

EARTHQUAKE LOCATION VIEWED AS AN INVERSE PROBLEM

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The earth is continuously being subjected to both internal and external stresses of varying magnitudes. If the stresses are not too large, elastic or plastic deformation may occur. However if in some region of the earth they are allowed to accumulate over a period of time to the point exceeding the strength of the material, then according to Reid's elastic rebound theory (1911) this will rapidly lead to fracture. Fracture inside the earth involves the sudden release of elastic strain energy; such an event is commonly known as an earthquake. An earthquake generates shear and compressional kinetic energy which both radiate from the source and travel through the earth as elastic (or seismic) waves. This energy may be detected by a seismic network of receivers at the earth's surface which record the ground motion caused by the passage of these waves. A major problem in seismology is to deduce the earthquake source parameters and seismic properties of the earth from a set of observations at the surface.

In the hypocentral location problem one wishes to determine only the four hypocentral parameters of an earthquake x_k , $k = 1 \dots 4$, i.e. the three spatial coordinates and the origin time of the event. [Note we essentially neglect the detailed structure of the source region and consider a point solution]. The data for the problem consists of the observed first arrival times at a network of seismic stations of

identifiable seismic phases t_{0i} , $i = 1 \dots N$ (usually body wave phases, see later discussion).

The inversion of seismic data for hypocentral parameters is actually an underdetermined problem. This is due to our lack of knowledge of the real earth and hence the infinite number of parameters upon which the arrival times of seismic waves depend. Nevertheless in practice most earthquake location procedures commonly used today treat the nonlinear hypocentral location problem as an overdetermined one. This is achieved by assuming knowledge of the compressional and shear wave velocity structure of the earth in the form of a fixed seismic velocity model, and thereby reducing the number of unknowns in the problem to the four hypocentral parameters of the event.

Location of these parameters is still a difficult problem, being both nonlinear and ill-conditioned. The nonlinearity arises due to the nonlinear dependence of the travel times of seismic waves on both the hypocentral and the velocity model parameters. The ill-conditioning arises due to measurements being made over a restricted network on the surface and therefore being poorly placed to resolve trade-offs between the depth and time location parameters, or to resolve the location of events outside the network.

The objective of this paper is to introduce the hypocentral location problem as an inverse problem, describe some commonly used approaches to its solution and the problems associated with them, and to suggest some possible alternatives that have, at least initially, proved fruitful. (The paper by P. R. Williamson in this Proceedings contains a

discussion of the related ill-posed problem of determining the velocity model parameters from seismic data.)

Before we can attempt to solve this inverse problem, however, we need first to gain some understanding of the forward problem, i.e. of the propagation process of elastic waves in the earth (or, more precisely, in our limited approximation of the earth).

2. The Forward Problem

A full treatment of elastic wave propagation in both homogeneous and heterogeneous media may be found in Keller and Karal (1959). For our purposes we need only be able to calculate the travel times between source and receiver of a few 'easily observed' phases. This is usually achieved by making a high frequency approximation, i.e. we assume that the velocities with which seismic waves propagate through a medium vary on a distance scale which is much larger than their wavelength. We may then appeal to ray theory in which it is assumed that seismic energy propagates along rays (which are always normal to wave fronts). The ray equations (2.1) may then be solved to determine the travel time and ray path between two fixed endpoints through a given earth model.

$$\begin{aligned} & \frac{d}{ds} \left[\frac{1}{v} \frac{dx}{ds} \right] - \frac{d}{dx} \left[\frac{1}{v} \right] = 0 \\ (2.1) \quad & \frac{d}{ds} \left[\frac{1}{v} \frac{dy}{ds} \right] - \frac{d}{dy} \left[\frac{1}{v} \right] = 0 \\ & \frac{d}{ds} \left[\frac{1}{v} \frac{dz}{ds} \right] - \frac{d}{dz} \left[\frac{1}{v} \right] = 0 \end{aligned}$$

Here $v = v(x,y,z)$ is the local seismic velocity at a point in the medium, and ds is an element along a ray path.

Note that (2.1) is given in a Cartesian coordinate system. In microearthquake studies this coordinate system is appropriate since the space under consideration is small. However, in regional network studies, such as those of South-Eastern Australia, it is usual to perform some type of transformation of latitude and longitude co-ordinates onto a rectangular plane, e.g. the transfer Mercator projection.

The derivation of the ray equations is quite straight forward and will not be treated here. They may be derived from Fermat's principle together with the use of Euler's equation for the extremals of (2.2) in the calculus of variations. Fermat's principle states that the travel time T

$$(2.2) \quad T = \int_A^B \frac{ds}{v(x,y,z)}$$

for a physical ray path between two fixed end points A and B is extremised. An outline of this derivation is given in Lee & Stewart (1981).

Several different techniques have been developed to solve the ray equations numerically in a three dimensional heterogeneous structure, including Julian & Gubbins (1977) and Pereyra et. al. (1980). However, in practice, for routine earthquake locations one is usually confined to a rather simplistic one dimensional model, i.e. one in which we specify

a variation of velocity with depth only, so that $v = v(z)$. This is due to the fact that we are usually rather ignorant of lateral velocity variations beneath a seismic network, and also because three-dimensional ray-tracing schemes are computationally very expensive.

If we consider the two-point seismic ray tracing problem in the xz plane with velocity as a function of depth only, the ray equations reduce to

$$(a) \quad \frac{1}{v(z)} \frac{dx}{ds} = \text{const}$$

(2.3)

$$(b) \quad \frac{d}{ds} \left[\frac{1}{v(z)} \frac{dz}{ds} \right] = \frac{-1}{v^2} \frac{dv}{dz} .$$

By definition, the direction cosines of the ray are

$$(2.4) \quad \cos \alpha \equiv \frac{dx}{ds} , \quad \cos \beta \equiv \frac{dy}{ds} , \quad \cos \gamma \equiv \frac{dz}{ds}$$

If θ is the angle the ray makes with the z axis (defined as positive downwards) then in the planar example

$$\cos \alpha = \sin \theta , \quad \cos \gamma = \cos \theta .$$

Then (2.3) (a) becomes,

$$(2.5) \quad \frac{\sin \theta}{v(z)} = \text{const} = p$$

which is just Snell's law. p is a constant along any ray and is known as the ray parameter. (2.3) (b) now reduces to (see Officer 1958)

$$(2.6) \quad \frac{d\theta}{ds} = p \frac{dv}{dz} .$$

This equation simply states that the curvature of the ray path $\frac{d\theta}{ds}$ is directly proportional to the velocity gradient $\frac{dv}{dz}$. Using these equations for any continuous velocity profile one may trace rays from entrance point to exit point and evaluate the travel time from (2.2). The most common type of velocity profile used in these models are layered profiles with the velocity constant across a layer, i.e.

$$v = v_0 \quad z_0 \leq z \leq z_1$$

or changing linearly, with

$$v = v_0 + v_1 z \quad z_0 \leq z \leq z_1$$

for a layer between z_0 and z_1 .

Most velocity models used to locate earthquakes are made up of a combination of constant and/or linear layers, with a discontinuity in the profile at the crust/mantle boundary (usually near the base of the model at a depth of approximately 30-50 km). Note that the integral (2.2) reduces to a simple expression for either of these types of velocity profiles.

The discontinuity in velocity at the crust/mantle boundary (the Moho) causes a variety of different phases to be generated. The only one seen usually on seismograms from a regional network is the so-called headwave (often called P_h). This is due to a critically refracted ray

from the Moho which travels along the interface for a large proportion of its travel time. Including the discontinuity in the velocity model allows derivation of a simple expression for the travel-time of the ray in term of the source and receiver co-ordinates and the velocities at either side of the discontinuity (see Appendix).

Thus with ray theory and our simple velocity model we may consider the forward problem to be solved. However, this is only true if the observed phases can really be interpreted in the simple manner described, and, more importantly, if our model earth together with the travel times we generate from it are representative of the real earth to within observational error.

3. The data

The source/receiver separation for a regional network varies from a few kilometres to several hundred kilometres. Over these distances only a few phases are clearly visible on a seismogram. One may usually identify the direct longitudinal P wave (known as P_g) and the corresponding shear or transverse wave (known as S_g). At larger source/receiver separations the headwave phase (known as P_h) arrives before the direct P_g wave, and it is this which is observed as the initial P wave. Generally the S wave is more difficult to distinguish from the preceding wavetrain. Consequently S waves are observed at approximately only one half of the receiver stations. The P wave itself may also be difficult to pick accurately if one is faced with a high level of background noise. For a regional seismic network observational errors of the order of 0.3-0.8 sec are typical for these

phases. We must be cautious in using arrival times of later phases, such as S or P_h for locating earthquakes as they may easily be misidentified on the seismograms and their theoretical arrival times may be difficult to model. In particular the S wave velocity structure is usually less well known than that of the P wave.

The data then consist of the initial arrival times of the first observed P , and sometimes S , waves at approximately 10-100 seismic stations in the network, depending on the magnitude and position of the event.

4. The inversion of seismic data

The earthquake location problem has in recent times been solved using Geiyer's method (1912), or variations of it. This is simply a Gauss-Newton type method and may be summarized as follows.

Given an initial guess for the hypocentral parameters x_k , $k = 1, \dots, 4$ and some previously determined velocity model of the earth, one may calculate theoretical arrival times of the first P and S waves $t_{ci}(\mathbf{x})$, $i = 1, \dots, N$ (where \mathbf{x} represents a vector of hypocentral parameters) at a set of seismic stations whose positions are known. These N arrival times may be compared with the corresponding observed times $t_{oi}(\mathbf{x})$, $i = 1, \dots, N$, and so some measure of the suitability of the initial guess at the hypocentre may be gained by defining a residual function $F(\mathbf{x})$

$$(4.1) \quad F(\mathbf{x}) = \sum_{i=1}^N [r_i(\mathbf{x})]^2$$

where r_i is the i th data residual

$$r_i(\mathbf{x}) = t_{oi} - t_{ci}(\mathbf{x}) \quad i = 1, \dots, N .$$

This may be rewritten as

$$r_i(\mathbf{x}) = t_{oi} - [T_{ci}(\mathbf{x}) + x_4]$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4)^T$ is the vector of hypocentral parameters, x_4 is the origin time parameter, and $T_{ci}(\mathbf{x})$ is the travel time of the ray to the i th station.

We may consider the residuals $r_i(\mathbf{x})$ as components of a vector in N -dimensional Euclidean space and write

$$\mathbf{r} = [r_1(\mathbf{x}), r_2(\mathbf{x}), \dots, r_n(\mathbf{x})]^T .$$

If the adjustment vector $\delta\mathbf{x}$ is defined as

$$\delta\mathbf{x} = (\delta x_1, \delta x_2, \delta x_3, \delta x_4)^T$$

then the least squares minimization of (4.1) yields

$$(4.2) \quad \delta\mathbf{x} = -[A^T A]^{-1} A^T \mathbf{r}$$

where A is the $(n \times 4)$ Jacobian matrix

$$A = \begin{bmatrix} \frac{\partial r_1}{\partial x_1} & \frac{\partial r_1}{\partial x_2} & \frac{\partial r_1}{\partial x_3} & \frac{\partial r_1}{\partial x_4} \\ \frac{\partial r_2}{\partial x_1} & \frac{\partial r_2}{\partial x_2} & \frac{\partial r_2}{\partial x_3} & \frac{\partial r_2}{\partial x_4} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial r_n}{\partial x_1} & \frac{\partial r_n}{\partial x_2} & \frac{\partial r_n}{\partial x_3} & \frac{\partial r_n}{\partial x_4} \end{bmatrix}$$

Due to the nonlinearity of the travel time expressions (see Appendix) A is itself a function of the hypocentral parameters and so varies with position. The problem is therefore solved iteratively, with each stage providing an improvement to the existing vector. The stability problems associated with this local linearization procedure are well known. A damped least squares (or Levenberg-Murquardt) approach is often used, so that,

$$(4.3) \quad \delta \mathbf{x} = -[A^T A + \lambda I]^{-1} A^T \mathbf{r}$$

where λ is a step control parameter.

The entries of the Jacobian matrix A describe the way in which the travel times are related to the hypocentral parameters. The various forms of the entries of A for a few simple velocity models may be found in the Appendix. More precisely, they are only an estimation of this relationship, since A is entirely a function of our assumed velocity model. In any case the matrix $A^T A$ can still become rank defective when events are poorly constrained by the seismic network. For example, if an event occurs outside the network of stations then geometrical considerations imply that some columns of A may become nearly proportional to one another and so cause it to become rank

defective. Another problem is the trade-off between depth and origin time of an event. Since most receivers are located on the surface of the earth and most earthquakes occur at some depth, the depth parameter is usually rather poorly constrained.

It can be easily shown (see Buland 1976) that the least squares optimization of (4.1) is equivalent to assuming a Gaussian probability distribution for the observational errors on the arrival times with no cross-correlation of errors, i.e. the error distribution is proportional to

$$\exp \left\{ -\frac{1}{2} \sum_{i=1}^N \frac{(t_{oi} - t_{ci})^2}{\sigma_i^2} \right\}$$

where σ_i is the standard deviation of the i^{th} data being treated as a random variable with mean at the 'true' value of the data. However we defined this allegedly 'true' value by our calculated arrival time t_{ci} which is itself likely to be erroneous. In fact in this approach all errors in our theoretical arrival times are inherently being treated as errors in our observations. So we are actually assuming a Gaussian probability distribution for the velocity model errors with respect to the real earth. This assumption is very unlikely to be justified in practice, with the result that the solution obtained from the inversion will be dependent on the velocity model used.

Attempts to use other statistics or combinations of error distributions are very uncommon. Jeffreys (1932) put forward an alternative to the straightforward Gaussian distribution which he noted to be representative of teleseismic travel time residuals. The

corresponding objective function for optimization becomes

$$(3.4) \quad F(\mathbf{r}) = - \sum_{i=1}^N \log \left\{ \frac{(1-f)}{(\sigma_i(2\pi))^{1/2}} \exp \left[\frac{-r_i^2}{2\sigma_i^2} \right] + \frac{f}{v(2\pi)^{1/2}} \exp \left[\frac{-r_i^2}{2v^2} \right] \right\} .$$

This error distribution is basically a superposition of two Gaussians. The constant f determines the relative scale of the distributions of the errors at each observation, which have standard deviation $\sigma_i, i=1, \dots, N$, and the smaller, and broader background distribution which has standard deviation v . The distribution is known to be more robust than the Gaussian; i.e. if we seek the hypocentral parameters which minimize this objective function then they are less biased by the occasional outlying residual. By this we mean a residual which is large compared to the standard deviation of observational errors. This may be due to a misidentification of a phase (large observational error) or a region of strong lateral heterogeneity which is not accounted for in the simple velocity model (large theoretical error). These and other distributions are discussed in Anderson (1982).

A greater understanding of the effect of velocity model errors on the solution of the inversion would seem to be required. The current use of a Gaussian statistic is rather misleading in that little account is taken of systematic errors occurring at the forward modelling stage.

A relatively simple scheme has been produced by Sambridge and Kennett (1986) which is both independent of the type of residual function minimized and velocity model used. Essentially this independence is achieved by eliminating the need for any derivative

information at the expense of function evaluations. The algorithm belongs to the class of grid search methods. Due to the special form of the hypocentral location problem such methods are computationally feasible as only a small number of parameters are to be determined. The algorithm performs a search on a three-dimensional spatial grid for the minimum of the residual function at a particular origin time, t . If this minimum is denoted by

$$\hat{F}(\mathbf{x}_t, t)$$

where $\mathbf{x}_t = (x_1, x_2, x_3, t)$, then we must find the value of t for which

$$\min_t \hat{F}(\mathbf{x}_t, t) = \min_{\mathbf{x}} F(\mathbf{x})$$

i.e. the global minimum. By determining $\hat{F}(\mathbf{x}_t, t)$ for any desired t , we essentially reduce the problem to a one-dimensional optimization problem which maybe solved by a variety of means. The technique employed here is based on the Golden Section Search (Whittle (1971)). Essentially we consider values of $\hat{F}(\mathbf{x}_t, t)$ at initial upper and lower estimates of the origin time, say t_1 and t_2 , and then generate progressively decreasing upper and lower estimates by comparing the corresponding spatial minima $\hat{F}(\mathbf{x}_t, t)$. The iteration is considered to have converged when the variation of the two estimates is less than some preset tolerance level.

This procedure essentially identifies a region of parameter space within which the solution lies. We may then sample a rather larger

region surrounding the estimated solution and determine a nonlinear confidence region based on an estimated level of observational and theoretical error. We are thus not restricted to estimating ellipsoidal confidence regions as is the usual case when applying direct matrix inversion methods. Full details of this algorithm may be found in Sambridge and Kennett (1986).

An obvious point raised by the use of a grid search algorithm is whether such an approach can be made computationally efficient. An important point to remember is that when we search for local spatial minima of $\hat{F}(\mathbf{x}_t, t)$ over varying values of t , we do so on an invariant spatial lattice. Thus when any lattice point is encountered for the second or later time the travel times of the rays to the stations need not be recalculated. This permits the residuals and hence the objective function (3.1) to be recomputed with very little computational effort compared to that required originally to trace rays through the velocity structure. In fact if we prescribe in advance the size of spatial lattice we require, then all ray tracing may be done beforehand. That is travel times and rays from each lattice point to each seismic station through any velocity model, even one incorporating strong lateral velocity gradients, may be calculated at the beginning and stored in a travel time table. In this way routine earthquake locations may be performed using the most detailed velocity model available for any particular region.

In this work an attempt has been made to remove the dependence of earthquake locations on both the particular velocity model employed and the assumptions made as to the appropriate error statistics involved.

This has been achieved by producing an algorithm which is independent of both, and so allows one to vary them at will. The algorithm is thus a useful tool in investigating the effect on earthquake locations of introducing realistic velocity models and more robust error distributions.

Appendix

The Appendix presents the travel time expressions derived from the ray equations (1.3) for a few simple laterally homogeneous velocity models.

1. Constant Velocity Model

Suppose that $v = v_0$, a constant independent of depth. If the earthquake source is at point A with co-ordinates (x_A, y_A, z_A) and a receiver is at B with co-ordinates (x_B, y_B, z_B) then the travel time is given by:

$$(A1) \quad T = \frac{S}{v_0}$$

where S is the path length of the straight line ray between A and B, i.e.

$$S = [(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2]^{1/2} .$$

The spatial derivatives of travel time at the source in this case are simply,

$$(A2) \quad \left. \frac{\partial T}{\partial x} \right|_A = \frac{-(x_B - x_A)}{v_0 S}, \quad \left. \frac{\partial T}{\partial y} \right|_B = \frac{-(y_B - y_A)}{v_0 S}, \quad \left. \frac{\partial T}{\partial z} \right|_A = \frac{-(z_B - z_A)}{v_0 S}$$

2. Continuous Velocity Model

Referring to Fig A1 and using equations (1.4), (1.6), the travel time T from point A to B is

$$(A.3) \quad \begin{aligned} T &= \int_A^B \frac{ds}{v(z)} \\ &= \int_A^B \frac{dz}{v(z) \cos \theta} \\ &= \int_{\theta_A}^{\theta_B} \frac{d\theta}{\left[\frac{dv}{dz} \right] \sin \theta} . \end{aligned}$$

For a simple linear velocity gradient

$$(A.4) \quad v = v_0 + v_1 z$$

from (2.6) we have

$$\frac{d\theta}{ds} = p v_1 = \text{const} .$$

Thus the curvature along a ray path is constant, so the rays are arcs of a circle with radius

$$\Omega = \left[\frac{d\theta}{ds} \right]^{-1} .$$

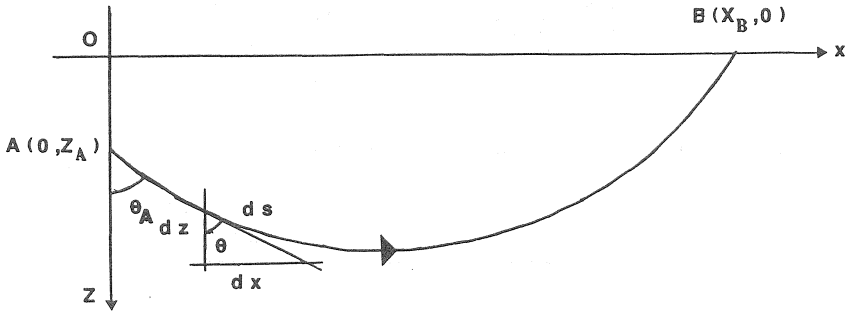


Fig A 1.

Geometry of ray tracing in the xz plane, with $v=v(z)$.

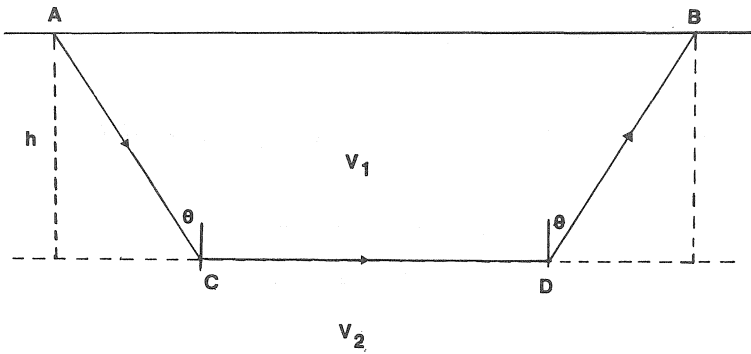


Fig A 2.

Diagram of refracted path for velocity model of a layer over a half space.

The travel time is given by (A.3) and may be integrated with the use of (A.4), (2.5) and (2.6) to give

$$(A.5) \quad T = \int_{\theta_A}^{\theta_B} \frac{d\theta}{v_1 \sin \theta} = \frac{1}{v_1} \log \left[\frac{\tan(\theta_B/2)}{\tan(\theta_A/2)} \right]$$

and spatial derivatives of T at the source are

$$\left. \frac{\partial T}{\partial x} \right|_A = -\frac{\sin \theta_A}{v_0 + v_1 z_A}, \quad \left. \frac{\partial T}{\partial z} \right|_A = \frac{-\cos \theta_A}{v_0 + v_1 z_A}.$$

3. Head wave phase (critically refracted ray)

A simple illustration of this type of ray is given in Fig A2. In this case we consider only a single layer of thickness h and velocity v_1 over a half space with velocity v_2 .

The travel time of the refracted ray T_r along \overline{ACDB} is given by

$$(A.6) \quad T_r = \frac{\overline{AC}}{v_1} + \frac{\overline{CD}}{v_2} + \frac{\overline{DB}}{v_1}$$

and using Snell's law at the boundary gives

$$(A.7) \quad \sin \theta = \frac{v_1}{v_2}$$

T_r may be written in the more convenient form

$$(A.8) \quad T_r = \frac{A}{v_2} + \frac{2h}{v_1} \cos \theta$$

where Δ is the epicentral distance \overline{AB} , i.e.

$$(A.9) \quad \Delta = [(x_A - x_B)^2 + (y_A - y_B)^2]^{1/2} .$$

A full derivation of (A.8) may be found in Officer (1958).

The extension of (A.8) for a source at depth z is simply

$$(A.10) \quad T_r = \frac{\Delta}{v_2} + \frac{(2h-z)\cos\theta}{v_1} .$$

From equations (A.8) and (A.9) the spatial derivatives of travel times at the source are

$$(A.11) \quad \begin{aligned} \left. \frac{\partial T_r}{\partial x} \right|_A &= \frac{-(x_B - x_A)}{\Delta v_2} \\ \left. \frac{\partial T_r}{\partial y} \right|_A &= \frac{-(y_B - y_A)}{\Delta v_1} \\ \left. \frac{\partial T_r}{\partial z} \right|_A &= \frac{-(v_2^2 - v_1^2)^{1/2}}{v_2 v_1} . \end{aligned}$$

Travel time and derivative expressions for multi-layer velocity models may be found in Lee and Stewart (1981).

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