimates are proved in a manner parallel to those of [C.N.S.], and are all global in nature. The only difficult estimate is the boundary second derivative estimate, which makes strong use of all the structure conditions. The lack of local estimates means that high powers of the gradient in the inhomogeneous term cannot be handled by these methods.

An important example of a function f satisfying the conditions above is given by the k-th root of the k-th elementary symmetric function on $n+1$ variables. Thus in the case $k = 1$, one obtains the heat equation, and in the case $k = n+1$, one obtains the Krylov equation. If one now asks the question what equation is the natural parabolic equation associated with the elliptic equation arising from the k-th elementary symmetric equation on n variables, the obvious answer generalizing from the relationship between Poisson's equation and the Heat equation is that arising from the k-th elementary symmetric function on $n+1$ variables. To return to the question of the 'correct' analogue of the Monge-Ampère equation, I would like to suggest that attention be given to the equation

$$14) \quad \det(D^2u) - D_t u . \sigma_{n-1}(D^2u) = f(x,t,u,Du)$$

where σ_{n-1} is the $n-1$-th elementary symmetric function, as the above reasoning suggests this as an appropriate analogue.

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