ON THE BEHAVIOUR OF SOLUTIONS TO A SEMILINEAR NEUMANN PROBLEM

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§1. Introduction

In this expository paper we wish to survey some recent results on the following semilinear Neumann problem (with the diffusion coefficient d varied as a parameter)

$$(1.1)\qquad \begin{cases} d\Delta u - u + u^p = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$, ν denotes the unit outer normal to $\partial\Omega$, $\Delta \equiv \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ and $d > 0$, $p > 1$ are two constants. Equation (1.1) arises naturally in various models in mathematical biology; for instance, it is equivalent to an elliptic chemotaxis system and is also known as the "shadow" system of an activator-inhibitor system of Gierer and Meinhardt.

Chemotaxis is the oriented movement of cells in response to chemicals in their environment. For example, cellular slime molds (amoebae) release a certain chemical, move toward places of its higher concentration and then form aggregates. In 1970 Keller and Segel [KS] proposed a model, which in particular includes the following system (see e.g. [S]), to describe the chemotactic aggregation stage of cellular slime molds:

$$(1.2)\quad \begin{cases} \varphi_t = D_1\Delta\varphi - \chi\nabla\cdot(\varphi\nabla\log\psi) & \text{in } \Omega \times \mathbb{R}^+ , \\ \psi_t = D_2\Delta\psi - a\psi + b\varphi & \text{in } \Omega \times \mathbb{R}^+ , \\ \frac{\partial\varphi}{\partial\nu} = \frac{\partial\psi}{\partial\nu} = 0 & \text{on } \partial\Omega \times \mathbb{R}^+ , \\ \varphi(x,0) = \varphi_o(x) \ , \ \psi(x,0) = \psi_o(x) & \text{in } \Omega , \end{cases}$$

where D_1, D_2, a, b, χ are all positive constants, $\varphi(x,t)$ denotes the population of amoebae at place x and at time t, and $\psi(x,t)$ is the concentration of the chemical. It is expected that (1.2) possess inhomogeneous spatial patterns under some appropriate hypotheses. We are thus led to study the stationary solutions of (1.2); i.e. the following elliptic system

$$(1.3)\quad \begin{cases} D_1\Delta\varphi - \chi\nabla\cdot(\varphi\nabla\log\psi) = 0 & \text{in } \Omega , \\ D_2\Delta\psi - a\psi + b\varphi = 0 & \text{in } \Omega , \\ \frac{\partial\varphi}{\partial\nu} = \frac{\partial\psi}{\partial\nu} = 0 & \text{on } \partial\Omega , \end{cases}$$

(In fact, since $\int_\Omega \varphi(x,t)dx = \int_\Omega \varphi_o(x)dx$ for all $t > 0$, we should perhaps add one more constraint to (1.3), namely,

$$\int_\Omega \varphi(x)dx = \text{a given constant} \quad (= \int_\Omega \varphi_o(x)dx).$$

But this can always be achieved once (1.3) is solved. See, e.g. [LNT].)

While it is clear that the parabolic system (1.2) is <u>not</u> equivalent to the following parabolic system

$$(1.4)\quad \begin{cases} u_t = d\Delta u - u + u^p & \text{in } \Omega \times \mathbb{R}^+ , \\ \frac{\partial u}{\partial\nu} = 0 & \text{on } \partial\Omega \times \mathbb{R}^+ , \\ u(x,0) = u_o(x) & \text{in } \Omega , \end{cases}$$

it is nevertheless just an easy exercise to show that their elliptic counterparts are equivalent. That is, (1.1) <u>is</u>

<u>equivalent to</u> (1.3) (see, for instance, [S]). Indeed, rewrite the first equation in (1.3) as $\mathrm{div}(D_1\nabla\varphi - \chi\varphi\nabla\log\psi) = 0$, i.e.

$$\mathrm{div}[D_1\varphi\nabla\log(\varphi/\psi^p)] = 0$$

where $p = \chi/D_1$. Setting $\rho = \varphi/\psi^p$, we obtain

$$\begin{cases} \mathrm{div}[\varphi\nabla\log\rho] = 0 & \text{in } \Omega\,, \\ \dfrac{\partial\rho}{\partial\nu} = 0 & \text{on } \Omega\,. \end{cases}$$

Thus ρ satisfies

$$\begin{cases} \Delta\rho + \sum\limits_{i=1}^{n} [(\frac{\varphi}{\rho})_{x_i}\frac{\rho}{\varphi}]\rho_{x_i} = 0 & \text{in } \Omega\,, \\ \dfrac{\partial\rho}{\partial\nu} = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Now, the Hopf Boundary Point Lemma implies that $\rho \equiv$ constant, say λ. We have thus obtained the following relation between (1.1) and (1.3):

(1.5) $\quad d = \dfrac{D_2}{a}\,,\quad p = \dfrac{\chi}{D_1}\,,\quad \varphi = \lambda\psi^p\,,\quad \mu = (\dfrac{b\lambda}{a})^{\frac{1}{p-1}}$ and $u = \mu\psi$.

Conversely, it is straightforward to verify that if u is a solution of (1.1), then (1.5) gives a solution of (1.3).

Equation (1.1) also arises in the theories of biological pattern formation. Following the idea of "diffusion driven instability" of A. Turing, Gierer and Meinhardt [GM] (see [M] also), in 1972, proposed to study several systems of activator-inhibitor type where stable nonconstant solutions are interpreted as spatially inhomogeneous state of cells. One of those systems, known as the non-saturated case, is as follows:

$$(1.6)\quad \begin{cases} d\Delta u - u + \dfrac{u^p}{u^q} = 0 & \text{in } \Omega, \\ D\Delta v - \zeta v + \dfrac{u^r}{v^s} = 0 & \text{in } \Omega, \\ u > 0, \quad v > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where d, D, ζ are positive constants, and the exponents p, q, $r > 0$, $s \geq 0$ satisfy the condition $0 < (p-1)/q < r/(s+1)$. Heuristically, v approaches a constant, say $\xi > 0$, as $D \longrightarrow \infty$ (this may be verified in various special cases, see [NT]), and we are led to the shadow system of (1.6):

$$(1.7)\quad \begin{cases} d\Delta u - u + u^p\xi^{-q} = 0 & \text{in } \Omega, \\ -\zeta\xi + \xi^{-s}|\Omega|^{-1}\displaystyle\int_\Omega u^r dx = 0, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Again, (1.7) is equivalent to (1.1) if we suitably rescale the quantities involved.

From the biological point of view, it is hoped that one may be able to find solutions of (1.1) which exhibit spiky patterns when d is sufficiently small. We shall see that this will be the case if $1 < p < (n+2)/(n-2)$. (In case Ω is a ball, we can actually obtain fairly precise information about the "desirable" solution, see Section 2 below.) A natural question arises: what happens if $p \geq (n+2)/(n-2)$? As far as the existence of nontrivial solutions of (1.1) is concerned, the primary parameter seems to be the diffusion coefficient d instead of the exponent p. However, the exponent p (and thus the critical power $(n+2)/(n-2)$) does seem to play an important role in studying the "spiky" behavior of nonconstant solutions of (1.1) especially when d is small. (See Section 2 below for more detailed descriptions.)

§2. Existence of Spiky Patterns

In this section we shall describe some results obtained in [LN], [LNT] and [NT] concerning (1.1). It would be helpful if we also consider the equation in (1.1) with homogeneous Dirichlet boundary condition

$$(2.1)\quad \begin{cases} d\Delta u - u + u^p = 0 & \text{in } \Omega , \\ u > 0 & \text{in } \Omega , \\ u = 0 & \text{on } \partial\Omega . \end{cases}$$

Problem (2.1) has long been studied and all the results stated below concerning (2.1) are well-known and standard except the uniqueness for (2.1) in a ball which was only recently proved by K. McLeod and J. Serrin [MS]. The following list compares results of (2.1) to that of (1.1) side-by-side.

$d\Delta u - u + u^p = 0$, $d > 0$, $p > 1$.
$u > 0$ in $\Omega \subset\subset \mathbb{R}^n$.
[Set $n^* = (n+2)/(n-2)$.]

Dirichlet: $u=0$ on $\partial\Omega$.

<u>Existence</u>

(i) <u>Ω General:</u> $\forall p < n^*$ & $\forall d > 0$, $\exists$ a solution.

(ii) <u>Ω star-shaped:</u> $\exists$ solution $\Leftrightarrow$ $p < n^*$.

(iii) <u>Ω annuli:</u> $\forall\, p > 1$ $\exists$ a solution.

<u>Remark:</u> The existence depends on the exponent p & the domain Ω, but is independent of d.

Neumann: $\partial u/\partial \nu = 0$ on $\partial\Omega$.

<u>Existence</u>

(i) <u>Ω General:</u> $\forall p < n^*$ $\exists\, d_0$ & d_1 (both may depend on p) s.t.
(a) $\forall d < d_0$ $\exists$ a solution $\not\equiv 1$.
(b) $\forall d > d_1$ $\exists$ no solution $\not\equiv 1$.

(ii) <u>Ω balls:</u> $\forall\, 1 < p \neq n^*$ $\exists\, d_0$ & d_1 s.t.
(a) $\forall d < d_0$ $\exists$ a radial solution $\not\equiv 1$.
(b) $\forall d > d_1$ $\exists$ no radial solution $\not\equiv 1$.

(iii) <u>Ω annuli:</u> $\forall\, p > 1$ $\exists\, d_0$ & d_1 s.t.
(a) $\forall d < d_0$ $\exists$ a radial solution $\not\equiv 1$.
(b) $\forall d > d_1$ $\exists$ no radial solution $\not\equiv 1$.

<u>Conjecture</u>: (i) holds for all $p > 1$.

<u>Remark:</u> The existence depends on the the diffusion coefficient d.

<u>An Estimate:</u> $d_0 \geq (p-1)/\lambda_2$ where λ_2 is the 2nd eigenvalue of Δ in Ω with zero Neumann boundary value.

<u>Uniqueness</u>

Let Ω be a ball. For $n < 9$, $\exists\, p_0(n) < n^*$ s.t. $\forall p < p_0(n)$ the solution is unique.

<u>Conjecture:</u> Uniqueness holds for all $p < n^*$, at least for balls.

<u>Nonuniqueness ("Point-Condensation")</u>

Let $p < n^*$. For each d, $\exists$ a solution u_d s.t. $u_d \to 0$ in measure as $d \to 0$ but $1 < |u_d| < C$ where C is independent of d.

<u>Question:</u> Let $p \geq n^*$. For each d, does there exist a solution u_d s.t. $u_d \to 1$ in measure & $\|u_d\|_\infty \to \infty$ as $d \to 0$?

We see from the above list that for $d > 0$ small and $1 < p < n^* = (n+2)/(n-2)$, (1.1) has a nontrivial solution. In fact, when $d > 0$ is small, there are lots of solutions of (1.1) (see Remark (4) below). However, what we would really like to know is that if any of those solutions exhibits "spiky" pattern as we have discussed in the Introduction. To answer this, we have the following

Theorem. *Let $p < (n+2)/(n-2)$. Then for each $d > 0$, there exists a solution u_d of (1.1) with the following properties:*

(i) $u_d \longrightarrow 0$ *in measure as* $d \longrightarrow 0$;

(ii) *there exists a constant* C, *independent of* $d > 0$, *such that*

$$1 < \|u_d\|_{L^\infty} < C$$

for all $d > 0$;

(iii) *for each* $q \in [1,\infty)$, *there exist constants* $C_i(q)$, $i=1,2$, *independent of* $d > 0$, *such that*

$$C_1(q)d^{n/2} \leq \int_\Omega u_d^q \leq C_2(q)d^{n/2}$$

for all $d > 0$;

(iv) $\lambda_1(u_d) < 0 \leq \lambda_2(u_d)$ *for all* $d > 0$, *where* $\lambda_j(u_d)$ *is the j-th eigenvalue of (1.1) linearized at* u_d ; i.e.

$$\begin{cases} (d\Delta - 1 + pu_d^{p-1})\varphi + \lambda_j(u_d)\varphi = 0 & \text{in } \Omega, \\ \dfrac{\partial\varphi}{\partial\nu} = 0 & \text{on } \partial\Omega ; \end{cases}$$

(v) $u_d \not\equiv 1$ *if* $d < (p-1)/\lambda_2$ *where* λ_2 *is the second eigenvalue of* Δ *on* Ω *with zero Neumann boundary data* ;

(vi) *for any* $\eta > 0$, *set* $\Omega_{\eta,d} \equiv \{x\epsilon\Omega \mid u_d(x) > \eta\}$. *Then there exists a positive integer* m *which depends only on* Ω, p *and* η *(but independent of* d*) such that* $\Omega_{\eta,d}$ *may be covered by at most* m *balls of radius* $\sqrt{d}$;

(vii) *there exist positive constants* $\tilde{C}$, γ , *independent of* $d > 0$, *such that*

$$\inf_{\Omega} u_d \leq \tilde{C} \exp(-\gamma/\sqrt{d})$$

for all $d > 0$.

The existence proof is based on the well-known Mountain-Pass Lemma of Ambrosetti and Rabinowitz [AR]. In $H_1(\Omega)$, we define the variational functional

$$J_d(u) = \frac{d}{2} \int_{\Omega} |\nabla u|^2 + \frac{1}{2} \int_{\Omega} u^2 - \frac{1}{p+1} \int_{\Omega} (u^+)^{p+1} ,$$

and we look for the critical points of J_d . It is standard to show that the value c_d defined by

$$c_d = \inf_{h\epsilon\Gamma} \max_{t\epsilon[0,1]} J_d(h(t))$$

where Γ is the class of all continuous paths connecting 0 and e in $H_1(\Omega)$ (here e is an arbitrary but fixed positive function with $J_d(te) \leq 0$, for all $t \geq 1$, in $H_1(\Omega)$), is a positive critical values of J_d , and thus gives a positive solution u_d of (1.1). Since (1.1) always has a constant solution, namely, $u \equiv 1$, and we know that it is the only solution of (1.1) if d is sufficiently large, we conclude easily that the "Mountain-Pass solution" $u_d \equiv 1$ for d large. To show that $u_d \not\equiv 1$ for d small, we first observe that $J_d(1) = (\frac{1}{2} - \frac{1}{p+1})|\Omega|$ which is independent of d. Then we estimate the value c_d, <u>we show that</u>

(2.2) $$c_d \sim d^{n/2}$$

<u>for</u> $d > 0$ <u>small</u>. This the the first crucial estimate in proving the theorem above. For the rest of the proof, since it is somewhat long and technical, we omit it here and just refer the interested readers to [LN] and [LNT] for details.

<u>Remarks</u>.

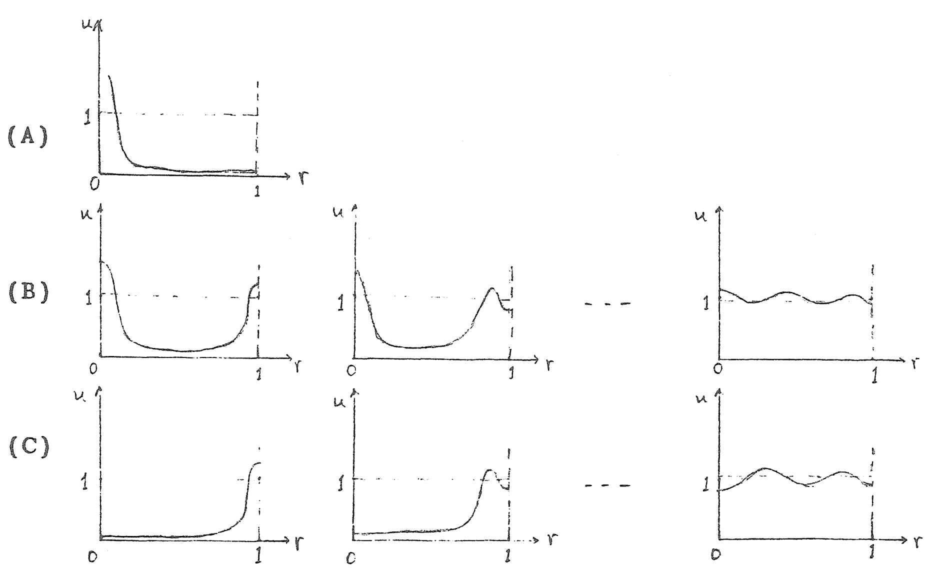
(1) It is easy to see, from integrating the equation (1.1) directly, that the set $\{x \in \Omega \mid u_d(x) > 1\}$ is non-empty for d small (since $u_d \not\equiv 1$ for d small). This, together with (i) in the theorem, imply that <u>the</u> C^1 <u>norm of</u> u_d <u>cannot possibly be bounded independent of</u> d, <u>for</u> d <u>small</u>. Nevertheless the L^∞ norm of u_d is bounded independent of d as is guaranteed by (ii). This indicates that u_d should have "peaks" of finite amplitude when d is sufficiently small and thus exhibits "spiky" pattern.

(2) Since $p > 1$ in (1.1), no nontrivial solution of (1.1) can be stable (although <u>we do expect</u> the systems (1.3) and (1.6) to have stable nontrivial solutions). In particular, u_d is unstable (which is equivalent to $\lambda_1(u_d) < 0$, guaranteed by (iv)). However, (iv) also says that $\lambda_2(u_d) \geq 0$ which, in some sense, seems to suggest <u>that</u> u_d, <u>although unstable, is the "most stable" nontrivial solution of</u> (1.1). Furthermore, (iv) may be strengthened as follows: $\lambda_1(u_d) < 0 < \lambda_2(u_d)$ <u>for all</u> $u_d \not\equiv 1$. This implies that the unstable manifold of u_d is of

dimension one only. Since there is one conserved quantity in the parabolic system (1.2), u_d seems to give rise to (via (1.5)) a stationary solution (φ, ψ) of (1.2) which is likely to be stable.

(3) The estimate given by (v) in the theorem is, in general, best possible. This follows from Theorem 4, p. 218 in [T].

(4) (vi) is actually very useful in determining the "shape" of u_d in the radial case. First we remark that the theorem above holds true without any change if Ω is a ball and if we restrict ourselves to the class of radial functions. If we examine all the possible radial solutions of (1.1) in a ball, it is not hard to see that they may be categorized as follows (see [N], [LN]):



Note that only (A) exhibits a spiky pattern, and the others exhibit either boundary layer phenomena or combinations of spikes and boundary layers. It is easy to see from (vi) that u_d in this case must be (A).

(5) In case Ω is an annulus, we claim that u_d is non-radial if d is sufficiently small. For, again if we restrict ourselves to radial functions in $H_1(\Omega)$, then the same arguments in proving the theorem may be carried through without change except now we have $n=1$. In particular, the estimate (2.2) now reads $c_d^* \sim d^{1/2}$, where c_d^* is the critical value of J_d given by the Mountain-Pass Lemma when restricted to the class of radial functions in $H_1(\Omega)$. Thus, for d small, $c_d \neq c_d^*$ and the corresponding critical points must also be different. In particular, this implies that u_d cannot be radial. Notice that this observation applies equally well to the Dirichlet problem (2.1). However, the existence of non-radial solutions to (2.1) (i.e. the Dirichlet problem), in case Ω is an annulus, was established earlier by C.V. Coffman [C].

(6) In the "super-critical" case $p > (n+2)/(n-2)$, our progress is rather limited. However, we do know that in the radial case (Ω is either a ball or an annulus) (1.1) possesses a nontrivial radial solution if d is sufficiently small, and that (1.1) has no nonconstant radial solution if d is sufficiently large. This part seems to agree with the "sub-critical" case $p < (n+2)/(n-2)$. We would also like to give some partial results just to indicate the difference

between these two cases.

Let Ω be the unit ball. After a change of scale, a radial solution of (1.1) satisfies

$$\begin{cases} u'' + \frac{n-1}{r} u' - u + u^p = 0 , & p > (n+2)/(n-2) , \\ u'(0) = u'(1/\sqrt{d}) = 0. \end{cases}$$

It is not difficult to show that there exists a positive constant α, independent of $d > 0$, such that

$$\inf_{\Omega} u \geq \alpha$$

for all radial solutions u of (1.1) with $u(0) > 1$. This marks a basic difference between the behavior of solutions of these two cases $p < n*$ and $p > n*$. It eliminates the possibility of the existence of a radial spiky solutions which approaches zero in measure as d approaches zero in the super-critical case $p > n*$.

The critical case $p = (n+2)/(n-2)$ is a bit more delicate. Some of our methods do carry over to this case; however, we shall not discuss this case here.

(7) Equation (1.1) may be viewed as a singular perturbation problem when d is sufficiently small. Methods and techniques developed in that field could be helpful here in locating the spikes of a particular solution of (1.1) for general domain Ω. However, we are not able to do this using singular perturbation techniques, even in the radial case (when Ω is the unit ball) which we already know from Theorem (vi) (see Remark (4) above) that (1.1) possesses a solution which has only one spike and it is located at the origin.

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