BOUNDARY BEHAVIOR OF SOLUTIONS OF ELLIPTIC EQUATIONS IN "BAD" DOMAINS

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The natural setting for the theory of nondivergence form second order elliptic equations is in the Hölder spaces $C^{k,\alpha}$. To explain this statement, consider the elliptic operator Δ, the Laplacian. Then, the map $u \to \Delta u$ is a bijection of $C^{2,\alpha}(\bar{\Omega})$ onto $C^{\alpha}(\bar{\Omega})$ provided the boundary values of u are fixed and $\partial\Omega \in C^{2,\alpha}$; however, this map is not a bijection of $C^2(\bar{\Omega})$ onto $C^0(\bar{\Omega})$ because it is never surjective. (We do not consider the mapping from $W^{2,p}(\Omega)$ to $L^p(\Omega)$ because the appropriate boundary conditions cannot be described intrinsically via the same sort of spaces.)

To pin down the boundary values, we consider the Dirichlet boundary condition,

$$u = u_0 \quad \text{on} \quad \partial\Omega \tag{1}$$

for some $u_0 \in C^{2,\alpha}(\partial\Omega)$, and the oblique boundary condition

$$\beta \cdot Du = g \quad \text{on} \quad \partial\Omega \tag{2a}$$

for some vector field $\beta \in C^{1,\alpha}(\partial\Omega)$ satisfying

$$\beta \cdot \gamma > 0 \quad \text{on} \quad \partial\Omega, \tag{2b}$$

where γ is the inner normal, and $g \in C^{1,\alpha}(\partial\Omega)$. With these boundary conditions, we ask how much the regularity of u_0, β, g, and $\partial\Omega$ can be relaxed without losing the desirable feature that the boundary condition still be satisfied classically. The relaxed conditions will be called "bad" for the reasons previously mentioned even though fairly strong regularity results are known for "bad" boundary conditions.

THE DIRICHLET PROBLEM

We first suppose that $\partial\Omega \in C^{1,\alpha}$ and $u_0 \in C^{1,\alpha}(\partial\Omega)$. Kellogg [8] showed that harmonic functions with such boundary values are globally $C^{1,\alpha}$. Giraud [5] extended this result to solutions of more general elliptic equations. For a slightly different class of equations, Gilbarg and Hörmander [3] proved not only that the solution are $C^{1,\alpha}$ but also that the operators set up a bijection between suitable weighted Hölder spaces involving second derivatives.

Previously Wiener [20] had studied the question of regularity for the Dirichlet problem for harmonic functions and provided a complete answer by introducing the capacity of a set Σ (which is the infimum over all compactly supported C^1 functions v with $v = 1$ on Σ of $\int|Dv|^2dx$). For $x_0 \in \partial\Omega$ and $\lambda > 0$, let $C_j(\lambda,x_0)$ denote the capacity of the set of points not in Ω but within a distance λ^j of x_0. Wiener proved that the continuity of a certain generalized solution of the Dirichlet problem at x_0 is equivalent to the divergence of the sum $\Sigma C_j(\lambda,x_0)\lambda^{j(2-n)}$. When this generalized solution (which is a classical solution of the elliptic equation) is continuous at x_0, we call x_0 a regular point. Hervé [7] verified Wiener's criterion for equations with Lipschitz coefficients; Krylov [9] showed that the coefficients need only be Dini.

When the coefficients of the equation are not Dini, the situation becomes more complicated. Miller [16], [17] showed that the divergence of Wiener's sum may be neither necessary nor sufficient for a point to be regular in this case; however, Alkhutov [1] introduced an ellipticity function whose Dini continuity implies this equivalence even if the coefficients themselves are discontinuous. Other conditions are known which guarantee the regularity of a boundary point for any equation with bounded coefficients. The first of these conditions is the well-known exterior sphere condition. In 1927, Zaremba [21] proved that an exterior cone condition gives regular boundary points, and Pucci [19] and Miller [15] extended this result to arbitrary operators. Ladyzhenskaya and Ural'tseva's condition A [10, p. 6], which requires Ω to have Lebesque upper density less than one at x_0, also suffices. Although this result is not stated explicitly, it follows easily from Gilbarg and Trudinger's Theorem 9.30 of [4]. Condition A is a measure theoretic version of the geometric cone condition; a geometric generalization of the cone condition is the flat cone condition, which was shown by Lieberman [13] to imply regularity of boundary points.

Landis [11] provided another sufficient condition for regularity points via a generalized capacity. Proceeding in part from Landis's work, Bauman [2] developed an analog of the Wiener criterion for elliptic equations with bounded coefficients. Her criterion is both necessary and sufficient for regularity of boundary points, but it has the drawback that the capacity she constructs is determined by the Green's function of the operator in question.

OBLIQUE DERIVATIVE PROBLEMS

Boundary condition (2) has not received nearly as much attention as (1). Nonetheless some results are known for "bad" domains.

Suppose first that $\partial\Omega \in C^{1,\alpha}$ and that β and g are in $C^{\alpha}(\partial\Omega)$. Giraud [6] showed that solutions of a large class of elliptic equations with boundary condition (2) are in $C^{1,\alpha}(\bar{\Omega})$. Analogs of Gilbarg and Hörmander's results have been established for the oblique derivative problem by Lieberman [12].

Now suppose Ω is merely Lipschitz, and write

$$Lu = a^{ij}D_{ij}u + b^{i}D_{i}u + cu \;.$$

(Here we follow the convention that repeated indices are to be summed from 1 to n.) Suppose also that the coefficients of L are sufficiently smooth, that $c \leq 0$, and that the vector $\beta(x_0)$ points into the interior of a cone lying in Ω with vertex x_0 for all $x_0 \in \partial\Omega$. Nadirashvili [18] asserted that the problem

$$(3) \qquad Lu = 0 \quad \text{on} \quad \partial\Omega, \quad \beta\cdot Du = g \quad \text{on} \quad \partial\Omega$$

has a unique solution for $\beta \in C^{2}(\partial\Omega)$ and $g \in C(\partial\Omega)$ if $c \not\equiv 0$; however, there is a flaw in his proof. Under slightly stronger smoothness hypotheses on the coefficients of L, this flaw has been corrected by Lieberman [14]. In case $c \equiv 0$ Nadirashvili also inferred (correctly) from his basic result that solutions of (3) are unique up to constants and that there is $\psi \in L^{2}(\partial\Omega)$ such that (3) is solvable if and only if

$$\int_{\partial\Omega} g\psi \, ds = 0.$$

He also concluded that $g \in C^{\delta}(\partial\Omega)$ implies $u \in C^{1,\sigma}(\bar{\Omega})$ for small enough $\delta > 0$. This final result is also proved, with $\beta \in C^{\delta}$, in [14].

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