# MAXIMAL SURFACES AND GENERAL RELATIVITY

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In this report I'd like to review the development of the regularity theory of maximal (and prescribed) mean curvature hypersurfaces and describe some of the main ideas that are involved. Such surfaces have long been an important tool in general relativity, but it is only recently that their regularity properties have been fully described [CY], [BS], [G], [B1], [B2]. These results, culminating in [B2] which showed that variational extremal hypersurfaces are smooth and spacelike, are described in some detail in the following sections; here I will give a brief summary of the main properties and applications of mean curvature hypersurfaces in general relativity, together with selected references.

Perhaps the most important reason for the utility of constant mean curvature hypersurfaces is their uniqueness property [BF], [G]; in a cosmological spacetime — i.e. a spatially compact, globally hyperbolic Lorentzian manifold satisfying the *timelike convergence condition*

$$(TCC) \qquad Ric(T,T) \; \geq \; 0 \quad \text{for all timelike vectors } T \quad ,$$

a constant (non-zero) mean curvature Cauchy surface is unique [BF] and a maximal (i.e. zero mean curvature) Cauchy surface is almost unique [G]. Thus constant mean curvature slicings provide a "canonical" choice of global time function in a cosmological spacetime, and maximal surfaces parameterised by the "time at infinity" play a similar role in asymptotically flat spacetimes [B1].

The uniqueness property is the main reason this time coordinate is the most common choice in theoretical numerical relativity [P], [Sm], [SmY1,2]. Indeed, it was precisely to investigate the initial value problem that Lichnerowicz first considered the maximal slicing gauge [L]. Explicit calculations for spherically symmetric spacetimes such as the Schwarzschild [R], [Eet], [D], [SmY2], [BCI] and Tolman-Bondi [ES] spacetimes indicate that maximal slices have good singularity avoidance features (c.f "crushing singularities" [ES], "collapse of the lapse" [SmY2]). There is however a significant computational cost in solving the resulting elliptic equations, especially if a spatial elliptic gauge such as the "minimal distortion" gauge of [SmY1] is used to prevent the spatial coordinates being swallowed by any black hole. For this reason various numerical alternatives have been proposed [BP], [MSNM], [Sw] but whether these have useful singularity avoidance features is not clear. We note that the obvious alternative gauges such as the De Dondeur and Gaussian coordinates become singular long before the spacetime develops singularities [CB1].

Constant mean curvature gauges have also been successfully applied to more theoretical questions involving Einstein's equations. Apart from the uniqueness property, they have the additional advantage that the (Gauss-Codazzi) constraint equations simplify [CBY]. There has been a lot of work investigating the structure of the space of solutions, particularly by Marsden and his collaborators [MI1,2], [AFMM], [EIMM], [FMM]. Topics they have considered include the linearisation stability of Einstein's equations in the presence of symmetries, existence of conformal Killing fields, Hamiltonian structures and slice theorems for the phase space. Another very interesting application is the program of Christodoulou and Klainerman [CK] to prove global existence for small data solutions of the vacuum equations.

Maximal surfaces in spacetimes satisfying the weak energy condition are 3-manifolds with non-negative scalar curvature and it was widely recognised [Ge2],

[CBFM], [CBM] that this was important for the positive mass conjecture [ADM]. This conjecture was finally settled by Schoen and Yau [SY1] and although later proofs [W], [SY2] removed the need for a maximal surface, it may well be that non-negative scalar curvature will yet be important in considering the structure of spacetime. This belief is motivated by the fact that non-negative scalar curvature imposes topological constraints.

An example of Brill [Br] of an asymptotically flat spacetime without any (complete) maximal slice exploits this by building a spacetime with spatial topology that cannot carry a metric of non-negative scalar curvature. Similar ideas also lead to a cosmological spacetime without any constant mean curvature Cauchy surfaces [B3]. Very briefly, Brill's construction starts with those pieces of the maximally extended Schwarzschild and k=0 Friedman dust solutions which are *not* used when constructing the Oppenheimer-Snyder stellar model. These two spacetimes can then be glued together along their boundary in the same way as in the construction of the Oppenheimer-Snyder model. The Friedman component has spatial topology $\mathbb{R}^3$–Ball and can be (spatially) compactified by identifying the faces of a large cube. The resulting spacetime has asymptotically flat spatial slices with one end, topologically $\mathbb{T}^3$–Ball. Such a slice cannot admit a metric of non-negative scalar curvature. The example of [B3] is a variation on this construction.

These examples are not as catastrophic for the theory of prescribed mean curvature surfaces as might first appear, since they are somewhat unrealistic physically. This follows from results of Schoen and Yau [SY2,3] which imply that an apparent horizon must exist in such examples. Reassuringly, the maximal [B1] and constant mean curvature [G], [Ga1], [B3] existence theorems have conditions which rule out the behaviour of these counterexamples – and these conditions are imposed for purely pde reasons. The paper [B3] contains a discusion of the conditions under which a cosmological spacetime should admit constant mean curvature Cauchy surfaces.

Another promising application is the study of spacetime singularities. The main result in this line is the celebrated Hawking singularity theorem [H], which shows that a cosmological spacetime with a Cauchy surface of strictly positive (or negative) mean curvature is necessarily timelike incomplete (i.e. singular). Using ideas due to Avez [A] and Geroch [Ge1] and the regularity results of [G], [B1,2], this leads to general conditions under which a cosmological spacetime is singular [Ga], [B3] (see section 4). We note that J.-H. Eschenburg has recently proved a spectacular splitting theorem using rather different methods [EJ].

## 1. The Problems.

First we outline the problem and briefly describe some notation. For rather more precise definitions and more detailed explanations than those given here, the reader is referred to the books of Hawking and Ellis [HE] and O'Neill [O'N] for causality theory, [GT] for pde background material and to [BS], [B1,2] for notation specific to these problems.

A hypersurface M in a Lorentz manifold $\mathcal{V}$ is said to be *weakly spacelike* if it is locally achronal, so that M can be written locally as the graph of a Lipschitz function. Since M then has a tangent plane almost everywhere, we can define the area of M ; in the special case where M is a graph in Minkowski space $\mathbf{R}^{n,1}$ , M = $\text{graph}_\Omega u$ , $u \in C^{0,1}(\Omega)$, $\Omega \subset \mathbf{R}^n$ , the area of M is given by

$$\text{area(M)} = \int_\Omega \sqrt{1 - |Du|^2}\, dx$$

The concavity of the area integrand means it is natural to look for hypersurfaces (or equivalently, Lipschitz functions $u$) which *maximise* the area. More generally we consider the variational problem

(VP) $\qquad \max_{M \in \mathcal{F}} \left\{ \text{area}(M) - \int_{V(M^*,M)} F \, dv \right\}$

where $F \in C^\infty(\mathcal{V})$ is the prescribed mean curvature function, $V(M^*,M)$ is an open set in the spacetime with boundary $\text{bd}(V(M^*,M)) = M \cup M^*$ and $M^*$ is a reference hypersurface, and $\mathcal{F}$ is a class of weakly spacelike hypersurfaces (for example, the set of surfaces spanning a given boundary set, or those passing through a given point).

We say that $M$ is a *regular hypersurface* if it is a smooth weakly spacelike hypersurface with everywhere spacelike tangent plane. For such surfaces we can define the mean curvature $H_M$ as the trace of the second fundamental form (extrinsic curvature): in Minkowski space this is given by

(1) $\qquad H_M \;=\; \dfrac{1}{\sqrt{1 - |Du|^2}} \left( \delta_{ij} + \dfrac{D_i u \, D_j u}{1 - |Du|^2} \right) D_{ij}^2 u \quad .$

There are a variety of geometric interpretations of this expression, the most natural being

$$H_M \;=\; \text{div}_M \, N \quad ,$$

where $N$ is the future unit normal to the hypersurface $M$ and $\text{div}_M$ is the divergence operator on $M$ [B1]. From this expression it is not hard to see that the prescribed mean curvature equation

$$\text{find } M \text{ such that } \quad H_M \;=\; F|_M$$

is the Euler-Lagrange equation of (VP). Although other boundary conditions are

possible, it is most natural to consider the prescribed boundary problem (Dirichlet problem)

(DP)    find M , a regular hypersurface, such that

$$H_M = F|_M \quad \text{and} \quad \partial M = S, \text{ a given boundary set.}$$

This corresponds to (VP) with $\mathcal{F}$ being the class of weakly spacelike hypersurfaces having boundary S.

The Euler-Lagrange equation (1) is a quasi-linear, non-uniformly elliptic equation with ellipticity governed by the important quantity

(2)
$$\begin{aligned}
v &= 1/\sqrt{1 - |Du|^2} \\
&= -\langle N, T \rangle
\end{aligned}$$

where $T = \partial_t$ is a future timelike reference vector. If an *a priori* bound for $v$ is known then the Leray-Schauder theorem and the De Giorgi-Nash estimates (see eg. [GT]) can be applied to derive existence and regularity results for (1). This rather standard argument is described in [BS], [G], [B1]. Thus, the first main problem is

**Problem 1.** Find an *a priori* bound for $v$ , valid for regular prescribed mean curvature hypersurfaces.

Since $v$ measures the hyperbolic angle between the normal vector and a reference timelike direction and thus blows up as the tangent plane approaches the light cone, this problem has been paraphrased as that of showing that prescribed mean curvature hypersurfaces don't "go null" [MT].

Notice that while the (classical) mean curvature (1) requires $u \in C^2(\Omega)$ and $v < \infty$ , the area functional is well-defined for $u \in C^{0,1}(\Omega)$ and $v \le \infty$ , so it is conceivable that an extremal for (VP) need not satisfy the Euler-Lagrange equation. This leads to

**Problem 2.** Show that solutions of (VP) are in fact classical solutions of the Euler-Lagrange equation (i.e. regular hypersurfaces with the prescribed mean curvature).

## 2. Preliminary Results.

The existence of variational solutions was first shown by Avez [A], with later modifications by Goddard [Go2] and Bancel [B], [AB]. This involves first imposing some geometric conditions to constrain any maximising sequence and then using the concavity of the area functional to show semicontinuity, so that the limit surface is maximal (see also [BS], [B2]). Avez also claimed the solution is regular, but he overlooked the non-uniform ellipticity of the Euler-Lagrange equation (1). This serious error was propagated in [Ge1], [HE] and to a lesser extent in [Cet].

The first correct existence results for classical solutions were based on the implicit function theorem, linearising about known solutions [CB2], [St], [CBFM], [MT]. If $M$ is a regular hypersurface with Laplace operator $\Delta_M$ and second fundamental form $A$ , the linearised mean curvature operator on $M$ is [CB2]

$$(3) \qquad L_M \;=\; -\Delta_M \;+\; |A|^2 \;+\; \mathrm{Ric}(N,N) \;,$$

and the implicit function theorem requires us to be able to invert $L_M \varphi = 0$ in order to find nearby maximal and/or constant mean curvature surfaces. The invertibility is

immediate if the timelike convergence condition holds and appropriate boundary conditions are imposed. Although this approach has the disadvantage that it needs M to be controlled *a priori* and hence can only be applied near known exact solutions, it does yield some physically useful results [MT], [CBM].

The timelike convergence condition appears also in the main uniqueness result, due to Brill and Flaherty [BF]. If $M_0$, $M_1$ are two compact Cauchy surfaces then there is a future timelike geodesic $\gamma$ which maximises the distance between $M_0$ and $M_1$. Assuming $\gamma(0) \in M_0$ and $\gamma(d) \in M_1$, by considering the second variation formula for geodesics they show that

$$(4) \qquad 0 \;\geq\; \int_0^d \mathrm{Ric}(\gamma',\gamma') \, ds \;+\; H_1(\gamma(1)) \;-\; H_0(\gamma(d)) \quad,$$

where $H_0$, $H_1$ are the mean curvatures of $M_0$, $M_1$ respectively. Coupled with the implicit function theorem, this shows uniqueness for regular constant (non-zero) mean curvature Cauchy surfaces [MT]. Later Claus Gerhardt [G] considered the borderline case and by combining the uniqueness results of [BF], [CB2] and his regularity estimates, he showed that if a cosmological spacetime has two maximal Cauchy surfaces then they are both totally geodesic and the region they bound is static. This can readily be sharpened to show that if there is just one maximal Cauchy surface then either there is a constant non-zero mean curvature Cauchy surface or the spacetime is globally static [B3].

## 3. PDE results.

The first result from outside the relativity community is due to E. Calabi [C]. Using the Lorentzian analogue of Simons' identity in minimal surface theory [SJ], he showed that maximal surfaces in Minkowski space have the Bernstein property

(i.e. entire solutions are linear) in dimensions $n \leq 4$. This was extended to all dimensions by Cheng and Yau [CY], using a maximum principle argument. This argument has been simplified by Schoen [S] and Ecker [E] and is worth describing:

Let $M = \text{graph}_\Omega u$ be a maximal hypersurface in $\mathbf{R}^{n,1}$ with $0 \in M$ and define the functions

$$z = |x|^2 - u^2(x) \quad , \quad w = \langle X, N \rangle^2 \quad ,$$

where $X = (x, u(x))$ is the position vector and $N$ is the future-directed unit normal to $M$. Denoting the gradient and Laplace operators of $M$ by $\nabla, \Delta$ respectively, we have the formulae

$$\Delta z = 2n \quad , \quad |\nabla z|^2 = 4(w + z) \quad ,$$
$$\Delta w = 2w|A|^2 + 2|\nabla \langle X, N \rangle|^2$$

where $A(\ ,\ )$ is the second fundamental form of $M$. Denoting by $X^T$ the tangential component of $X$ and $\lambda$ the maximum modulus eigenvalue of $A$, we can estimate

$$|\nabla \langle X, N \rangle|^2 = A^2(X^T, X^T) \leq (w + z) \lambda^2 \ .$$

Since $H = \text{tr } A = 0$ (maximal), by the Schwarz inequality,

$$|A|^2 \geq n/(n-1) \lambda^2 \ ,$$

so that

$$\Delta \log w \geq 2(w/(n-1) - z) |\nabla \log w|^2 / |\nabla z|^2 \ .$$

We now apply the maximum principle to the function

$$f(z) + \log w \; ,$$

where $f(z)$ is chosen so $f(Z) = -\infty$ and $Z > 0$ is such that $M \cap \{\, z \leq Z \,\} \subset\subset M$. This will be satisfied for every $Z > 0$ if $M$ is regular and entire. In fact, we take

$$f(z) \;\; = \;\; K \log(Z - z) + \log z \; , \qquad K > 0 \text{ to be fixed,}$$

and note that $M$ regular ensures $f(z) + \log w$ is $C^2$ near $0$ and hence has an interior maximum. At the maximum point we have

$$f'(z)\, \nabla z + \nabla \log w \;\; = \;\; 0 \quad \text{and} \quad 0 \; \geq \; \Delta f(z) + \Delta \log w \; ,$$

so substituting gives

$$0 \;\; \geq \;\; 2n f'(z) + |\nabla z|^2 f''(z) + 2\,(w/(n{-}1) - z)\, f'^2 \; .$$

Collecting terms in $w$, substituting for $f(z)$ with $K = 2n$ and simplifying, we get

$$w \;\; \leq \;\; n(n+1)\, z \qquad \text{at the maximum point.}$$

Thus at every point in $M \cap \{\, z < Z \,\}$ we have

$$f(z) + \log w \;\; \leq \;\; \log(n(n{+}1)Z^{2n}) \; ,$$

which gives the estimate

$$|\nabla z|^2 \;\; \leq \;\; 4z\,(1 + n(n{+}1)\,(1 - z/Z)^{-2n}) \; .$$

If $M$ is entire then sending $Z \to \infty$ shows

$$|\nabla z|^2 \leq 4(n+1)^2 z$$

and in terms of the Lorentzian distance function $\rho = \sqrt{z}$, this is

$$|\nabla \rho| \leq (n+1) .$$

This shows $M$ is a complete Riemannian manifold, but more is needed to prove the Bernstein theorem. Following [E] we use the Calabi-Simons identity

$$\Delta |A|^2 = 2|A|^4 + 2|\nabla A|^2 \geq 2|A|^4 + 2|\nabla |A||^2 ,$$

together with the maximum principle for $\log(|A|(Z{-}z))$. This gives

$$0 \geq |A|^2 - |\nabla \log |A||^2 - 2n(Z-z)^{-1} - |\nabla z|^2 (Z-z)^{-2}$$

and since at the maximum point also $\nabla \log |A| = -\nabla \log(Z{-}z)$, we see that

$$|A|^2 \leq 2n(Z-z)^{-1} + 2|\nabla z|^2 (Z-z)^{-2} .$$

Thus at every point of $M \cap \{z < Z\}$ we have

$$|A|^2 (Z-z)^2 \leq 2nZ + 2\sup|\nabla z|^2 ,$$

so for an entire $M$ we send $Z \to \infty$ to find that $|A|^2 \equiv 0$, which is the Bernstein result.

Another application of the Calabi-Simon identity was given by Nishikawa [N], whilst Treibergs [T] extended the estimates of [CY] to study constant mean curvature hypersurfaces in Minkowski space. He classified the possible "blow downs" and showed there is a constant mean curvature hypersurface asymptotic to any $C^2$ cut of null infinity, contrary to an earlier conjecture [Go1] (see also [St]).

We note that the $|\nabla\rho|$ estimate implies an interesting interior bound for the ellipticity parameter $\nu$ [B2] but this is of limited use for the general prescribed mean curvature problem, since the argument does not extend to non-constant mean curvature or non-flat spacetimes. However, with hindsight we can see the Cheng - Yau argument is closely related to the interior estimate for $\nu$ of [B2]. This will be described later.

From the formula for the variation of mean curvature [CB2], [B1]

$$(5) \qquad \Delta\nu \;=\; \nu\,(\,|A|^2 + \mathrm{Ric}(N,N)\,) \;+\; T(H_T) \;-\; \langle T,\nabla H\rangle \,,$$

where $\mathrm{Ric}(\,,\,)$ is the Ricci curvature of the spacetime and $T$ is the reference unit timelike vector field used to define $\nu$, we see that for constant mean curvature hypersurfaces in Minkowski space,

$$\Delta\nu \;=\; \nu\,|A|^2 \;\geq\; 0 \,.$$

Of course, this can also be derived by a direct calculation. This implies $\nu$ is bounded by its value on the boundary and thus solvability of the Dirichlet problem for constant mean curvature in flat space follows from suitable boundary gradient estimates. Such estimates were given by Flaherty [F] for $C^2$ domains $\Omega \subset \mathbf{R}^n$ with non-negative mean curvature, and Bancel [Ba1] for convex domains with small data.

The paper [BS] essentially settled the main problems in Minkowski space. A Moser iteration argument based on the $\Delta\nu$ identity and the Sobolev inequality gives a

bound for $v$ in terms of a boundary estimate (for merely bounded measurable mean curvature) – this boundary estimate follows for general $C^2$ domains by a spherical barrier construction. This settles Problem 1. A comparison lemma for (DP) and (VP) solutions shows that a variational extremal is the limit of classical solutions and a mean value type estimate for $\int |D^2 u|^2$ shows that the approximating sequence satisfies uniform estimates, so the limit surface is regular. Thus the Dirichlet problem is solvable for arbitrary domains, provided only that the boundary data admit a spacelike extension. The "contained light ray" lemma shows that a null ray segment within a variational extremal surface extends within the surface to the boundary, and this permits a description of the solution if the boundary data admits only a weakly spacelike extension; namely, the surfaces is regular except on the contained null rays. With the proviso about contained null rays, this settles Problem 2.

Independently, Gerhardt [G] derived the $v$ bound and showed it can be extended to non-flat spacetimes. Like [BS], Gerhardt's estimate requires only bounded mean curvature and depends on the size of the domain and on a boundary estimate for $v$. An immediate application is the existence of constant mean curvature surfaces in spacetimes with compact Cauchy surfaces and barriers (crushing singularities [ES]) to the past and future. A special case of this result in Gowdy spacetimes (i.e. spacetimes with a $T^2$ symmetry) had been obtained slightly earlier by Moncrief and Isenberg [IM]. Further, by constructing spherical barriers Gerhardt was able to solve the Dirichlet problem in spacetimes conformal to a product. As mentioned above, he also applied the estimate to show that maximal surfaces in cosmological spacetimes are either unique or the spacetime splits metrically.

A global estimate for $v$ which does not depend on an *a priori* boundary estimate and could be applied to unbounded domains was given in [B1]. Although the maximum principle argument is described in [B1], an integral argument

(distinct from [G], [BS]) gives the same estimate, including the dependence on the $C^1$ norm of H . The key idea of the argument is to use a time function which has been adapted to the boundary data and consider the maximum principle for the functions $\pm K u + \log v$ simultaneously, where $u$ is the time function restricted to the surface and K is a large constant. The condition $u = 0$ on the boundary is used to control the case where both $\pm K u + \log v$ have boundary maxima. Together with a coordinate bending result constructing a time function incorporating a given regular hypersurface as a level set, this solves (DP) for smooth boundary.

Another application of the estimate is to the problem of finding a maximal surface in an asymptotically flat spacetime. The method here is to solve the Dirichlet problem with $\partial M$ going to (spatial) infinity and try to take a limit. The difficulty is that the estimate for $v$ depends on the height $u$ , and thus an *a priori* estimate is needed for $u$ . Such an estimate is given in [B1] and involves two conditions of physical interest. Firstly, the mean curvature of the reference time function $H^0$ is required to decay as $O(r^{-3})$ and this condition is also needed to remove an arbitrariness in the structure of spatial infinity [A]. Secondly, a uniformity condition is needed in the interior region which ensures that weakly spacelike hypersurfaces have bounded height variation in the interior. Brill's example of an asymptotically flat spacetime not admitting any maximal surface extending through the interior region shows that some restriction is necessary, and the uniformity condition is sufficient to exclude this example.

The height estimate is obtained by first modifying the time function at infinity so $H^0 \leq -cr^{-3}$ (the first condition is essential here) and then applying a test function argument to the mean curvature formula

$$H v = \operatorname{div}_M(\alpha \nabla u) + H^0 + \text{junk} .$$

Some careful estimation of the error terms and a mysterious choice of test function

leads to the estimate for $u$ . This is described in [B1] section 5.

There are many open problems related to this result. The interior condition also implies that the "body" is stationary with respect to infinity: − this can be generalised to a body moving with bounded velocity (the "boosted slice" problem), but not immediately to the case of several bodies moving apart. Another interesting problem is that of showing existence of constant (non-zero) mean curvature surfaces asymptotic to a given cut at null infinity (see [St], [T]). In view of Brill's example, it may be useful to consider a Neumann boundary condition on the horizon.

The most recent results [B2] show that the Minkowski space existence theorems of [BS] hold in general Lorentzian manifolds, after taking into account the possibility of more complicated causal structure. A generalisation of the idea of "graph" is essential to the statement and proof of the (VP), (DP) results: we say that weakly spacelike hupersurfaces $M_0$, $M_1$ are *T-homotopic rel.* $\partial M_0$ (where T is a reference timelike vector field and $\partial M_0 = \partial M_1$) if they are connected by a family of weakly spacelike hypersurfaces $M_t$ , $0 \le t \le 1$ , $\partial M_t = \partial M_0$ , moving along the integral curves of T . Since there may be quite unrelated surfaces spanning a given boundary set (an (immersed) example in Minkowski space is described in [Q]), it is most natural to consider the existence problems for surfaces in a given T-homotopy equivalence class. Thus the basic (DP) existence theorem is:

**Theorem** [B2]   *Suppose* S *is a weakly spacelike hypersurface with* $cl(D(S))$ *globally hyperbolic, and* $F \in C^1(\mathcal{V})$ . *Then there is a weakly spacelike* M *with* $M \approx S$ *rel.* $\partial S$ *and a singular set* $\Sigma \subset M$ *such that* $M - \Sigma$ *is regular and* $H_M = F$ *on* $M - \Sigma$ .

Here the singular set $\Sigma$ is entirely analogous to the "contained light rays" of [BS] :

$$(6) \qquad \Sigma \;\; = \;\; \bigcup \{ \gamma \mid \gamma : (0,1) \to M \text{ is a null geodesic, } \{\gamma(0), \gamma(1)\} \subset \partial M \} \, ,$$

and the $\gamma$ are disjoint and without conjugate points. Thus, any weakly spacelike hypersurface spanning $\partial M$ (in the T-homotopy class) must contain $\Sigma$. The theorem can be extended to allow for immersed surfaces and to allow a class of $C^1$ spacetime metrics [B2].

The proof starts by showing an interior gradient bound from a maximum principle similar to [B1]. A simple form of this estimate in Mikowski space is reminicent of [CY] and the interior gradient estimate for minimal surfaces [K] – we briefly describe this :

Let $M \subset \mathbf{R}^{n,1}$ be a regular hypersurface with mean curvature $H_M = F|_M$ where $F \in C^1(\mathbf{R}^{n,1})$. Set $\tau = \sqrt{t^2 - |x|^2}$, $T = \partial_t$, $T^* = \tau^{-1}X$ and

$$\nu = -\langle T, N \rangle \quad , \quad \zeta = -\langle T^*, N \rangle \ ,$$

where $N$ is the future unit normal to $M$. Now suppose $M$ satisfies

(i) $\qquad \tau_0 = u(0)/2 > 0$ ,

(ii) $\quad M \cap \{\tau \geq \tau_0\} \subset\subset M \cap \{(x,t): |x| \leq R, 0 \leq t \leq R\}$ .

Clearly, by shifting the origin we can arrange that (i) and (ii) hold, for some $\tau_0 > 0$. Since $\langle T, T^* \rangle = -t/\tau$, from (ii) and the triangle inequality ([B1] lemma 3.3) we have

$$\nu \leq 2R\zeta/\tau \quad , \quad \zeta \leq 2R\nu/\tau \ .$$

We have the identities [BS]

$$\Delta\tau = H\zeta - n/\tau - |\nabla\tau|^2/\tau \quad , \quad |\nabla\tau|^2 = \zeta^2 - 1 \ ,$$
$$\Delta\nu = \nu|A|^2 + \delta_{n+1}H \ ,$$

and the estimates

$$|A|^2 \geq (1 + 1/n)\lambda^2 - H^2 \ ,$$
$$|\nabla \log v|^2 \leq \lambda^2 \ ,$$
$$|\delta_{n+1}H| \leq v^3|DH| \ ,$$

from which we see that, in the region $M \cap \{\tau \geq \tau_0\}$,

$$\Delta \log v \geq 1/n|\nabla \log v|^2 - C_1\zeta^2 \ ,$$
$$\Delta \tau \geq -C_2\zeta - |\nabla \log v|^2/\tau \ ,$$

where $C_1$, $C_2$ depend only on $F$, $DF$, $\tau_0$ and $R$. Now we can apply the maximum principle to $f(\tau) + \log v$, where $f(\tau) = K \log(\tau - \tau_0)$ and $K$ is some large constant to be fixed. At the maximum point we have $f'(\tau)\nabla \tau = -\nabla \log v$ and

$$0 \geq f'(\tau)\Delta \tau + f''(\tau)|\nabla \tau|^2 + \Delta \log v$$
$$\geq (f''(\tau) - f'(\tau)/\tau + f'(\tau)^2/n)|\nabla \tau|^2 - C_1\zeta^2 - C_2f'(\tau)\zeta$$

and substituting for $f$ gives

$$K^2(\tau - \tau_0)^{-2}|\nabla \tau|^2 \leq nC_1\zeta^2 + nC_2K(\tau - \tau_0)^{-1}\zeta \ .$$

Since $\zeta^2 - 1 = |\nabla \tau|^2$, for $K$ sufficiently large (depending on $R$, $\tau_0$, $F$, $DF$) we see that at the maximum point,

$$v \leq R\zeta/\tau_0 \leq C(R, \tau_0, F, DF) \ .$$

This gives the estimate

$$v \leq (\tau - \tau_0)^{-K}$$

in $M \cap \{\tau \geq \tau_0\}$ and in particular, $v(0) \leq C(R, \tau_0, F, DF)$.

To generalise this argument it is necessary to find a time function $\tau$ such that $\varnothing \neq M \cap \{\tau > 0\} \subset\subset M$, for any $M \approx S$ rel. $\partial S$. Such time functions are constructed by smoothing the Lorentzian distance from $S_+ = H^-(I^+(D^-(S)))$ and $S_- = H^+(I^-(D^+(S)))$ — the singular set $\Sigma$ arises naturally here since $S_+ \cap S_- = \partial S \cup \Sigma$ and the regions where the Lorentzian distance gives a time function cover $M - \Sigma$.

To describe the variational regularity result, let $I_F(M)$ denote the variational functional (VP). We say $M$ is *locally extremal* at $p$ if for any $S$ such that $S = M$ outside a neighbourhood of $p$, we have $I_F(M) \geq I_F(S)$. Clearly this is weaker than $M$ being maximal for $I_F$.

**Theorem** [B2]   *Suppose $M$ is a weakly spacelike hypersurface which is locally extremal for $I_F(M)$. Then $M$ is regular except for a singular set $\Sigma$, defined as above (6) except that the null geodesic $\gamma \subset M$ may have no endpoints (eg. a closed null loop).*

We note that the hypotheses are completely local, so the result holds regardless of the causal structure of the spacetime — even time-orientability can be dropped if a suitable variational functional can be defined. The thorem reduces the problem of finding a smooth spacelike extremal to a variational problem to two steps:

(i)    show the existence of a Lipschitz hypersurface, extremal for the variational problem — by the semi-continuity of the area mentioned above, this amounts to showing that any sequence of weakly spacelike hypersurfaces which is maximising for $I_F$ is *a priori* uniformly bounded,

(ii)    show that the limiting hypersurface does not contain any entire null geodesics (i.e. the singular set $\Sigma$ is empty). For example, this follows immediately if the spacetime is globally hyperbolic.

The regularity of the limit surface then follows from the theorem.

The proof is based on a foliation uniqueness identity : suppose $\tau$ is a local time function with level sets $Q_\tau$ having mean curvature $F$ , and $M$ is a weakly spacelike hypersurface, T-homotopic to $Q_0$ . Then by applying Stokes' theorem to the identity $\text{div}_M T = F$ , where $T$ is the unit normal vector of the foliation, we get

$$(7) \qquad I_F(M) \quad = \quad I_F(Q_0) \; - \; \int_M (v-1)\,dv_M \quad \leq \quad I_F(Q_0) \; ,$$

with equality exactly when $M$ coincides with a level set of $\tau$ . Here $v$ is derived from the foliation normal vector. The difficulty in applying this result lies in constructing a time function/foliation with mean curvature $F$ . In Minkowski space this is easy (if we also assume $\partial_t F = 0$ ) – just t-translate . This gives a special case of (7),

$$\int_\Omega (H(u) - H(v))\,(u-v)\,dx \quad \leq \quad 0 \quad , \quad u = v \quad \text{on } \partial\Omega \; ,$$

where $u,\, v \in C^{0,1}(\Omega)$ are variational solutions with mean curvatures $H(u)$, $H(v)$ respectively. This identity is similar to that used in the proof of the height estimate for asymptotically flat maximal surfaces. To construct local foliations in general we use the implicit function theorem and this requires a new estimate for the first eigenvalue of the linearisation $L_M$ (see (3)) for all regular prescribed mean curvature surfaces in a small cylinder-neighbourhood $\mathcal{U}_R$ . By estimating the Raleigh quotient directly we show that

$$\lambda_1(L_M; \mathcal{U}_R) \quad \geq \quad \lambda_1(\Delta; B_R)/2 \quad > \quad 0 \; ,$$

where $\lambda_1(\Delta; B_R)$ is the first Dirichlet eigenvalue for the standard Laplacian on the ball of radius $R$ .

This foliation argument also gives local uniqueness, without any further conditions on the spacetime. Uniqueness also holds if the timelike convergence

condition is satisfied or if there is a timelike isometry. Clearly the optimal conditions for uniqueness have not been found — there are large gaps between these conditions.

## 4. Some Applications.

As mentioned in the introduction, there is an interesting application of these results to cosmological spacetimes. If the spacetime $\mathcal{V}$ also satisfies

(G)        $\mathcal{V} - I(p)$ is compact, for one point $p \in \mathcal{V}$,

then there is a regular constant mean curvature Cauchy surface in $\mathcal{V}$. This is proved by considering the family of constant mean curvature surfaces passing through $p$, and showing that exactly one of these surfaces is regular at $p$. It is interesting that although the regularity results described above make no assumptions on the curvature apart from boundedness, this result definitely needs the timelike convergence condition. By Hawkings singularity theorem and Gerhardt's splitting result, the spacetime is then either timelike incomplete (i.e. singular) or it is static. This is Galloway's splitting theorem [Ga], which however assumed (G) holds at every point in $\mathcal{V}$. We note that the condition (G) was first introduced by Geroch [Ge1] with the aim of proving exactly this singularity result.

The necessity of some condition like (G) is shown by the example [B3] of a cosmological spacetime admitting *no* constant mean curvature Cauchy surface. Using the regularity theory and the nature of the spacetime singularities of the example, we can show however that there are complete noncompact constant mean curvature surfaces, which are not Cauchy surfaces.

More recently J.-H. Eschenburg [EJ] has used geodesic methods to prove Yau's splitting conjecture [Y] — a timelike geodesically complete, globally hyperbolic spacetime satisfying the timelike convergence condition and having a *line* (i.e. a doubly infinite timelike geodesic which realises the distance between any two of its points) is necessarily a metric product. This does not require any compactness condition and generalises the result of [B3], since a cosmological spacetime satisfying (G) is either timelike geodesically complete or admits a line. Using the regularity theory of [B2], G. Galloway [Ga2] has shown the assumption of timelike geodesic completeness can be removed from this result.

The argument of [Ge1], [B3] leads to surfaces which are singular at an isolated point. In Minkowski space such singularities have been classified by Ecker [E], using barrier arguments and results from [CY], [BS]. We note that numerous examples of singular surfaces in $\mathbf{R}^{2,1}$ have been given by Kobayashi [Kb1], using the Weierstrass representation for maximal surfaces (this was also known to Calabi), whilst [Kb2], [AN] used this representation to study the Gauss map. In [B4] the results of [E] are extended to non-flat spacetimes and the interior gradient bound of [B2] is adapted to prove a removeable singularity result. It may be possible to improve this result to allow for singular sets larger than just points. A similar question is that of determining the regularity of a maximal surface near a null ray — L. Simon has conjectured the surface is $C^{1,1}$.

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