EIGEN-EXPANSIONS OF SOME SCHRÖDINGER OPERATORS AND NILPOTENT LIE GROUPS

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This note is a summary of the results previously obtained by the authors, and of a number of new results and problems.

In papers [8] - [10] the authors studied Schrödinger operators $H = L + V$, on $\mathbb{R}^d$,.there

(1) $$-L = \sum_{j=1}^{d} (-1)^{n_j} D_j^{2n_j} , \quad n_j \geq 1$$

(2) $$V(x) = \sum_{j=1}^{k} P_j^2(x) \text{ , where } P_j \text{ are real polynomials.}$$

We say that the family of polynomials is *irreducible* if there is no linear change of variables in $\mathbb{R}^d$, such that all the polynomials depend on a smaller number of variables. By restriction to a lower dimensional subspace of $\mathbb{R}^d$, we consider only operators H for which the polynomials $P_1,\ldots,P_k$ are irreducible.

THEOREM 1 [9] *For some* N *and* $\lambda > 0$, $(\lambda + H)^{-N}$ *is a Hilbert-Schmidt operator on* $L^2(\mathbb{R}^d)$.

Let $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be the eigenvalues and $\varphi_1,\varphi_2,\ldots$ the corresponding eigen-functions of H .

THEOREM 2 [9] *There is an* N *such that if* $K \in C^N(\mathbb{R}^+)$ *and*

(3) $$(1 + \lambda)^N |K^{(j)}(\lambda)| \leq C \quad \textit{for} \quad \lambda > 0 \text{ , } j = 0,\ldots,N$$

then the operator

$$\varphi \longrightarrow \sum_{n=1}^{\infty} K(\lambda_n)(\varphi,\varphi_n)\varphi_n$$

is bounded on all $L^p(\mathbb{R}^d)$, *including* $p = 1$.

THEOREM 3 [9] *There is an* N *such that if* $K \in C^N(\mathbb{R}^+)$, $K(0) = 1$ *and* K *satisfies* (3), *then*

$$\varphi(x) = \lim_{t\to 0} \sum_{1}^{\infty} K(t\lambda_n)(\varphi,\varphi_n)\varphi_n(x)$$

a.e. and in norm for φ *in* $L^p(\mathbb{R}^d)$ *and all* p, $1 \le p < \infty$.

Using the methods of the proof of theorem 2 and 3 , cf. below, the ideas of the first author and E.M. Stein [5], chapter 6, and an application of the "transference" methods of R. Coifman and G. Weiss [1], [2] and C. Herz [7], [8], the following theorem can be proved.

THEOREM 4 (Dlugosz [3]). *There exists* N *such that if* $K \in C^N(\mathbb{R}^+)$ *and* K *satisfies the Marcinkiewicz condition*

$$\lambda^j |K^{(j)}(\lambda)| \le C \quad \text{for} \quad \lambda > 0 \ , \ j = 0,\dots,N,$$

then the operator

$$M\colon \varphi \longrightarrow \sum_{n=1}^{\infty} K(\lambda_n)(f,\varphi_n)\varphi_n$$

is bounded on $L^p(\mathbb{R}^d)$, $1 < p < \infty$ *and* $\|M\|_{L^p,L^p} \le C$.

Let us mention also the following unsatisfactory generalization of theorem 2 . This is proved only under the following additional condition

(4) In (1), all $n_j = 1$, i.e. L is the ordinary Laplacian.

THEOREM 5 *If* V *is an arbitrary locally integrable function such that*

(5) $$V(x) \geq |x|^a \quad \textit{for} \quad |x| > C \quad \textit{for some } a, C > 0 ,$$

then there is an N *such that if* $K \in C^N(\mathbb{R}^+)$ *and (3) holds, then the conclusion of theorem 2 remains true.*

The proof is based on the Feynman-Kac formula and an easy functional calculus of the type used in [8] within the Banach algebra $\mathrm{lin}\{T_t : t > 0\}^-$ in $B(L^p(\mathbb{R}^d))$, where $T_t = e^{-tH}$.

The main idea of the proofs of theorems 1 - 4 is to consider the Lie algebra spanned by the partial derivatives D_j and multiplications by iP_j on $\mathcal{S}(\mathbb{R}^d)$. This of course, is a finite dimensional Lie algebra, nilpotent of step c , say. We pass to the free nilpotent Lie algebra $\underline{g}$ of step c generated by $X_1, \ldots, X_d$, $Y_1, \ldots, Y_k$ and we consider the representation

$$\partial\pi : \underline{g} \ni \begin{array}{l} X_j \longrightarrow D_j \\ Y_j \longrightarrow \text{multiplication by } iP_j \end{array} .$$

It is easy to verify that $\partial\pi$ is the derivative of a unitary representation π of $G = \exp \underline{g}$ which is induced from the normal subgroup and, if $P_1, \ldots, P_k$ are irreducible, then π is an irreducible representation.

Let

(6) $$\underline{L} = \sum_{j=1}^{d} (-1)^{n_j} X_j^{2n_j} + \sum_{j=1}^{k} Y_j^2 .$$

Then, by (1) and (2), $\pi(\underline{L}) = H$.

By a theorem of Olejnik and Radkievitch (or Helffer-Nourigat) $\underline{L}$ is a subelliptic symmetric operator on $L^2(G)$, so it is essentially self-adjoint. Let

$$\underline{L}f = \int_0^\infty \lambda dE(\lambda)f$$

be the spectral resolution of $\underline{L}$ on $L^2(G)$. Let

$$T_t f = \int_0^\infty e^{-t\lambda} dE(\lambda)f \quad .$$

By a theorem of G. Folland and E.M. Stein [5], we have

(7) $$T_t f = f * p_t \ , \ p_t \in L^1(G) \cap L^2(G) \ ;$$

in fact

(8) $$p_t \in \mathcal{S}(G) \ .$$

Let δ_r be the one-parameter group of dilations of G such that

(9) $$\delta_r \underline{L} = r\underline{L}$$

Such a group exists, since $\underline{g}$ is freely generated by $X_1,\dots,X_d$, $Y_1,\dots,Y_k$.

Now the proofs involve the following ingredients.

(a) A functional calculus such that if $F \in C_c^N(\mathbb{R}^+)$, then for $K(\lambda) = F((1+\lambda)^{-s})$ for some s , the operator $T_K f = \int_0^\infty K(\lambda)dE(\lambda)f$ is of the form $T_K f = f * k$, where $k \in L^1(G) \cap L^2(G)$, cf. [12]. Moreover, to prove theorem 3 we show that $|k(x)| \leq C(1 + |x|)^{-m}$, where $|x|$ is the homogenous gage on G and $m > Q$, the homogenous dimension of G.

(b) A method to carry the operator T_K by the representation to obtain $\pi(T_K)\varphi = \sum_1^\infty K(\lambda_n)(\varphi,\varphi_n)\varphi_n$ and prove that it has the desired properties.

This is done by a very substantial use of homogeniety of $\underline{L}$. Indeed, we have

(10) $$T_{K_t} f = \int_0^\infty K(t\lambda)dE(\lambda) = \varphi * k_t ,$$

where

$$k_t(x) = k(\delta_{t^{-1}}x)/xt^Q .$$

This allows us to use dilations in the study of the maximal function

$$\sup_{t>0} |T_{K_t} f(x)|$$

as in the proof of the Marcinkiewicz multiplier theorem on G, and carry it down by the representation π.

GENERALIZATIONS AND PROBLEMS

One would like to generalize the above theorems to more general potentials allowing, perhaps, for assumption (4). The first step should certainly be to include potentials which are non-negative polynomials, not only sums of squares of polynomials. However, positivity of V does not seem to have an interpretation in terms of properties of operators on G, i.e. it seems hopeless to us to define such a property of an operator Y on G which would imply that $\pi(Y)$ is multiplication by a non-negative polynomial. However, if for Y in $\underline{g}$

we write

$$|Y|^{2a}f = -\int_0^\infty t^{1+a}(f*h_t-f)dt \ ,$$

where $\{h_t\}_{t>0}$ is the semi-group of probability measures whose infinitesimal generator is Y^2 , and if $\pi(Y)\varphi = iP\varphi$, then

$$\pi(|Y|^a)\varphi = |P|^a\varphi \quad .$$

On the other hand, the operator

$$\underline{L} = -X_1^2-\ldots-X_d^2 + |Y_1|^{a_1} +\ldots+|Y_k|^{a_k} \tag{11}$$

is essentially self-adjoint on $L^2(G)$ and, if

$$\underline{L}f = \int_0^\infty \lambda dE(\lambda)f$$

is the spectral resolution of $\underline{L}$, then

$$T_tf = \int_0^\infty e^{-t\lambda}dE(\lambda)f$$

has the property that

$$T_tf = f * \mu_t \ ,$$

where μ_t is a probability measure. Thus, since again $\underline{L}$ can be made homogenous of degree 1 on G we have $\mu_t = \mu_1\circ\delta_t$.

However, the crucial property for the functional calculus as described in (a) is that μ_t is absolutely continuous with respect to the Haar measure, moreover, the density of μ_t should belong to $L^2(G)$. Since $\underline{L}$ is not a differential operator any more and has a very non-smooth "symbol", it is not hypoelliptic and so the methods of Olejnik and Radkievitch or Helffer and Nourigat are not available. The first step to handle this more general situation has been made by the authors in [10] where it is proved that if X and Y generate the Lie algebra $\underline{g}$ and all $ad_X^j Y$, $j=0,\dots,c$, commute, then the semi-group $u_t = e^{t(X^2-|Y|^a)}$ consists of measures with $L^2(G)$ densities. The proof goes via a study of the representations and the Plancherel measure of the nilpotent group $G = \exp \underline{g}$ and uses estimates of the lowest eigenvalue of some Schrödinger operators established by C. Fefferman in [4]. This already yields some summability results for operators of the form

$$-\frac{d^2}{dx^2} - |x|^a \quad a > 0 \text{ on } \mathbb{R} .$$

Very recently a real break through has come from P. Glowacki who proved the following

THEOREM (Glowacki). *Let* G *be a homogenous group and let* $\underline{L}$ *be a symmetric homogenous left-invariant operator on* $\mathcal{S}(G)$ *such that* $-\underline{L}$ *generates a semi-group of probability measures. If for every irreducible non-trivial representation* π *of* G *the operator* $\pi(\underline{L})$ *is injective, then the semi-group consists of measures which have densities in* $L^2(G)$.

It is fairly easy to verify that the operators of the form (11) satisfy the hypothesis of Glowacki's theorem.

This opens a way to good C^N functional calculus in the sense of (a) for operators of the form (11). In the absence of (8), however, we have succeeded only for step 2 nilpotent groups, as yet. Also to prove theorems like theorem 3 or 4 for potentials $V(x) = |P(x)|^a$, P a polynomial, we have to obtain estimates for the maximal functions like (10). The classical methods of comparing our maximal function to the Hardy-Littlewood maximal function on $\mathbb{R}^d$ do not work.

Another more satisfactory version of theorem 5 seems to be within reach.

THEOREM (?) *If for* $|x| \geq C$ *we have* $V(x) \geq |P_j(x)|^{a_j}$ *for some irreducible family of polynomials* $P_1, \ldots, P_k$, $a_j > 0$, *then the conclusion of theorem* 2 *holds.*

To illustrate the strength of Glowacki's theorem, we mention that one can deduce from it that V satisfies the assumption of the above hypothetical theorem; the operator $-\Delta + V$ is invertible and has discrete spectrum. It is very tempting to compare the smallest eigen-value of the operator $-\Delta + V$, V being the sum of absolute values of polynomials, using Glowacki's result and [11] and C. Fefferman's estimates in [4].

REFERENCES

[1] R. Coifman and G. Weiss, Operators associated with representations of amenable groups, singular integrals induced by ergodic flows, rotation method and multipliers, Studia Math. 47 (1973), 285-303.

[2] R Coifman and G. Weiss, Transference methods in analysis, AMS regional conference series in mathematics 31 (1976).

[3] J. Dlugosz, Marcinkiewicz multiplier theorem for Laguerre expansions (to appear).

[4] C. Fefferman, The uncertainty principle, Bull.AMS 9 (1983), 129 - 206.

[5] G. Folland and E.M. Stein, Hardy spaces on homogenous groups, Princetown University Press (1982).

[6] C. Herz, The theory of p-spaces with application to convolution operators, Trans. AMS 154 (1971), 62-82.

[7] C. Herz, Harmonic synthesis for subgroups, Ann. Inst. Fourier (Grenoble) 23 (1972), 91-123.

[8] A. Hulanicki and J.W. Jenkins, Almost everywhere summability on nilmanifolds, Trans. AMS 278 (1983), 703 - 715.

[9] - Nilpotent Lie groups and summability of eigenexpansions of Schrödinger operators, Studia Math. 80 (1984), 235-244.

[10] - Nilpotent Lie groups and summability of eigenexpansions of Schrödinger operators II, Studia Math.87 (to appear).

[11] - and J. Ludwig, Minimum eigenvalues for positive Rockland operators, Proc. AMS 94 (1985), 718-720.

[12] A. Hulanicki, A functional calculus for Rockland operators on nilpotent Lie groups, Studia Math. 78 (1984), 253-266.

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