ABOUT SOME ILL-POSED PROBLEMS

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1. INTRODUCTION

The purpose of this note is to give an informal account of recent results, obtained jointly with A. Friedman [1,2] about some non-linear ill-posed problems arising in fluid dynamics.

Before describing them, we wish to make some remarks about ill-posed problems, as an attempt to single out the mathematical issues we are interested in.

There is not a "canonical" definition of ill-posed problems; they are a set of facts, diverse in scope and motivation, and are difficult to unify in a group of issues and methods. Efforts towards an organic theory have been made only in recent years. We mention in particular the books of Tikhonov-Arsenin [17] and Lavrentiev [9]; the monograph of L. Payne [10], and a beautiful review article of G. Talenti [16].

The following way of looking at ill-posed problems is partial and incomplete; however it is adequate for the purpose of this note.

A boundary value problem associated with a partial differential equation is ill-posed if

(a) a solution can be found (if at all!) only for a very narrow class of data, or

(b) if a solution exists, it exhibits instabilities with respect to small variations of the data.

The meaning of "narrow class of data", "instabilities", "small variation", has to be made precise in each single problem, through the specification of appropriate spaces and topologies.

If a problem comes from physics, then it is reasonable to "measure" the data (or their variation) in a topology sufficiently coarse to reflect experimental approximations. Thus the well or ill-posedness of a problem depends on the topology chosen.

The following two examples, by now classical, typify the situation.

Example 1   Let $Q \equiv \{0 < x < 1\} \times \{0 < y < b\}$. Find $u \in C^2(\bar{Q})$ satisfying

$$(P) \qquad \begin{cases} \Delta u = 0 & \text{in } Q \\ u(x,0) = \phi(x) \; ; & u_y(x,0) = \psi(x) \; , \quad 0 < x < 1 \; . \end{cases}$$

This problem has been posed by Hadamard [4]. The discussion to follow is taken from Talenti [16]. A solution of (P) exists if and only if the function

$$(0,1) \ni x \to \phi(x) - \frac{1}{\pi} \int_0^1 \psi(t) \, \ln|x - t| \, dt$$

is analytic. Indeed if u solves (P) write $u = v + w$ where

$$v(x,y) = \frac{1}{2\pi} \int_0^1 \psi(t) \, \ln[(x - t)^2 + y^2] \, dt \; .$$

Since $\Delta v = 0$ in $\{y > 0\}$ and $v_y(x,0) = \psi(x)$, $x \in (0,1)$, we must have $\Delta w = 0$ in Q and $w_y(x,0) = 0$. Therefore, by the reflection principle $(x,y) \to w(x,|y|)$ is harmonic in $\{0 < x < 1\} \times \{-b < y < b\}$ and $x \to w(x,0)$ is analytic.

Thus a solution exists if and only if the data are in a quite narrow class. If we insist on solutions $u \in C^2(\bar{Q})$, then the natural assumption on the data is

$$\phi, \psi \in C^2([0,1]) \times C^1([0,1]) = \mathcal{C} \; .$$

The previous remarks show that those data ϕ,ψ for which a solution

exists are very few relatively to C .

Also solutions of (P) exhibit instability. In (P) set $\psi(x) \equiv 0$,

$\phi(x) = \phi_n(x) = \exp(-\sqrt{n}) \cos(nx)$, $x \in (0,1)$, $n \in \mathbb{N}$. Then (P) has a

solution $u(x,y) = \exp(-\sqrt{n}) \cos(nx) \cdot \cosh(ny)$, such that $u(0,1)$ can

be made arbitrarily large, in spite of the fact that

$$\left\| \frac{d^k}{dx^k} \phi_n \right\|_{\infty,[0,1]} \to 0 \quad \text{as} \quad n \to \infty , \quad \forall k \in \mathbb{N} .$$

This example lends itself to a number of general considerations about

ill-posed problems, especially about choices of topologies. We refer

to [ ] for a complete and clear discussion.

Example 2 (The backward heat equation.)

Let Ω be a bounded open set in $\mathbb{R}^N$, $N \geq 1$ with smooth boundary

$\partial\Omega$. Let $0 < T < \infty$, set $\Omega_T \equiv \Omega \times (0,T)$ and consider the problem of

finding

$$u \in C(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) ,$$

such that

$$(H) \qquad \begin{cases} u_t + \Delta u = 0 & \text{in} \quad \Omega_T \\ u(x,0) = u_0(x) & \text{in} \quad \Omega . \end{cases}$$

Part of the problem is to give conditions on u_0 to insure the

existence of a solution. If (H) were a physical problem, it would seem

natural to assume $u \in H_0^1(\Omega)$, to account for experimental errors. It

turns out that very few elements of $H_0^1(\Omega)$ (in fact very few elements

of the space of analytic functions in Ω), yield the existence of a

solution. It is in this sense that (H) is improperly-posed.

The following result is due to Showalter [14].

Problem (H) has a (unique) solution if and only if $u_0 \in D[(-\Delta)^n]$,
$\forall n \in \mathbb{N}$. Here $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is the Riesz-map and $D(T)$ denotes
the domain of the operator T .

In fact Showalter's result is more general in that $-\Delta$ could be
replaced by any linear unbounded operator A in a complex Hilbert
space H , provided $D(A)$ is dense in H and A^2 is accretive.

The proof employs the method of quasi-reversibility (see Lattes-
Lions [8]). First changing $t \to -t$ (H) reduces to the final value
problem

(1.1)
$$u_t - \Delta u = 0 \qquad \text{in} \quad \Omega_T$$
$$u(x,T) = u_0(x) , \qquad x \in \Omega .$$

Then such a problem is approximated by the pseudo-parabolic
approximations

(1.2)
$$\begin{cases} \dfrac{\partial}{\partial t} \theta_\alpha + (-\Delta)_\alpha \theta_\alpha = 0 & \text{in} \quad \Omega_T \\[2mm] \theta_\alpha(x,T) = u_0(x) \end{cases}$$

where $(-\Delta)_\alpha = -\Delta \circ (I - \alpha\Delta)^{-1}$ is the Yoshida approximation of $-\Delta$ in
$L^2(\Omega)$. Since $(-\Delta)_\alpha : L^2(\Omega) \to L^2(\Omega)$ is Lipschitz-continuous (and
maximal monotone), (1.2) has a unique solution θ_α , which can be
represented by $S_\alpha(-t) u_0(x)$, $t \le T$, where $S_\alpha(\cdot)$ is the group
generated by $(-\Delta)_\alpha$ in $L^2(\Omega)$. The value $S_\alpha(-T) u_0$ is then used as
an initial datum to solve (1.1) forward in time. If $t \to S(t)$ is the
semigroup generated by $-\Delta$ in $L^2(\Omega)$ then this process yields an
approximate solution

$$u_\alpha(t) = S(t) [S_\alpha(-T) u_0(x)] ,$$

and it remains to prove that $u_\alpha(T) \to u_0(x)$ in $L^2(\Omega)$ as $\alpha \to 0$. It

is shown in [14] that such a convergence in fact occurs if and only if

$u_0 \in D[(-\Delta)^n]$, $\forall n \in \mathbb{N}$.

2. THE HELE-SHAW FLOW

Consider a slow incompressible viscous fluid, say oil, moving together with a light fluid, say air, between slightly separated plates. Such a device is called a Hele-Shaw cell and it is studied in connection with flow of fluids in porous media. The objective is to remove the heavy fluid from the cell. We will consider two cases.

HS1   The fluid occupies a bounded domain D and surrounds a "core" G through which it is removed. The suction along ∂G is at constant rate Q and causes the fluid blob to recede.
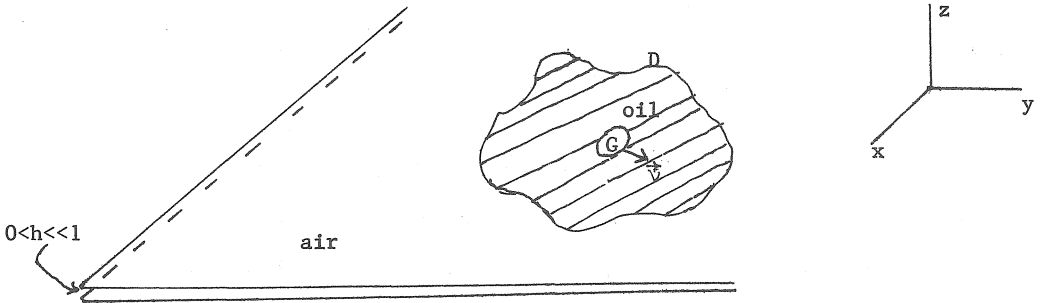

Fig. 1.

Let $P = p(x,y,z,t)$ be the pressure of the heavy fluid, and let $\vec{\nu}$ denote the normal to ∂G directed towards the fluid. Then

(2.1) $\dfrac{\partial p}{\partial \nu} = Q > 0$ along ∂G (pressure decreased)

where Q is a given positive constant.

It is observed in experiments that the suction process ends with the formation of irregular "fingers". Moreover repeating the experiment with the same configuration of the initial blob D ("same" within the limits of experimental errors), it is observed that the extinction time and the shape and size of the fingers are dramatically different. The problem is physically ill-posed in this sense.

A natural question is then to find the configurations of the initial blob (if any) for which the suction ends smoothly with no fingers. More generally, given a configuration of fingers, is there an initial blob for which the suction process ends with such a configuration?

HS2   The fluid occupies the whole plane $\mathbb{R}^2$. A light fluid is injected and forms a bubble D(t) at each time t . The oil is incompressible and the extraction at $|x| \to \infty$ is at constant rate Q given by

(2.2) $\left| \nabla_x \left[P(x,y,z,t) + \frac{Q}{2\pi} \ln \sqrt{(x^2 + y^2)} \right] \right| = 0(|x|^{-2})$,

where ∇_x denotes the gradient with respect to the space variables (x,y) only and $|x| = (x^2 + y^2)^{\frac{1}{2}}$.
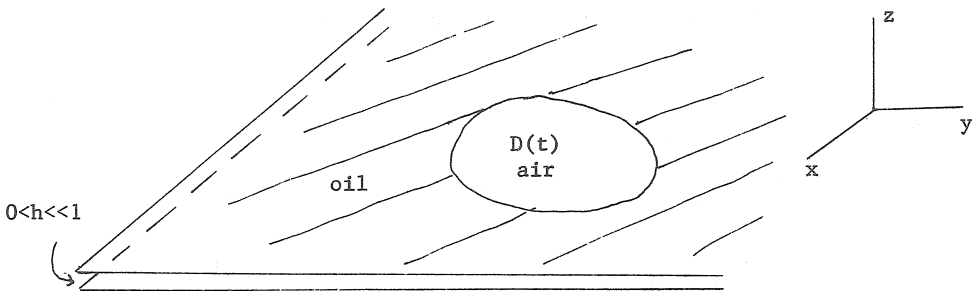

Fig. 2.

Also in this case it is observed that as the bubble increases the light fluid escapes through irregular fingers. The problem is then to determine the shape of those initial bubbles for which the oil can all be removed (i.e. for which a solution exists for all times).

2-(a) The Model

In both cases we assume the pressure of the air is neglectible and the viscous fluid is Newtonian so that the motion is governed by Navier-Stokes equations for the velocity

$$\vec{v}(x,y,z,t) \equiv (v^{(1)}, v^{(2)}, v^{(3)})(x,y,z,t).$$

However the motion is slow ($|\vec{v}|$ small) so that we may assume that $\vec{v}$ satisfies the steady-state Stokes equations

(2.3)
$$\nu \Delta \vec{v} = \nabla p \; ; \quad \text{div } \vec{v} = 0$$

where the kinematic viscosity ν is proportional to h^{-2}. Since $0 < h \ll 1$ one can also assume

$$v^{(3)}(x,y,z,t) = 0 \quad ; \quad p(x,y,z,t) = p(x,y,t)$$

$$\Delta v^{(i)} = v_{zz}^{(i)} \; , \quad i = 1,2 \; .$$

The term $\Delta \vec{v}$ accounts for internal friction in the fluid [7]. Therefore since the motion takes place on planes parallel to $z = 0$, the terms $v_{xx}^{(i)}$, $v_{yy}^{(i)}$ are neglectible when compared to $v_{zz}^{(i)}$, $i = 1,2$.

From (2.3) we have

(2.4)
$$\nu v_{zz}^{(1)} = p_x \; ; \quad \nu v_{zz}^{(2)} = p_y \; , \quad 0 \leq z \leq h$$

and $v^{(i)}(x,y,0,t) = v^{(i)}(x,y,h,t) = 0$, $i = 1,2$ (the fluid is viscous). Multiplying (2.4) by z and integrating twice in z we find

(2.5) $$\vec{v} = -c \nabla p \qquad , \qquad c = c(h)$$

where

$$v^{(i)} = \frac{1}{h} \int_0^h v^{(i)} \, dz \quad , \quad i = 1,2 \; .$$

Equation (2.5) is Darcy's law and explains the connection between the Hele-Shaw flow and flow of fluids in a porous medium.

Incompressibility of the oil fields

(2.6) $$\text{div } \vec{v} = 0 \quad \text{and} \quad \Delta p = 0 \quad \text{in the fluid .}$$

Suppose the interface Γ separating air from oil is smooth and has an intrinsic representation $\Phi(x,y,z,t) = 0$, where $\Phi \in c^2(\mathbb{R}^4)$, $\Phi < 0$ in the fluid , $\Phi > 0$ in the air .

Then on Γ we have $x = x(t)$, $y = y(t)$ and the velocity of the fluid coincides with the velocity of Γ so that

$$(\mathring{x},\mathring{y}) = \vec{v} \quad \text{on} \quad \Gamma \; .$$

Computing the total time derivative of Φ on Γ we have

$$\nabla_x \Phi \cdot (\mathring{x},\mathring{y}) = -\Phi_t \quad \text{on} \quad \Gamma \; .$$

This and (2.5) give

(2.7) $$\nabla_x p \cdot N_x = N_t \; ,$$

where

$$N_x = \frac{\nabla_x \Phi}{|\nabla \Phi|} \; ; \quad N_t = \frac{\Phi_t}{|\nabla \Phi|} \; , \quad |\nabla \Phi| \neq 0 \quad \text{on} \quad \Gamma \; .$$

These remarks yield the following classical formulation of the problems at hand. The problems presented are in 2 space variables. We will however present a general N-dimensional formulation of them. This will give quite general results especially in the case of HS2.

2-(b) Mathematical Formulation of HS1

Let D and G be two open sets in $\mathbb{R}^N$ with $C^{2+\alpha}$ boundaries ∂D and ∂G (for some $0 < \alpha < 1$). Assume $\bar{G} \subset D$ and set $\Omega = D \backslash \bar{G}$, $\Omega_T \equiv \Omega \times (0, T]$ for any $0 < T < \infty$.

We wish to find $T \in (0, \infty)$, $u : \bar{\Omega}_T \to \mathbb{R}$ and $\Phi : \bar{\Omega}_T \to \mathbb{R}$ such that setting

$$\Omega(t) \equiv \{x \in \Omega : \Phi(x, t) < 0\} : \partial_0 \Omega(t) \equiv \partial \Omega(t) \backslash \partial G ,$$

the following conditions are satisfied

(2.8) $-\Delta u = 0$ in $\Omega(t)$, $\quad 0 < t \leq T$

(2.9) $u = 0$ on $\partial_0 \Omega(t)$, $\quad 0 < t \leq T$

(2.10) $\dfrac{\partial u}{\partial \nu} = Q$ on $\partial G \times \{0 < t \leq T\}$,

(2.11) $\nabla_x u \cdot N_x = N_t$ on $\partial_0 \Omega(t)$, $\quad 0 < t \leq T$,

(2.12) $u \leq 0$ in Ω_T, $\quad u(x, 0) < 0$ in Ω

(2.13) $u(x, T) \equiv 0$.

If u is a classical solution of (2.8)-(2.13) then by the maximum principle $|\nabla_x u| \neq 0$ on $\partial_0 \Omega(t)$ and we can take $\Phi = u$. Then (2.11) becomes

(2.14) $$|\nabla_x u|^2 = u_t .$$

By the maximum principle it then follows that $u_t \geq 0$; hence the free boundary is decreasing in the sense that

$$\Omega(t_1) \subset \Omega(t_2) \quad \text{if} \quad t_1 > t_2 .$$

We note that (2.12) need not be required; it is in fact a consequence of the maximum principle.

2-(c) Mathematical Formulation of HS2

Let D be an open set in $\mathbb{R}^N$ with $C^{2+\alpha}$ boundary ∂D. D will be the initial bubble.

Find $\Phi : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ be C^2, $\Phi(x,0) > 0$, $x \in D$, and $u : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ such that setting

$$\Omega(t) \equiv \{x \in \mathbb{R}^N : \Phi(x,t) < 0\} ,$$

the following conditions are satisfied

(2.15) $-\Delta u = 0$ in $\Omega(t)$, $0 < t < \infty$

(2.16) $u = 0$ on $\partial\Omega(t)$, $0 < t < \infty$

(2.17) $\nabla_x u \cdot N_x = N_t$ on $\partial\Omega(t)$, $0 < t < \infty$

(2.18) $u \leq 0$ in $\Omega(t)$, $0 < t < \infty$

(2.19) $u = 0$ on $D(t) \equiv \mathbb{R}^N \backslash \Omega(t)$

(2.20) $\nabla_x(u(x,t) - Q\Gamma_N(|x|)) = O(|x|^{-N})$ as $|x| \to \infty$,

where Q is a given positive constant and, denoting with ω_N the area of the unit sphere in $\mathbb{R}^N$,

$$\Gamma_N(|x|) = \begin{cases} -\dfrac{1}{2\pi} \ln|x| & \text{if } N = 2 \\[3mm] \dfrac{1}{(N-2)\omega_N} |x|^{2-N} & \text{if } N \geq 3 . \end{cases}$$

Condition (2.20) is equivalent to

$$u(x,t) = Q\Gamma_N(x) + \gamma(t) + O(|x|^{1-N})$$

as $|x| \to \infty$ (see Lemma 2.1 of [ ] page 10).

For further information on the model and classical formulation we refer to S. Richardson [11,12], P.G. Saffman, G.I. Taylor [13], Lamb [7].

3. HS1, THE INITIAL BLOB

In this section we will characterize all the initial domains D

for which HS1 has a solution. First we reformulate (2.8)-(2.13) in a

suitable weak form.

3-(a) Weak Formulation

Assume $u \in C^{2,1}(\Omega_T)$ is a classical solution of (2.8)-(2.13) with

free boundary $\Phi(x,t) = 0$ of class $C^{2,1}$. Let $\phi \in C_0^\infty(\Omega_T)$ be such

that supp $\phi \cap (\bar{\Omega} \times \{T\})$ is empty and denote by $\langle \cdot, \cdot \rangle$ the

distribution pairing in Ω_T . Then

$$\langle -\Delta u, \phi \rangle = -\iint\limits_{\Omega_T} u \Delta \phi = \iint\limits_{[u<0]} \nabla_x u \cdot \nabla_x \phi \, dxdt$$

$$= \int_0^T \int_{\partial_0 \Omega(t)} \nabla_x u \cdot N_x \phi \, d\sigma - \iint\limits_{[u<0]} \Delta u \, \phi \, dxdt .$$

Using (2.8) and (2.11) we get

(3.1) $\langle -\Delta u, \phi \rangle = \int_\Gamma N_t \phi \, d\sigma$

where

$$\Gamma \equiv \bigcup_{0<t\leq T} ((x,t) : x \in \partial_0 \Omega(t)) .$$

Introduce the graph

$$H(s) = \begin{cases} 1 & \text{if } s < 0 \\ [0,1] & \text{if } s = 0 \\ 0 & \text{if } s > 0 , \end{cases}$$

and compute

$$\langle \frac{\partial}{\partial t} H(u), \phi \rangle = \langle H(u), -\phi_t \rangle = -\iint\limits_{[u<0]} \phi_t \, dxd\tau = -\int_\Gamma N_t \phi \, d\sigma .$$

Comparing with (3.1) we find

(3.2) $$\frac{\partial}{\partial t} H(u) - \Delta u \ni 0 \quad \text{in} \quad \mathcal{D}^1(\Omega_T) \ .$$

Introduce a new unknown function

$$v(x,t) = \int_t^T u(x,\tau) \ d\tau$$

and observe that if u is a classical solution then,

$$H(u) = H(v) \ .$$

From (3.2) one obtains

(3.3) $$-\Delta v \in H(v) - \xi(x) \ , \quad \forall t \in [0,T]$$

where $\xi(x) \subset H(u(x,T))$ is to be determined.

    To find T integrate (3.3) over $\Omega(t)$ for $0 < t < T$ to obtain

(3.4) $$Q(T-t) \ |\partial G| = \int_{\Omega(t)} [H(v(x,t)) - \xi(x)] \ dx \ .$$

Letting $t \searrow 0$

$$QT \ |\partial G| = \int_\Omega (1 - \xi(x)) \ dx \ .$$

These quantities will now be used to define the concept of weak

solution.

    We seek $T \in (0,\infty)$, $\xi : \Omega \to [0,1]$ and $v : \bar{\Omega}_T \to \mathbb{R}$ such that the

following conditions hold:

(3.5) $$v \leq 0$$

(3.6) $$v \in C^0([0,T]; H^2(\Omega))$$

(3.7) $$-\Delta v \in H(v) - \xi \quad \text{a.e. in} \quad \Omega \ , \quad \forall t \in [0,T]$$

(3.8) $$v = 0 \quad \text{on} \quad \partial D \times (0,T)$$

(3.9) $$\frac{\partial v}{\partial \nu} = Q(T-t) \quad \text{on} \quad \partial G \times \{t\} \ , \quad 0 < t < T$$

(3.10) $$v(x,T) \equiv 0$$

(3.11) $$Q|\partial G|T = \int_\Omega (1 - \xi(x)) \ dx$$

(3.12) $v(x,0) < 0$ in Ω .

Equation (3.7) has to be interpreted in the sense that

$-\Delta v = \eta(x,t) - \xi(x)$ a.e. Ω_T , where η is a selection out of the graph

$H(v)$. Here (3.9) is meant in the weak sense; however it is shown in

[1] that $\nabla_x v$ is Hölder continuous in $\bar{\Omega} \times (0,T)$ and thus (3.9) will

hold in the classical sense.

A triple (T,ξ,v) satisfying (3.5)-(3.12) is a weak-solution of

HS1 and ξ is the terminal phase; in particular ξ might be used to

represent the fingers.

Suppose we assign ξ *a priori* in the following way. Let Σ be a

disjoint union of open sets Σ_i in $\mathbb{R}^N \backslash \bar{G}$ such that $\partial \Sigma_i$ contains an

open subset of ∂G , and set

$$\xi = \chi_\Sigma \ .$$

If $\Sigma = \phi$ we set $\xi = 0$ (final extinction with no fingers).

3-(b) Existence of Solutions

We now establish under what conditions on the domain D , occupied

by the blob at time $t = 0$, the problem HS1 has a solution.

Lemma 3.1   *If* (T,ξ,v) *is a weak solution then*

$$\frac{\partial v}{\partial \nu} (x,0) = 0 \quad \text{on} \quad \partial D \ .$$

Proof   Integrating (3.7) for $t = 0$ over Ω we find

$$\int_{\partial D} \frac{\partial v}{\partial \nu} (x,0) \, d\sigma = 0 \ .$$

On the other hand since $v \leq 0$ in Ω_T and $v = 0$ on ∂D we have

$\frac{\partial v}{\partial \nu} \geq 0$ on ∂D and the lemma follows.

From this we deduce

Corollary 3.2   *If there exists a weak solution* (T,ξ,v) *of* HS1 *then the function* $V(x) = v(x,0)$ *is a solution of the variational inequality*

(3.13)
$$
\left.
\begin{cases}
-\Delta V \leq 1 - \xi \\[4pt]
\quad V \leq 0 \\[4pt]
V(-\Delta V - 1 + \xi) = 0 \\[4pt]
\dfrac{\partial V}{\partial \nu} = QT
\end{cases}
\right\}
\quad \text{a.e. in } \mathbb{R}^N \backslash G
$$
$$
\frac{\partial V}{\partial \nu} = QT \qquad \text{on } \partial G
$$

(3.14)
$$
\Omega \equiv \{x : V(x) < 0\} .
$$

For any ξ *of the form* χ_Σ *and* $\lambda > 0$ *, denote by* $\Omega(\xi,\lambda)$ *the open set* $\{V < 0\}$ *, where* V *is the unique solution of* (3.13) *with* $QT = \lambda$.

The Corollary says that if HS1 has a weak solution then the initial blob must be concentrated in the support of the solution of (3.13). Note that given (3.13) the solution V will determine its own support (3.14) so that such a domain cannot be assigned in an arbitrary way.

The following theorem, proved in [ ] establishes the converse.

Theorem 3.3   *Let* $\xi = \chi_\Sigma$ *and* $\Omega(\xi,\lambda)$ *be given. There exists a unique weak solution* (T,ξ,v) *of* HS1 *with* $Q = \lambda/T$.

These facts characterize completely the initial domains D for which there is a solution to HS1.

Given $\xi = \chi_\Sigma$ and the problem (3.13) we may say that HS1 has a solution if and only if the blob is confined in the complement of the coincidence set of the obstacle problem (3.14).

In particular if a specific configuration of fingers is prescribed, say Σ , then setting $\xi = \chi_\Sigma$ the arguments above give a way of finding an initial blob which will terminate with fingers Σ .

Analogously if $\Sigma = \phi$ (i.e. $\xi \equiv 0$) then (3.13) says how to choose the initial blob to have extinction without fingers.

The results in [1] are more general. In particular more general structures for the final phase ξ are considered and regularity results concerning the free boundary are established.

In [1] we also stress that the phenomenon of smooth extinction without fingers is only linked to the variational structure of (3.13) and not to the regularity of ∂D. We give an example that shows that any finger configuration (however irregular) can develop from an initial blob D whose boundary is nearly spherical and analytic.

4. HS2. THE INITIAL BUBBLES

We let Σ be a set in $\mathbb{R}^N$ prescribed to be the final phase of the fluid (at $t \to \infty$); Σ consists of "fingers" created at time $t = \infty$ in the viscous fluid, or equivalently we may say that the bubble increases to $\mathbb{R}^N \backslash \Sigma$ as $t \to \infty$.

We will give conditions both on Σ and the initial bubble so that the fluid reaches the configuration Σ after a large time. In particular if we impose $\Sigma = \phi$ then we will find conditions to extract the fluid without fingers.

We start by noting that the distributional formulation (3.2) holds also in the present situation. Therefore at least formally we have to solve the following problem. Given an open set $D(0)$ (the initial bubble), find u such that

$$(4.1) \quad \begin{cases} \dfrac{\partial}{\partial t} H(u) - \Delta u \ni 0 \quad \text{in } \mathcal{D}^1(\mathbb{R}^N \times (0,\infty)) \\[2mm] H(u(x,0)) = \chi[\mathbb{R}^N \backslash D(0)] \\[2mm] \nabla_x (u - Q\Gamma_N(|x|)) = O(|x|^{-N}) \quad \text{as } |x| \to \infty \\[2mm] H(u(x,0)) \sim \chi_\Sigma \ . \end{cases}$$

The key idea in section 3 was to introduce the new unknown

$$(4.2) \qquad v(x,t) = \int_t^T u(x,\tau) \, d\tau \, ,$$

and define the concept of weak solution in terms of v . In our case $T = \infty$ so that it is difficult to work directly with v . Instead we shall define a sequence of truncated problems corresponding to fictitious final phases $\mathbb{R}^N \backslash \Sigma_n$ (Σ_n will decrease to the terminal phase Σ as $n \to \infty$), with "pressures"

$$(4.3) \qquad v_n(x,t) = \int_t^{T_n} u_n(x,\tau) \, d\tau \, .$$

To start and describe the process, we truncate (4.1) at time $t = T$ and rewrite it in terms of v defined by (4.2). We let Σ_0 be an open set in $\mathbb{R}^N$ containing a neighbourhood of infinity and set $K_0 \equiv \mathbb{R}^N \backslash \Sigma_0$. To the truncated problem we impose a fictitious final phase $\xi_0 \equiv \chi_{\Sigma_0}$. Thus we have to find $(x,t) \to v(x,t)$, $t \in [0,T]$ satisfying

$(4.4) \qquad v \leq 0 \, , \qquad v \in C^0(0,T; H^2_{loc})$

$(4.5) \qquad -\Delta v \in H(v) - \xi_0 \quad$ a.e. $\mathbb{R}^N \quad \forall t \in (0,T)$

$(4.6) \qquad \nabla_x (v - Q(T-t) \Gamma_N(|x|)) = O(|x|^{-N}) \quad$ as $|x| \to \infty \quad \forall t \in (0,T)$

$(4.7) \qquad v(x,T) \equiv 0 \, .$

The problem (4.4)-(4.7) is different in nature with respect to (3.5)-(3.12). From (4.6) by integration over $\mathbb{R}^N$ and letting $t \searrow 0$ we find

$$(4.8) \qquad QT = \int_{K_0 \backslash D} (1 - \xi_0(x)) \, dx$$

where D is the initial bubble. Also from (4.5) and (4.7) we have

$\xi_0 \in H(v(x,T))$ (in the sense of the graphs).

Thus we are given ξ_0 , Q , T and we are asked to find a function $v(x,t)$ *and a set* D such that (4.4)-(4.8) hold with

(4.9) $\mathbb{R}^N \backslash D \equiv \{x : v(x,0) < 0\}$.

To this end we observe that if v is a weak solution, setting

          $D(t) \equiv \{x \in \mathbb{R}^N;\ v(x,t) = 0\}$; $V(x) = v(x,0)$,

we have by (4.8)

(4.10) $\lambda = \text{meas}(K_0 \backslash D)$ $(K_0 \equiv \mathbb{R}^N \backslash \Sigma_0)$,

and v is the solution of the variational inequality

(4.11)
$$\begin{cases} -\Delta v \leq 1 - \xi_0 \\ v \leq 0 \\ v(-\Delta v - 1 + \xi_0) = 0 \end{cases} \text{a.e. in } \mathbb{R}^N$$
$$\nabla_x(v - \lambda \Gamma_N) = O(|x|^{-N}) \quad \text{as } |x| \to \infty .$$

The following theorem establishes existence of time-truncated problems.

Theorem 4.1   *Given* Σ_0 *as above and* Q > 0 *, for any* $\lambda \in (0,\ \text{meas } K_0)$ *there exists a unique solution of* (4.4)-(4.10) *with* $T = \lambda/Q$.

The proof of the theorem is in [2] page 15-23 and makes use of potential representation of solutions by means of the kernel Γ_N . The proof shows also the following additional facts

(4.12) $v(x,t) = \displaystyle\int_{K_0 \backslash D(t)} \Gamma_N(x-y)\ dy + \gamma(t)$

where $t \to \gamma(t)$ is a bounded function of t and $v(x,t) = 0$ $\forall x \in D(t)$. Moreover

(4.13) $\text{meas } D(t) = \text{meas } D(0) + Qt$.

Using this existence result we now define a sequence of truncated problems as follows.

(i) Definition of truncated final phases

Consider the spheres $S_n^0 \equiv \{x \in \mathbb{R}^N : |x| < n\}$ and the exterior of S_n^0 given by

$$S_n^\infty \equiv \{x \in \mathbb{R}^N : |x| > n\} \ .$$

If Σ is the (prescribed) final phase of the fluid, introduce fictitious truncated phases Σ_n

(4.14)
$$\begin{cases} \Sigma_n \equiv S_n \cup \Sigma ; \quad \xi_n = \chi_{\Sigma_n} \ , \\[2mm] K_n \equiv S_n^0 \backslash \Sigma \\[2mm] \lambda_n = \text{meas } K_n - \mu \ . \end{cases}$$

(ii) Definition of truncated times

Having given λ_n by the last of (4.14) and the positive number Q being fixed, we may define T_n according to theorem 4.1, by

$$T_n = \lambda_n/Q \ .$$

(iii) Definition of truncated problems

For $n = 0,1,2,\ldots$, solve the problems

(4.15)
$$\begin{cases} v_n \le 0 , \quad v_n \in C^0(0,T_n,H_{loc}^2) \\[2mm] -\Delta v_n \in H(v_n) - \xi_n \quad \text{a.e. } \mathbb{R}^N , \quad \forall t \in (0,T_n) \\[2mm] \nabla_x(v_n - Q(T_n - t)\Gamma_N(x)) = 0 \ (|x|^{-N}) \quad \text{as } |x| \to \infty \quad t \in (0,T_n) \\[2mm] v_n(x,T_n) = 0 \ . \end{cases}$$

We briefly comment on the parameter μ in the last of (4.14). The sets Σ and S_n^0 being given, then $K_n \subset K_{n+1}$ is a well defined sequence of sets and $\{\text{meas } K_n\}$ is an increasing sequence. Then

motivated by (4.10) we assign a value μ for the measure of the
initial bubble and define λ_n by the last of (4.14). Thus when the
construction of the solution will be completed, it will have an
indeterminate parameter μ . This is not surprising in view of (4.13)
and essentially it corresponds to fixing the initial time.

By virtue of Theorem 4.1, problem (4.15) has a unique solution
and moreover

(4.16) $$v_n(x,t) = \int_{K_n \setminus D_n(t)} \Gamma_N(x-y)\,dy + \gamma_n(t)$$

(4.17) $$v_n(x,t) = 0 \quad \text{if} \quad x \in D_n(t)$$

(4.18) $$\text{meas } D_n(t) = \mu + Qt$$

where

(4.19) $$D_n(t) \equiv \{x \in \mathbb{R}^N : v_n(x,t) = 0\} .$$

Next we let $n \to \infty$ and define constructively a suitable concept
of weak solution.

For this we will need a priori estimates on $\{v_n\}$ and $\{D_n\}$.
Such estimates are derived under the assumptions that the final phase
Σ is made out of fingers which are not "too thick". Precisely we
assume

(4.20) $$\int_1^\infty \text{meas}\{\Sigma \cap (|x| = 2)\}\,d\nu < \infty .$$

(a) A priori estimates on the bubbles

Let (4.20) hold and let $D_n(t)$ be the approximate bubbles
arising from (4.15). Then $D_n(t)$, for finite time are all confined
within a finite ball. Precisely $\forall t_0 > 0$ there exists $R_0 > 0$,
$R_0 = R_0(t_0)$ such that

(4.21) $\qquad D_n(t) \subset B_{R_0}$, $\quad \forall 0 < t < t_0$, $\quad \forall n \in \mathbb{N}$

where $B_{R_0} \equiv \{x \in \mathbb{R}^N ; |x| < R_0\}$. For the proof we refer to Lemma 4.1

and Corollary 4.3 of [2] pp.35-39.

(b) A priori estimate on $\{v_n\}$

Let (4.20) hold. The following estimates are local. Fix a box $|x| < R_0$, $0 < t < R_0$. There is a constant $C = C(R_0)$ such that

(4.22) $\qquad \left| v_n(x,t) - v_n(0,t) \right| \leq C$

for all $(x,t) \in \text{box}\{|x| < R_0; 0 < t < R_0\}$ and for all $n \in \mathbb{N}$.

From (a) it follows that, given R_0 , within the box $\{|x| < R_0, 0 < t < R_0\}$, there is a sequence $\{x_n\}$ such that

$$v_n(x_n,0) = 0 .$$

This implies by (4.22) that $\left| v_n(0,t) \right| \leq C$ and

(4.23) $\qquad v_n \in L^\infty_{\text{loc}}(\mathbb{R}^N \times (0,\infty))$, uniformly in n .

Estimate (4.23) is the key fact to derive founds in stronger norms (for example $C^{1+\alpha}_{\text{loc}}$). Therefore by a suitable selection of a subsequence

$$v_n \to v \quad \text{uniformly on compact sets.}$$

Letting $n \to \infty$ in (4.15) we find

(4.24) $\qquad v \leq 0$, $\quad v \in C^0(0,T; H^2_{\text{loc}})$, $\quad \forall T > 0$

(4.25) $\qquad -\Delta v \in H(v) - \chi_\Sigma$ a.e. $\mathbb{R}^N \times (0,\infty)$.

The last of (4.15) loses meaning as $n \to \infty$ in view of the fact that we have only local estimates on $\{v_n\}$. We will discuss later the case of $\Sigma = \phi$ when $v(x,\infty) = 0$ is recovered in some sense.

Finally the third of (4.15) has to be interpreted in the limit as

(4.26) $\qquad |v(x,t)| \leq C|x|^2 , \quad$ for $\quad |x| \to \infty$.

This last fact is proven by suitable expansion of Γ_N in harmonic

polynomials for $|x| \to \infty$. For the proof see Theorem 4.5 of [2] pp.39.

Finally note that (4.24)-(4.26) is a family of variational

inequalities, parametrized with $t \in (0,\infty)$ and with non-coincidence

sets

$$D(t) \equiv \{x \in \mathbb{R}^N; \ v(x,t) = 0\} .$$

Then by a classical stability result for variational inequalities we

have

(4.27) $\qquad d(D_n(t), D(t)) \to 0 \quad$ as $\quad n \to \infty , \quad \forall t > 0 ,$

where $d(A,B)$ denotes the Hausdorf distance between two sets A , B .

Moreover from (4.18) and the mentioned stability result, it follows

(4.28) $\qquad$ meas $D(t) = \mu + Qt$.

Now we have completed the construction of solutions of HS2.

2-(i) Conditions of the initial bubbles

The construction shows that, given Σ , the final phase, then the

bubbles $D(t)$, $t \geq 0$ that generate the solution are the non-

coincidence sets $D(t)$ of the variational inequality (4.24)-(4.26).

2-(ii) The case of Σ empty

Let us return to the solution v_n , D_n of (4.15) and set

$\Sigma = \phi$. Then from (4.16) and the convergence arguments just discussed

we deduce

(4.29) $\qquad \lim_{n\to\infty} \int_{S_n^0 \setminus D(t)} [\Gamma_N(x-y) - \Gamma_N(y)] \ dy = v(x,t)$

(4.30) $\qquad v(x,t) = 0 \quad$ if $\quad x \in D(t)$,

so that

(4.31) $\lim\limits_{n\to\infty} \int_{S_n^0\backslash D(t)} [\Gamma_N(x-y) - \Gamma_N(y)]\, dy = 0\ ,\qquad \forall x \in D(t)\ .$

For $\lambda > 0$ consider the surfaces λS_n^0 and the corresponding weak

solutions

$$v_{n,\lambda}(x,t) = v_n\left(\frac{x}{\lambda}\ ,\ \frac{t}{\lambda}\right)$$

$$D_{n,\lambda}(t) = \lambda\, D_n\left(\frac{t}{\lambda}\right)\ ;\ T_{n,\lambda} = \lambda T_n\ .$$

Analogously to (4.31) we obtain

(4.32) $\lim\limits_{n\to\infty} \int_{\lambda S_n^0\backslash \lambda D(t/\lambda)} [\Gamma_N(x-y) - \Gamma_N(y)]\, dy = 0$

if $x \in \lambda D(t/\lambda)$. Taking $t = 0$ and comparing (4.31) and (4.32) we

obtain

(4.33) $\int_{\lambda D\backslash D} [\Gamma_N(x-y) - \Gamma_N(y)]\, dy = 0\quad$ if $\quad x \in D\ ,$

and where $D = D(0)$.

   Equation (4.33) is a geometric condition on D , i.e. on the

initial bubble. It says that the Newtonian potential generated by a

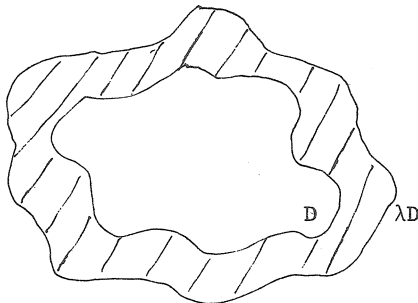uniform distribution of masses on the "shell" $\lambda D\backslash D$

Fig. 3

is constant inside the cavity D . Such domains are called homeoids

(see P. Dive [3]). A theorem of Newton asserts (see Kellogg [6]) that ellipsoids are homeoids.

The converse is also true. In dimension N = 2 a proof is in Kellogg [6] and in dimension N = 3 the proof is given in P. Dive [3]. Both these proofs rely heavily on calculation procedures. We have demonstrated that the statement is in fact true in every dimension N , via an indirect argument (see Theorem 5.1 of [2] p.45). Therefore we have

Every N-dimensional homeoid is an ellipsoid and vice versa.

The discussion above shows that if the initial bubble is a homeoid, then a solution of HS2 with $\Sigma = \phi$ exists and it is classical. The fact that it is classical follows from the smoothness of the free boundary $\partial D(t)$.

In fact also the converse is true. Every classical solution of HS2 has a homeoid as initial bubble. Therefore we have

A classical solution of the Hele-Shaw problem with empty final phase Σ exists if and only if the initial bubble is an ellipsoid.

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