

EIGENSTRUCTURE SPECIFICATION VIA STATE BOUNDARY FEEDBACK  
FOR LINEAR SYSTEMS IN HILBERT SPACE

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Recently Clarke and Holland [1] investigated the problem of eigenstructure specification for linear systems in Hilbert space with Distributed Control. By the construction of a spectral representation of the closed loop system it was shown that spectral specification is possible under certain critical conditions depending on the dimension of the control space being sufficiently large in relation to the dimension of the eigenspaces of the linear system operator, and, subject to an asymptotic condition on the closed loop spectrum.

In this paper the same spectral specification problem is investigated but with Boundary Control rather than Distributed Control. This is an important problem since, in practice, it is often more practical to exert control by means of the boundary conditions, particularly for systems governed by linear partial differential equations [2]. Fattorini [3] determined certain controllability conditions by replacing the Boundary Control by Distributed Controls which have the same effect on the system and then applying known results for distributed parameter systems. This method has been further pursued by Curtain [4] and is the underlying approach used here also.

Invoking Fattorini's controllability result it is shown that the possibility of spectral specification again depends critically on the dimension of the control space being sufficiently large in relation to the dimension of the eigenspaces of the linear system operator. However,

the asymptotic requirement for the spectrum of the closed loop system is found to be weaker than for the case of Distributed Control considered by Clarke and Holland previously [1]. This indicates that the effect of the implementation of Boundary Control on the system is more powerful than the effect of Distributed Control on the system.

Following Fattorini ([3] pp.350-352) we consider the linear system

$$(1.1) \quad \begin{aligned} \dot{x} &= ax \\ Tx &= B_S u \end{aligned}$$

where  $x(\cdot): [0, \infty) \rightarrow X$ , ( $X$  a complex, separable Hilbert space) and  $u(\cdot): [0, \infty) \rightarrow U$ , ( $U$  a finite dimensional complex inner product space) are functions,  $\dim U = m$ ,  $a: X \rightarrow X$  is a closed linear operator,  $T: X \rightarrow Z$  is a linear boundary operator with  $D(T) \supseteq D(a)$ ,  $Z$  is a finite dimensional space and  $B_S: U \rightarrow Z$  is a non-singular bounded linear operator. Here  $S$  denotes the boundary of some domain  $I$  of Euclidean space  $\mathbb{R}^r$ . Typically  $a$  would be a linear partial differential operator acting in  $X$  and  $T$  a linear differential operator acting on the boundary  $S$  of  $I$ .

We define another linear operator  $A$  on

$$D(A) = \{x \in D(a) \mid Tx = 0\}$$

by  $Ax = ax$ ,  $x \in D(A)$ . We assume that  $A$  is a discrete, spectral operator of scalar type [1], with  $\sigma(A) = \{\lambda_i: i = 1, 2, \dots\}$  satisfying

$$01 \quad \inf_i \delta_i = \delta > 0 \quad \text{where} \quad \delta_i = \inf_{k \neq i} |\lambda_k - \lambda_i|, \quad i = 1, 2, \dots$$

Remark 1.  $A$  is in fact the infinitesimal generator of a strongly continuous semi-group on  $X$  and consequently  $A$  and therefore  $A^*$  have dense domains in  $X$ .

Using the notation of [1], the eigenvectors  $\{\phi_j^i; i=1,2,\dots, j=1,\dots,v_i\}$  corresponding to the eigenvalues  $\{\lambda_i; i=1,2,\dots\}$  form a Riesz basis for  $X$  with corresponding biorthogonal basis  $\{\psi_j^i; i=1,2,\dots, j=1,\dots,v_i\}$  (the eigenvectors corresponding to the eigenvalues  $\{\bar{\lambda}_i; i=1,2,\dots\}$  of  $A^*$ ). Hence, any  $x \in X$  has the unique representation

$$x = \sum_{i=1}^{\infty} \sum_{j=1}^{v_i} \langle x, \psi_j^i \rangle \phi_j^i$$

where

$$(1.2) \quad c \sum_{i=1}^n \sum_{j=1}^{v_i} |\langle x, \psi_j^i \rangle|^2 \leq \left\| \sum_{i=1}^n \sum_{j=1}^{v_i} \langle x, \psi_j^i \rangle \phi_j^i \right\|^2 \leq C \sum_{i=1}^n \sum_{j=1}^{v_i} |\langle x, \psi_j^i \rangle|^2$$

for all  $n = 1, 2, \dots$ , and positive constants  $c, C$ , independent of  $x$  [6] (Ch.VI, Theorem 2.1).

Fatterini [3] (p.351, Assumption 4) made the following important additional assumption on the linear boundary operator  $T$ .

O2: There exists a bounded linear operator  $B: U \rightarrow X$  such that for all  $u \in U$ ,  $Bu \in D(A)$  and  $T(Bu) = B_s u$ .

Since  $\dim U = m$  and  $\text{Ker } B_s = \{0\}$ , there exists a basis  $\{z_i; i = 1, \dots, m\}$  for  $Z$  of dimension  $m$ . Assumption O2 then implies that there exist linearly independent  $b_1, \dots, b_m \in X$  such that

$$T(b_i) = z_i$$

for  $i = 1, \dots, m$ .

Remark 2. The importance of this assumption lies in the fact that any element of the form  $x - Bu$  ( $x \in D(a)$ ) is actually in  $D(A)$  since

$$\begin{aligned} T(x - Bu) &= T(x) - T(Bu) \\ &= B_S u - B_S u \\ &= 0. \end{aligned}$$

Upon introduction of state boundary feedback of the form  $u = Fx$ , the following spectral assignment problem arises. Given a countable sequence of complex numbers  $\{u_i; i = 1, 2, \dots\}$  does there exist  $F: X \rightarrow U$  bounded and linear such that the restriction  $a_F$  of  $A$  defined on

$$D(a_F) = \{x \in D(a) \mid (T - B_S F)x = 0\}$$

by  $a_F x = ax$ ,

is spectral, discrete and of scalar type with  $\sigma(a_F) = \{\mu_i; i = 1, 2, \dots\}$ .

## 2. Controllability Results

Fattorini [3] (pp.355-358, Theorem 3.3) showed that controllability of the linear system (1.1) is equivalent to the controllability of the linear system

$$(2.1) \quad \dot{x} = Ax + (R_{\lambda_0} a - aR_{\lambda_0})Bu, \quad x \in D(A),$$

(where  $R_{\lambda_0}$  is the resolvent operator  $(\lambda_0 I - A)^{-1}$  for all  $\lambda_0 \in \rho(A)$ ).

Letting  $Q = AB$ , which is a bounded linear operator by the Closed Graph Theorem, we are able to deduce the following Lemma.

**Lemma 2.1.** *The system (1.1) is controllable if and only if, for each  $i = 1, 2, \dots$ ,  $(\bar{\lambda}_i - \bar{\lambda}_0)^{-1}(Q^* - \bar{\lambda}_i B^*)$  is an injection on each of the eigenspaces  $F_i = \text{Ker}(\bar{\lambda}_i - A^*)$  for all  $\lambda_0 \in \rho(A)$ .*

**Proof.** By another result of Fattorini [5] (Corollary 3.3), (2.1) is controllable if and only if  $B'^*$  (where

$$B'_{\lambda_0} = (R_{\lambda_0} a - aR_{\lambda_0})B = (R_{\lambda_0} a - AR_{\lambda_0})B$$

is an injection on each of the  $F_i$  ( $i = 1, 2, \dots$ ) for all  $\lambda_0 \in \rho(A)$ .

Letting  $f \in F_i$  we have

$$\begin{aligned} B'^*f &= (Q^*R_{\lambda_0}^* - B^*R_{\lambda_0}^*A)f \\ &= (\bar{\lambda}_0 - \bar{\lambda}_i)^{-1}Q^*f - (\bar{\lambda}_0 - \bar{\lambda}_i)^{-1}\bar{\lambda}_i B^*f \\ &= (\bar{\lambda}_0 - \bar{\lambda}_i)^{-1}(Q^* - \bar{\lambda}_i B^*)f, \end{aligned}$$

for all  $\lambda_0 \in \rho(A)$  whence the result follows.

From this the ensuing corollary is trivial.

**Corollary 2.2.** *A necessary condition for controllability of the system (1.1) is that  $\sup v_i \leq m = \dim U$ .*

Henceforth we assume that the system (1.1) is controllable and that  $\lambda_0 = 0 \in \rho(A)$ .

Taking an orthonormal basis  $\{u_1, \dots, u_m\}$  for  $U$  it follows that the matrices  $Q_i^*$  ( $i = 1, 2, \dots$ ),  $Q_i^*: \mathbb{C}^V \rightarrow \mathbb{C}^m$  given by

$$(2.2) \quad Q_i^* = \begin{pmatrix} \langle (\bar{\lambda}_i^{-1} Q_i^* - B^*) \psi_1^i, u_1 \rangle & \dots & \langle (\bar{\lambda}_i^{-1} Q_i^* - B^*) \psi_{v_i}^i, u_1 \rangle \\ \langle (\bar{\lambda}_i^{-1} Q_i^* - B^*) \psi_1^i, u_m \rangle & \dots & \langle (\bar{\lambda}_i^{-1} Q_i^* - B^*) \psi_{v_i}^i, u_m \rangle \end{pmatrix}, \quad i = 1, 2, \dots,$$

are all of rank  $v_i$  (by Lemma 2.1). Thus, the matrices  $Q_i$  ( $i = 1, 2, \dots$ ),  $Q_i: \mathbb{C}^m \rightarrow \mathbb{C}^{v_i}$  defined by

$$(2.3) \quad Q_i = \begin{pmatrix} \langle (\bar{\lambda}_i^{-1} Q - B) u_1, \psi_1^i \rangle & \dots & \langle (\bar{\lambda}_i^{-1} Q - B) u_m, \psi_1^i \rangle \\ \langle (\bar{\lambda}_i^{-1} Q - B) u_1, \psi_{v_i}^i \rangle & \dots & \langle (\bar{\lambda}_i^{-1} Q - B) u_m, \psi_{v_i}^i \rangle \end{pmatrix}, \quad i = 1, 2, \dots,$$

(all being of rank  $v_i$ ) enable us to construct bases  $\{\gamma_j^i, j = 1, \dots, m\}$  ( $i = 1, 2, \dots$ ) for  $U$  by putting

$$(2.4) \quad \gamma_j^i = \sum_{r=1}^m (\omega_j^i)_r u_r, \quad j = 1, \dots, v_i,$$

where  $\omega_j^i = [(\omega_j^i)_1 \dots (\omega_j^i)_m]^T$  ( $j = 1, \dots, v_i$ ) is given by

$$(2.5) \quad \omega_j^i = Q_i^* (Q_i Q_i^*)^{-1} e_j^i,$$

$\{\varepsilon_j^i; j = 1, \dots, v_i\}$  is the standard basis for  $\mathbb{C}^{v_i}$  and extending each linearly independent set  $\{\gamma_j^i; j = 1, \dots, v_i\}$  to a basis  $\{\gamma_j^i; j = 1, \dots, m\}$  for  $U$  by taking  $\{\gamma_j^i; j = v_{i+1}, \dots, m\}$  as a basis of  $U_i^\perp$  where

$$U_i^\perp = \text{span}((\bar{\lambda}_i^{-1} Q_i^* - B^*) \psi_1^i, \dots, (\bar{\lambda}_i^{-1} Q_i^* - B^*) \psi_{v_i}^i).$$

From (2.4), (2.5) and (2.6) it follows that

$$(2.7) \quad \langle (\lambda_i^{-1} Q - B) \gamma_j^i, \psi_\sigma^i \rangle = \delta_{j\sigma} \quad j = 1, \dots, m, \quad \sigma = 1, \dots, v_i$$

for each  $i = 1, 2, \dots$ .

Moreover,

$$(2.8) \quad \sum_{j=1}^{v_i} \|\gamma_j^i\|^2 = \|(Q_i Q_i^*)^{-\frac{1}{2}}\|^2, \quad i = 1, 2, \dots,$$

where

$$(2.9) \quad \sum_{i=1}^{\infty} \|(Q_i Q_i^*)^{-\frac{1}{2}}\|^{-2} \leq \sum_{i=1}^{\infty} \|Q_i\|^2 < \infty.$$

### 3. Main Results

Let  $\{\mu_i; i = 1, 2, \dots\}$  be a countable collection of distinct complex numbers and  $\{\tilde{v}_i; i = 1, 2, \dots\}$  an associated countable collection of positive integers subject to the conditions

$$C1 \quad \sum_{i=1}^{\infty} |\mu_i - \lambda_i|^2 \|(Q_i Q_i^*)^{-\frac{1}{2}}\|^2 (|\lambda_i|^{-1} + \delta_i^{-1})^2 < \infty,$$

$$C2 \quad (i) \quad \tilde{v}_i \leq m \quad i = 1, 2, \dots,$$

(ii) for some positive integer  $K$ ,  $\tilde{v}_i = v_i$  for  $i > K$ , and

$$(iii) \quad \sum_{i=1}^k \tilde{v}_i = \sum_{i=1}^k v_i.$$

We now define a countable collection of vectors  $\{\xi_j^i; i = 1, 2, \dots, j = 1, \dots, \tilde{v}_i\}$  in  $X$  by

$$(3.1) \quad \xi_j^i = \begin{cases} \lambda_i^{-1}(\mu_i - \lambda_i) [R_{\mu_i}(Q - \mu_i B)\gamma_j^i + B\gamma_j^i], & j \leq v_i \\ R_{\mu_i}(Q - \mu_i B)\gamma_j^i + B\gamma_j^i, & j > v_i \end{cases}$$

whenever  $\mu_i \in \rho(A)$ .

Now, observing that  $R_{\mu_i}(Q - \mu_i B)\gamma_j^i \in D(A)$  and using Assumption O2 it is easy to see that

$$(\mu_i - A)[R_{\mu_i}(Q - \mu_i B)\gamma_j^i + B\gamma_j^i] = (Q - \mu_i B)\gamma_j^i + \mu_i B\gamma_j^i - Q\gamma_j^i = 0$$

and

$$T[R_{\mu_i}(Q - \mu_i B)\gamma_j^i + B\gamma_j^i] = T(B\gamma_j^i) = B_S\gamma_j^i$$

Hence, for each  $i = 1, 2, \dots$

$$(3.2) \quad \left\{ \begin{array}{l} A\xi_j^i = \mu_i \xi_j^i \quad \text{and} \\ T\xi_j^i = \begin{cases} \lambda_i^{-1}(\mu_i - \lambda_i) B_S\gamma_j^i, & j \leq v_i \\ B_S\gamma_j^i, & j > v_i. \end{cases} \end{array} \right.$$

Moreover, the sets  $\{\xi_j^i; j=1, \dots, v_i\}$  are linearly independent for each  $i = 1, 2, \dots$ . For example, if  $\tilde{v}_i > v_i$  and

$$0 = \sum_{j=1}^{\tilde{v}_i} \alpha_j \xi_j^i$$

for some choice of scalars  $\alpha_1, \dots, \alpha_{\tilde{v}_i}$ , then



$$0 = T_0 = \sum_{j=1}^{v_i} \alpha_j \lambda_i^{-1} (\mu_i - \lambda_i) B_s \gamma_j^i + \sum_{j=v_i+1}^{\tilde{v}_i} \alpha_j B_s \gamma_j^i .$$

Since  $\text{Ker } B_s = \{0\}$ ,  $\mu_i \in \rho(A)$  and  $\{\gamma_j^i; j=1, \dots, \tilde{v}_i\}$  are linearly independent we are able to conclude  $\alpha_1 = \alpha_2 = \dots = \alpha_{\tilde{v}_i} = 0$ . In fact:

Lemma 3.1. *The set  $\{\xi_j^i; i = 1, 2, \dots, j = 1, \dots, \tilde{v}_i\}$  forms a Riesz basis for  $X$ .*

Proof. Follows from [6] (Ch.VI, Theorem 2.3, Remark 2.1).

We are now able to define a linear operator  $F: X \rightarrow U$  (on the Riesz basis  $\{\xi_j^i\}$ ) by

$$(3.5) \quad F\xi_j^i = \begin{cases} \lambda_i^{-1} (\mu_i - \lambda_i) \gamma_j^i, & j \leq v_i \\ \gamma_j^i, & j > v_i . \end{cases}$$

Thus the operator  $F: X \rightarrow U$  is well defined by the action (3.5). Moreover, it is a bounded operator since for any  $x \in X$ ,

$$\begin{aligned} \|Fx\|^2 &= \left\| \sum_{i=1}^{\infty} \sum_{j=1}^{v_i} \langle x, \eta_j^i \rangle F\xi_j^i \right\|^2 \\ &\leq \left( \sum_{i,j} |\langle x, \eta_j^i \rangle|^2 \right) \left( \sum_{i,j} \|F\xi_j^i\|^2 \right) \\ &\leq c \|x\|^2 \left( \sum_{i,j} \|F\xi_j^i\|^2 \right) \\ &\leq c \|x\|^2 \sum_{i \in K} \sum_{j=1}^{\tilde{v}_i} \|F\xi_j^i\|^2 + \sum_{i > K} |\lambda_i|^{-2} |\mu_i - \lambda_i|^2 \| (Q_i Q_i^*)^{-\frac{1}{2}} \|^2 \\ &\leq c' \|x\|^2 \quad (\text{from C1}) \end{aligned}$$

for some positive constant  $C'(\{\eta_j^i\})$  being the biorthogonal basis of  $\{\xi_j^i\}$ .

The restriction operator  $a_F$  of  $a$  is now defined since by (3.2),

$$(3.6) \quad (T - B_S F)\xi_j^i = 0,$$

for each  $i = 1, 2, \dots$  and  $j = 1, \dots, \tilde{v}_i$ . Moreover

$\sigma(a_F) = \{\mu_i; i = 1, 2, \dots\}$  from (3.2) and (3.6).

We now state the main result.

**Theorem 3.2.** *Let  $A$  be a discrete, spectral operator of scalar type on a Hilbert space  $X$  with  $\sigma(A)$  satisfying condition O1. Let  $B: U \rightarrow X$  be a bounded linear operator satisfying O2. Further, suppose the linear system (1.1) is controllable. Then, for any countable collection of distinct complex numbers  $\{\mu_i; i = 1, 2, \dots\}$  and any countable collection of positive integers  $\{\tilde{v}_i; i = 1, 2, \dots\}$  which satisfy conditions C1 and C2, there exists a bounded linear operator  $F: X \rightarrow U$  such that the closed loop operator  $a_F$  is discrete, spectral and of scalar type and satisfies  $\sigma(a_F) = \{\mu_i; i = 1, 2, \dots\}$  and  $\dim \text{Ker}(\mu_i I - a_F) = \tilde{v}_i$  ( $i = 1, 2, \dots$ ).*

**Remark 3.** In practice it often happens that  $\sup(\| (Q_i Q_i^*)^{-1/2} \| (|\lambda_i|^{-1} + \delta_i^{-1})) < \infty$ , so that the asymptotic condition C1 simply becomes

$$\sum_{i=1}^{\infty} |\mu_i - \lambda_i|^2 < \infty$$

which is a vast improvement on the result given in [1] for eigenstructure specification via linear state feedback.

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