WHEN ARE SINGULAR INTEGRAL OPERATORS BOUNDED?

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The aim of this talk is to survey some results concerning the L_2-boundedness of singular integral operators, and in particular to present the $T(b)$ theorem.

Let us consider one-dimensional singular integral operators T of the following type:

$$(Tu)(x) = \text{p.v.} \int_{-\infty}^{\infty} K(x,y)u(y)dy$$

where, for $x,y \in \mathbb{R}$ with $x \neq y$,

$$(1) \qquad \begin{cases} |K(x,y)| \leq c_0|x-y|^{-1} \\ \left|\frac{\partial K}{\partial x}(x,y)\right| \leq c_1|x-y|^{-2} \\ \left|\frac{\partial K}{\partial y}(x,y)\right| \leq c_2|x-y|^{-2} . \end{cases}$$

Such T are called Calderón-Zygmund operators if $\|T\varphi\|_2 \leq c\|\varphi\|_2$ for all $\varphi \in C_0^\infty(\mathbb{R})$. We note first that an L_2-estimate of this type is sufficient to prove a variety of bounds.

THEOREM 1 (Calderón, Zygmund, Cotlar, Stein) *Suppose* T *is a Calderón-Zygmund operator. If* $u \in L_p$, $1 < p < \infty$, *then* $Tu(x)$ *is defined for almost all* x, *and* $\|Tu\|_p \leq c_p\|u\|_p$, $1 < p < \infty$. *If*

$u \in L_\infty$, *then* $\|Tu\|_* \le c_* \|u\|_\infty$, *where* $\|\cdot\|_*$ *denotes the* BMO *norm and* Tu *is only defined modulo the constant functions.*

In addition one has maximal-function estimates.

It has been a long-term program, initiated by Calderón, to determine whether certain classes of naturally occurring singular integral operators are Calderón-Zygmund operators. The best known case is when $K(x,y) = k(x-y)$ with $\hat{k} \in L_\infty(\mathbb{R})$, where $\hat{k}$ denotes the Fourier transform of k . In this case, $T = \hat{k}(D)$ where $D = -i\frac{d}{dx}$ and $\|Tu\|_2 \le \|\hat{k}\|_\infty \|u\|_2$. In particular, if $K(x,y) = i\pi^{-1}(x-y)^{-1}$, then $T = \mathrm{sgn}(D)$, which is the Hilbert transform on $\mathbb{R}$, appropriately scaled.

Another well-known class of kernels K_j give rise to the commutator integrals T_j . These are defined by

$$K_j(x,y) = \frac{i}{\pi} \frac{(g(x)-g(y))^j}{(x-y)^{j+1}}$$

where g is a Lipschitz function. It was shown by Calderón that T_1 is bounded, and then by Coifman and Meyer that T_j is bounded for $j > 1$. Subsequently the bound

$$\|T_j u\|_2 \le c(1+j)^4 \|g'\|_\infty{}^j \|u\|_2$$

was obtained by Coifman, McIntosh and Meyer [1].

It follows from these estimates for T_j that T_h is bounded, where T_h has kernel

$$K_h(x,y) = \frac{i}{\pi}(h(x)-h(y))^{-1} ,$$

with h a Lipschitz function such that $\mathcal{Re}\, h'(x) \geq \lambda > 0$ almost everywhere. For we can write $h(x) = \rho(x-g(x))$ with $\rho > 0$ and $\|g'\|_\infty < 1$, and then

$$K_h(x,y) = \rho^{-1} \sum_{j=0}^{\infty} K_j(x,y) .$$

So

$$\|T_h u\|_2 \leq \rho^{-1} \sum_{j=0}^{\infty} \|T_j u\|_2 \leq c_h \|u\|_2 .$$

The operator T_h arises as follows. The Cauchy integral on the Lipschitz curve γ parametrized by $z = h(x)$ is

$$C_\gamma U(z) = \frac{i}{\pi} \text{ p.v.} \int_\gamma (z-\varsigma)^{-1} U(\varsigma)d\varsigma .$$

On writing $U(z(x)) = u(x)$, we get

$$C_\gamma u(x) = \frac{i}{\pi} \text{ p.v.} \int_{-\infty}^{\infty} K_h(x,y)u(y)h'(y)dy .$$

i.e.

$$C_\gamma = T_h B$$

where B denotes multiplication by $b = h'$. So C_γ is L_2-bounded (though not itself a Calderón-Zygmund operator).

The original (unpublished) proof of the L_2-boundedness of C_γ was quite different from that indicated above. It was shown that

$$\| |D|^s C_\gamma u \|_2 \le C_s \| |D|^s u \|_2$$

when $0 < s < 1$, and hence that

$$\| |D|^s T_h u \|_2 \le c_s \| |D|^s B^{-1} u \|_2 .$$

Also, taking the dual of the above estimate with b replaced by $\bar{b}$, we have

$$\| |D|^{-s} B T_h u \|_2 \le c_s \| |D|^{-s} u \|_2 .$$

It was then shown that T_h is L_2-bounded by interpolating these inequalities. This interpolation was achieved via a theorem of Kato which states that the domains of fractional powers of maximal accretive operators interpolate [4], and by proving a variant of the Kato square root problem, namely that

$$\| (|D|^s B^{-1} |D|^s)^{\frac{1}{2}} u \|_2 \le c \| |D|^s u \|_2 .$$

Once the square root problem was solved, however, it was realized that the estimates used in its proof gave directly the boundedness of T_j and hence of T_h and C_γ.

Let us make some remarks about C_γ. Let $D_\gamma = \frac{1}{i} \frac{d}{dz}\Big|_\gamma = B^{-1} D$. Then D_γ has spectrum in the double sector

$$S_\omega = \left\{ z \in \mathbb{C} \;\middle|\; |\arg z| \le \omega \quad \text{or} \quad |\arg(-z)| \le \omega \right\}$$

where ω is large enough that $S_\omega \supset \{\zeta_1 - \zeta_2 \mid \zeta_1, \zeta_2 \in \gamma\}$. If the signum function is defined on S_ω by

$$\operatorname{sgn} z = \begin{cases} 1 , & \mathcal{Re} z > 0 \\ 0 , & z = 0 \\ -1 , & \mathcal{Re} z < 0 , \end{cases}$$

then $C_\gamma = \operatorname{sgn}(D_\gamma)$.

We remark that, for analytic functions φ on $\overset{\circ}{S}_{\omega+\epsilon}$ (the interior of $S_{\omega+\epsilon}$) which decay suitably at ∞ , $\varphi(D_\gamma)$ can be defined using resolvent integrals. On the other hand, if φ has inverse Fourier transform $\check{\varphi}$ which extends analytically to $\overset{\circ}{S}_{\omega+\epsilon}$ and decays suitably at ∞ , then

$$\varphi(D_\gamma)U(z) = \int_\gamma \check{\varphi}(z-\zeta)U(\zeta)d\zeta \ .$$

Let us go on. Subsequently to the operators T_j and T_h having been shown to be L_2-bounded, David and Journé proved an intriguing theorem. We see from theorem 1 that if T is a Calderón-Zygmund operator then $T(1) \in BMO$ and $T^*(1) \in BMO$. It is also clear that T satisfies the following weak boundedness property:

(2) there exists $m \ge 0$ and $c \ge 0$ such that

$$|\langle Tu_1, u_2 \rangle| \le cd$$

for all u_1 , $u_2 \in C_0^\infty(\mathbb{R})$ such that u_1 , $u_2 \in C_0^\infty(\mathbb{R})$ where u_1 and u_2 have support in an interval of length d and satisfy $|u_j^{(r)}| \le d^{-r}$ for all $r \le m$.

THEOREM 2.[2] *Suppose* K *satisfies* (1). *Then* T *is a Calderón-Zygmund operator if and only if* $T(1) \in BMO$, $T^*(1) \in BMO$ *and* T *satisfies* (2).

As noted above, the "only if" part of this result is straightforward. But the "if" part is quite striking. We note that if $K(x,y) = -K(y,x)$ and (1) is satisfied, then (2) holds automatically. So in this case the L_2-boundedness is equivalent to $T(1) \in BMO$.

Theorem 2 can be used inductively to show that the commutator operators T_j are bounded, but the bounds are not strong enough to imply that T_h and C_γ are bounded except when h has a small Lipschitz constant.

Another interesting recent result is that of Lemarié. He proved a more general version of the following:

THEOREM 3.[5] *Suppose that* (1) *is satisfied and that* $T(b) = 0$ ($\in$ BMO) *for some function* $b \in L_\infty(\mathbb{R})$ *Define* W *by* $W(u) = T(bu)$, *and suppose that* (2) *holds with* T *replaced by* W . *Then, for each* $s \in (0,1)$, *there exists* c_s *such that*

$$\| |D|^s Wu \|_2 \le c_s \| |D|^s u \|_2 \ .$$

As a corollary of this, Meyer and the author proved the following variant of David and Journé's theorem [6].

THEOREM 4. *Suppose that* $b_1, b_2 \in L_\infty(\mathbb{R})$ *with* $\mathcal{R}e\, b_j(x) \geq \kappa > 0$, *that* $T(b_1) = 0$ *and* $T^*(\overline{b_2}) = 0$, *that* (1) *holds, and that* (2) *holds with* T *replaced by both* TB_1 *and* B_2T (*where* B_j *is multiplication by* b_j). *Then* T *is a Calderón-Zygmund operator.*

This was proved by appealing to the square root problem in the same way as was originally done for the Cauchy integral.

Theorem 4 is a general theorem which includes the boundedness of the Cauchy integral as a special case, since $T_h(h') = C_\gamma(1) = 0$ ($\in$ BMO) and C_γ satisfies (2). A more general result again, which includes both theorem 4 and theorem 2 as special cases, was subsequently proved by David, Journé and Semmes [3].

THEOREM 5. *If the hypotheses of theorem* 4 *are weakened by replacing* $T(b_1) = 0$ *and* $T^*(\overline{b_2}) = 0$ *by* $T(b_1) \in \mathrm{BMO}$ *and* $T^*(\overline{b_2}) \in \mathrm{BMO}$, *then the conclusion remains valid.*

Theorem 5 can be reduced to theorem 4 if, given $\beta_1, \beta_2 \in \mathrm{BMO}$, we can find Calderón-Zygmund opereators L and M such that $L(b_1) = \beta_1$, $L^*(\overline{b_2}) = 0$, $M(b_1) = 0$ and $M^*(\overline{b_2}) = \beta_2$. To do this, let γ and δ be the curves parametrized by $z = h_1(x)$ and $z = h_2(x)$, where $h_j' = b_j$. Then define L by

$$Lu = 2\int_0^\infty \psi(tD_\delta)\{\psi(tD_\delta)\beta_1\}\varphi(tD_\gamma)b_1^{-1}u\frac{dt}{t}$$

and define M* similarly. In this formula, φ and ψ denote the following functions:

$$\varphi(\lambda) = (1+\lambda^2)^{-1} \quad \text{and} \quad \psi(\lambda) = \lambda(1+\lambda^2)^{-1} .$$

So, if $\varsigma \in S_\omega$, where S_ω was defined previously, then

$$\check{\varphi}_t(\varsigma) = \begin{cases} \frac{1}{2t} e^{-\varsigma/t} , & \mathcal{R}e\ \varsigma > 0 \\ \frac{1}{2t} e^{\varsigma/t} , & \mathcal{R}e\ \varsigma < 0 , \end{cases}$$

and

$$\varphi(tD_\gamma)U(z) = \int_\gamma \check{\varphi}_t(z-\varsigma)U(\varsigma)d\varsigma ,$$

or

$$\varphi(tD_\gamma)u(x) = \int_{-\infty}^{\infty} \check{\varphi}_t(h_1(x)-h_1(y))b_1(y)u(y)dy .$$

The operator $\psi(tD_\delta)$ is defined similarly. Square function estimates for $\psi(tD_\delta)$ can be obtained from the expansion for $\psi(tD_\delta) = \psi(tB_2^{-1}D) = \psi(t\rho^{-1}(I-F)^{-1}D)$ in powers of F using the techniques of [1], where ρ is chosen so that $\|F\| < 1$. Proceeding in this way it can be shown that L is a Calderón-Zygmund operator. In doing this, we are generalizing the proof of the T(1) theorem given in [2] rather than following [3].

We conclude with the remark that theorems 1-5 remain valid in higher dimensions if the appropriate dependence on the dimension is included in (1) and (2). However many of the intervening comments are specifically one-dimensional.

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