HOMOGENEOUS BANACH SPACES OF TYPE C

J.A. Ward

Let G denote a compact abelian group and B a Banach algebra of continuous functions defined on G with pointwise multiplication. G.E. Silov called B of type C if its norm is equivalent to that defined by

$$\|b\|^{C} = \sup_{x\in G} \inf\{\|c\|_B : c \in B \ , \ c(x) = b(x)\} \ ,$$

and gave a complete classification of those algebras which are homogeneous and of type C. In this paper, we first replace pointwise multiplication by convolution, before generalizing the notion of type C to homogeneous Banach spaces. Again a complete classification is obtained.

1. INTRODUCTION

We begin by reviewing a problem which was solved by G.E. Silov in a rather lengthy paper [3] in the early 1950's. The basic theory of commutative Banach algebras can be found in many texts available on the subject; see for example [1], from which all unexplained notation is taken. A commutative unital Banach algebra B, without radical, has a maximal ideal space $\mathbf{M}$ which is both Hausdorff and compact. By the Gelfand Representation Theorem there is an isomorphism $b \to \hat{b}$ of B onto a subalgebra $\hat{B}$ of $C(\mathbf{M})$, where $\mathbf{M}$ carrys the usual Gelfand topology. It is often convenient to identify elements of B with continuous functions on $\mathbf{M}$.

An ideal I of B *belongs to* a non-empty subset M of $\mathbf{M}$ if $f(m) = 0$ for all $m \in M$, $f \in I$, while for each $m \in \mathbf{M}\setminus M$ there exists an $f \in I$ such that $f(m) \neq 0$. If B is regular then for each $m \in \mathbf{M}$ there exists a smallest non-zero closed primary ideal, $J(m)$ say, belonging to $\{m\}$. The quotient algebras $B/J(m)$ then each have a unique maximal ideal.

In [3] Silov introduces the notion of the *norm of an element at a point*. For a regular commutative unital Banach algebra B, without radical, the norm $\|b\|_m$ of $b \in B$ at $m \in \mathbf{M}$ is

$$\|b\|_m = \|\bar{b}\|_{B/J(m)} , \tag{1}$$

where $\bar{b}$ is the coset containing b . Clearly, if $\| \ \|_B$ is the original norm on B then for each $m \in \mathbf{M}$,

$$\| \ \|_B \geq \| \ \|_m ,$$

and so

$$\| \ \|_B \geq \sup_{m \in M} \| \ \|_m . \tag{2}$$

Silov calls B of *type C* if its norm is equivalent to $\| \ \|^c$ defined by

$$\| \ \|^c = \sup_{m \in \mathbf{M}} \| \ \|_m .$$

In view of inequality (2), B is of type C if and only if there exists a positive constant K for which $\|b\|_B \leq K\|b\|^c$ for all $b \in B$.

EXAMPLE 1 For any compact set S , the Banach algebra $C(S)$ of continuous functions defined on S , with the usual pointwise operations, is of type C .

EXAMPLE 2 For each integer $n \geq 1$, the pointwise algebra $C^{(n)}(\mathbf{T})$, of all functions defined on the circle which have n continuous derivatives, with norm defined by

$$\|f\|_{(n)} = \sup_{t \in \mathbf{T}} \sum_{k=0}^{n} \frac{1}{k!} |f^{(k)}(t)| ,$$

is also of type C .

In particular, Silov considered the case where the maximal ideal space $\mathbf{M}$ could be endowed with the structure of a compact abelian group, in such a way that $\hat{B}$ is homogeneous; that is, $\hat{B}$ is translation invariant, each translation operator and each shift is continuous. (See [4] for a more detailed discussion of homogeneous Banach algebras.) Note that Silov actually used a different definition of homogeneity, but then required the functions in $\hat{B}$ to be continuous under translation, that is, to have continuous shift, which means that $\hat{B}$ is homogeneous in the commonly used sense.

Starting with a compact group G (which will play the role of $\mathbf{M}$), a Banach algebra K with unique maximal ideal Q (to play the role of $R/J(0)$ and a homomorphism ω of $X = \hat{G}$ into the multiplicative group of elements of K which lies in the coset of the unit element in K/Q , Silov constructs a new Banach algebra $K_\omega(G)$ which has maximal ideal space G , no radical and is homogeneous in the wide sense mentioned above. $K_\omega(G)$ is called the *continuous sum of the primary ring* K *over the group* G . While it need not always be regular, when it is it is of type C . Further, a homogeneous commutative regular unital Banach algebra, without radical, is of type C only if it can be constructed in this way.

2. A NEW PROBLEM

Suppose that the set S considered in Example 1 is a compact abelian group. Then, since each of $C(S)$ and $C^{(n)}(\mathbf{T})$ is homogeneous, each must be a continuous sum.

However, each space has another multiplicative structure, namely that provided by convolution. With respect to this operation, neither algebra has an identity, although the maximal ideal spaces are easily identified as the discrete spaces $\hat{S}$ and $\mathbf{Z}$ respectively. Since neither of these is compact, Silov's clasification cannot be applied. We can, however, still consider the question of whether or not each algebra is of type C . (As we shall see in section 3, the answer on both cases is no!)

We can, in fact, discuss the problem in a more general context. Let G denote a compact abelian group with dual group X . Then $PM(G)$ and $PF(G)$ denote the usual spaces of pseudomeasures and pseudofunctions defined on G (see [5]). If $(B,\| \|_B)$ is a Banach space of pseudo-measures on G , then for each $b \in B$ and $\chi \in X$ define

$$\|b\|_\chi = \inf\{\|c\|_B : c \in B,\ \hat{c}(\chi) = \hat{b}(\chi)\} . \tag{3}$$

Then $\| \|_\chi \le \| \|_B$. We say that B is of *convolution type* C if its norm is equivalent to that defined by

$$\|b\|^c = \sup_{\chi \in X} \|b\|_\chi .$$

Trivially, both PM(G) and PF(G) are of convolution type C .

If B is a convolution algebra with maximal ideal space X , as for example when B is one of the usual algebras C(G) or $L^p(G)$ $1 \le p < \infty$, then this essentially coincides with Silov's notion of type C .

In the next section we give a classification of those Banach spaces which are of type C , under the one additional assumption that the Fourier transformation on B is continuous, that is, there exists a positive constant K with $K\|b\|_{PM} \le \|b\|_B$ for all $b \in B$.

3. A CLASSIFICATION OF CONVOLUTION TYPE C BANACH SPACES

Let $F = \{\chi \in X : \hat{b}(\chi) \neq 0$ for some $b \in B\}$; F is usually called the <u>spectrum</u> of B .

Now, for $\chi \in F$ and $b \in B$, we have

$$\begin{aligned} \|b\|_\chi &= \inf\{\|c\|_B : c \in B,\ \hat{c}(\chi) = \hat{b}(\chi)\} \\ &= \inf\{\|\hat{b}(\chi)\chi + c\|_B : c \in B,\ \hat{c}(\chi) = 0\} \\ &= |\hat{b}(\chi)| \inf\{\|\chi + c\|_B : c \in B,\ \hat{c}(\chi) = 0\} . \end{aligned}$$

Define, for each $\chi \in F$, $\omega_B(\chi)$ by

$$\omega_B(\chi) = \inf\{\|\chi + c\|_B : c \in B,\ \hat{c}(\chi) = 0\} .$$

We know that $\|\chi + c\|_B \ge K\|\chi + c\|_{PM} \ge K|(\chi + c)^\wedge(\chi)| = K$ if $\hat{c}(\chi) = 0$, and so ω_B is bounded below by K on F . Hence we have the following proposition.

PROPOSITION 1 *There exists a function* $\omega_B : F \to \mathbb{R}^+$ *with* $\inf_{\chi \in F} \omega_B(\chi) \ge K$ *and* $\|b\|_c = \sup_{\chi \in F} \omega_B(\chi)|\hat{b}(\chi)|$ *for each* $b \in B$.

It follows from Proposition 1 that

$$B \subseteq \{S \in PM_F(G) : \omega_B\hat{S}|_F \in \ell^\infty(F)\}$$

where $PM_F(G)$ is the set of pseudomeasures whose transforms are supported by F and $\hat{S}|_F$ is the restriction of $\hat{S}$ to F. Since $\hat{S}$ is zero off F it is convenient to identify the restriction $\hat{S}|_F$ with $\hat{S}$, and to write $\omega_B\hat{S}|_F$ as $\omega_B\hat{S}$, the pointwise product of ω_B and $\hat{S}$ on F. We use this convention throughout the paper. The inclusion will usually be proper - for example, if $B = PF(G)$, then $\omega_B(\chi) = 1$ for all $\chi \in X = F$; hence, the set on the right is $PM(G)$, which properly contains $PF(G)$ if G is infinite.

On the other hand, we can start with a positive function ω, defined on some subset F of X, satisfying $\inf_{\chi\in F} \omega(\chi) \geq K$ for some positive constant K. Then we can construct the normed space P_ω where

$$P_\omega = \{S \in PM_F(G) : \omega\hat{S} \in \ell^\infty(F)\},$$

and

$$\|S\|_{P_\omega} \stackrel{\text{df}}{=} \|S\| = \|\omega\hat{S}\|_{\ell^\infty(F)} \quad \text{for each } S \in P_\omega .$$

Clearly P_ω is translation invariant, with each translation operator an isometry. (It is certainly not the case that each convolution type C Banach space is translation invariant - for instance, take the closed subspace of $PM(G)$ spanned by $\chi + \eta$ where χ and η are distinct elements of X.) We prove in Lemma 1 that P_ω is complete with respect to $\|\ \|_\omega$, and so is a Banach space, and in Proposition 2 that it is of convolution type C.

LEMMA 1 $(P_\omega, \|\ \|_\omega)$ *is complete.*

Proof For each $\chi \in F$ and $S \in P_\omega$, $\|S\|_\omega \geq \omega(\chi)|\hat{S}(\chi)|$ and

$$|\hat{S}(\chi)| \leq \|S\|_\omega (\omega(\chi))^{-1} \leq \|S\|_\omega (\inf_{\chi\in F} \omega(\chi))^{-1} \leq \frac{1}{K}\|S\|_\omega .$$

Hence, $K\|S\|_{PM} \leq \|S\|_\omega$ for each S.

The lemma now follows from a standard argument. Let (S_n) be a Cauchy sequence in P_ω. Then it is also Cauchy in $PM(G)$ and so converges to some pseudomeasure S. Clearly $\hat{S}$ must be supported by F as each $\hat{S}_n$ is supported by F. Now there exists a subsequence (S_{n_j}) with

$$\|S_{n_j} - S_{n_{j-1}}\|_\omega < 2^{-j}$$

for $j \geq 2$. Since

$$S_{n_N} = S_{n_1} + \sum_{j=2}^{N} (S_{n_j} - S_{n_{j-1}}) ,$$

we have

$$\begin{aligned} \|S_{n_N}\|_\omega &\leq \|S_{n_1}\|_\omega + \sum_{j=2}^{N} \|S_{n_j} - S_{n_{j-1}}\|_\omega \\ &\leq \|S_{n_j}\|_\omega + \sum_{j=2}^{N} 2^{-j} , \end{aligned}$$

which ensures that $S \in P_\omega$.

PROPOSITION 2 P_ω *is of convolution type* C.

Proof As a consequence of lemma 1, it is sufficient to verify that P_ω satisfies the required norm condition. Without loss of generality take $\chi \in F$, since $\|S\|_\chi = 0$ for each $S \in P_\omega$ and each $\chi \notin F$. Then for each $S \in P_\omega$,

$$\begin{aligned} \|S\|_\chi &= \inf\{\|T\|_\omega : T \in P_\omega , \hat{T}(\chi) = \hat{S}(\chi)\} \\ &= \inf\{\|\omega\hat{T}\|_{\ell^\infty(F)} : T \in P_\omega, \hat{T}(\chi) = \hat{S}(\chi)\} \\ &= \inf\{\|\omega(\hat{S}(\chi)\hat{\chi} + \hat{T})\|_{\ell^\infty(F)} : T \in P_\omega , \hat{T}(\chi) = 0\} \\ &= |\hat{S}(\chi)| \inf\{\|\omega(\hat{\chi} + \hat{T})\|_{\ell^\infty(F)} : T \in P_\omega , \hat{T}(\chi) = 0\} . \end{aligned}$$

However, if $T \in P_\omega$ with $\hat{T}(\chi) = 0$ then

$$\|\omega(\hat{\chi} + \hat{T})\|_{\ell^\infty(F)} \geq \omega(\chi)|\hat{\chi}(\chi) + \hat{T}(\chi)| = \omega(\chi) ,$$

and so $\|S\|_\chi \geq |\hat{S}(\chi)|\omega(\chi)$ and $\|S\|^c \geq \|S\|_\omega$. Since the reverse inequality is obviously true, the norms $\| \|^c$ and $\| \|_\omega$ are equal and P_ω is of type C .

We have already noted that a convolution type C Banach space need not be translation invariant; further, even if it is translation invariant, it need not be homogeneous in the usual sense - for example, PM(G) is translation invariant and of convolution type C but is not homogeneous. The next proposition gives a necessary condition for homogeneity of P_ω .

PROPOSITION 3 *If* P_ω *is homogeneous then* $\omega\hat{S} \in c_0(F)$ *for each* $S \in P_\omega$.

Proof The trigonometric polynomials contained in a homogeneous Banach space form a dense subspace. (See Theorem 2.17 of [4].) Hence $P_\omega \cap T(G)$ is dense in P_ω . Let

$$H = \{S \in P_\omega : \omega\hat{S} \in c_0(F)\} .$$

Then H is a closed subspace of P_ω which contains $P_\omega \cap T(G)$, and so $H = P_\omega$.

As ω is bounded away from 0 , it follows from Proposition 3 that if P_ω is homogeneous then it is a subspace of PF(G) . This suggests that we consider the subspace P^ω of P_ω which contains those elements S for which $\omega\hat{S} \in c_0(F)$.

PROPOSITION 4 $(P^\omega, \| \|_\omega)$ *is a homogeneous Banach space.*

Proof It is easy to see that P^ω is a translation invariant Banach subspace of P_ω . Hence, it is sufficient to prove that for each $S \in P^\omega$, the shift $x \to {}_xS$ is continuous from G to P^ω . In fact, since G is compact, it is sufficient to prove continuity at the identity. Now,

$$\|{}_x S - S\|_\omega = \sup_{\chi \in F} \omega(\chi) |{}_x\hat{S}(\chi) - \hat{S}(\chi)|$$

$$= \sup_{\chi \in F} \omega(\chi) |\chi(x) - 1| \; |\hat{S}(\chi)| \; . \tag{4}$$

For each $\epsilon > 0$, there exists a finite subset E of F with

$$\sup_{\chi \in F \setminus E} \omega(\chi) |\hat{S}(\chi)| < \frac{\epsilon}{2} \, . \tag{5}$$

Also, for each $\chi \in E$, there is a zero neighbourhood U_χ satisfying

$$x \in U_\chi \Rightarrow |\chi(x) - 1| < \frac{\epsilon}{\|S\|_\omega \, \mathrm{Max}\{\omega(\chi) : \chi \in E\}} \, . \tag{6}$$

Put $U = \cap \{U_\chi : \chi \in E\}$. Then combining 4,5 and 6 we see that for $x \in U$,

$$\|{}_x S - S\|_\omega \leq \max\{\max_{\chi \in E} \omega(\chi) |\chi(x) - 1| \; |\hat{S}(\chi)| \; ,$$

$$\sup_{\chi \in F \setminus E} \omega(\chi) |\chi(x) - 1| \; |\hat{S}(\chi)|\}$$

$$< \epsilon \, .$$

Hence, the shift is continuous at the identity of G .

All closed subspaces of convolution type C spaces are also of convolution type C . Hence P^ω is also of convolution type C . The classification theorem can now be proved.

THEOREM 1 *Let* B *be a homogeneous Banach space and* $F = \{\chi \in X : \hat{b}(\chi) \neq 0$ for some $b \in B\}$. *Let* $\omega = \omega_B$, *the positive function of Proposition* 1. *Then* B *is of type* C *if and only if* $B = P^\omega$.

Proof (<=) This is a consequence of Proposition 4 and the remark following it,

(=>) We have already observed, following Proposition 1, that $B \subseteq P_\omega$. An argument similar to that used in the proof of Proposition 3 proves that $\omega\hat{S} \in c_0(F)$ for each $S \in B$. That $B = P^\omega$ is a consequence of the facts that $P^\omega \cap T(G) = T_F(G)$ (see [6]), that $(T_F(G))^\wedge$ is dense in $c_0(F)$ and that ω is bounded away from 0. The equivalence of $\| \ \|_B$ and $\| \ \|_\omega$ on B is a consequence of the Closed Graph Theorem.

We have already noted that $PF(G)$ and $PM(G)$ are of type C; however, of these only $PF(G)$ is homogeneous. On the other hand, taking G equal to the circle group $\mathbb{T}$, neither $C(\mathbb{T})$ nor $C^{(n)}(\mathbb{T})$ for $n \geq 1$ is of type C. To see this, we recall that each of these is a homogeneous space and so can only be of type C if of the form P^ω for some ω defined on $\mathbb{Z}$. However, it is well-known that it is impossible to classify functions in $C(\mathbb{T})$, and so also those in $C^{(n)}(\mathbb{T})$, by looking at the rate of decay of their Fourier transforms. The same will also apply to other familiar homogeneous spaces such as $A(G)$, of functions with absolutely summable Fourier transforms, and $L^p(G)$ for $1 \leq p < \infty$. An alternative straightforward proof that none of $A(G)$, $C(G)$ or $L^p(G)$ $1 \leq p < \infty$, nor in fact any of their spectral subspaces $A(G)_F$, $C(G)_F$, etc., for F infinite, is of convolution type C follows from the next corollary.

Having decided that none of the *usual* Banach spaces is of type C, it is natural to ask are there many Banach spaces of type C? We consider this question in the next section. For the moment, note that, taking $G = \mathbb{T}$, $\hat{f} = o(n^{-m})$ for all $f \in C^{(m)}(\mathbb{T})$ and so if $\omega(n) = O(n^m)$ then $C^{(m)}(\mathbb{T}) \subseteq P^\omega$. On the other hand, as we shall see in section 4, P^ω is not the whole of $PF(G)$; in fact, for $n \geq 2$ $P^\omega \subseteq L^2(\mathbb{T})$.

4. BEHAVIOUR OF FOURIER TRANSFORMS

The closed subspaces $PF_F(G)$ of $PM(G)$ are the simplest examples of homogeneous Banach spaces of type C. It is easy to see that if $\omega \in \ell^\infty(F)$ then, in fact, $P^\omega = PF_F(G)$. The converse is also true.

PROPOSITION 5 $P^\omega = PF_F$ *if and only if* $w \in \ell^\infty(F)$.

Proof We only prove the necessary statement. Suppose that $P^{\omega} = PF_F(G)$ and define the linear operator $T : P^{\omega} \to c_0(F)$ by $T(S) = \omega\hat{S}$. Then T is a surjection since $\omega^{-1}\gamma \in c_0(F)$ for each $\gamma \in c_0(F)$. (Otherwise, given any finite subset E of F and any $\epsilon > 0$ there exists $\eta \in F\backslash E$ for which $[\omega(\eta)]^{-1}\gamma(\eta) > \epsilon$. Then $\gamma(\eta) > \epsilon \inf_{\chi \in F} \omega(\chi)$, which contradicts the choice of γ from $c_0(F)$.)

It is a consequence of the Closed Graph Theorem that T is continuous. To see this, let (S_n) be a sequence which converges to 0 in P^{ω}, and assume that $\omega\hat{S}_n$ converges to ψ in $c_0(F)$. If $\psi \neq 0$ then there exists $\eta \in F$ with $\psi(\eta) = \epsilon > 0$. There also exists $n \in \mathbb{N}$ such that for $n > N$,

$$\frac{\epsilon}{2} < \omega(\eta)|\hat{S}_n(\eta)| < \frac{3\epsilon}{2} .$$

Then for any $n > N$,

$$\|\hat{S}_n\|_{\ell^{\infty}(F)} \geq |\hat{S}_n(\eta)| > \frac{\epsilon}{2\omega(\eta)}$$

which means that the sequence (S_n) cannot converge to 0 in $PF_F(G)$, a contradiction.

It now follows from the continuity of T that there exists a positive constant, C say, with

$$\|\omega\hat{S}\|_{\ell^{\infty}(F)} \leq C\|S\|_{PF(G)}$$

for all $S \in P^{\omega} = PF_F(G)$. In particular, taking $S = \chi$, we see that $\omega \in \ell^{\infty}(F)$.

COROLLARY *Let* B *be a homogeneous Banach space of convolution type* C, *and put*

$$F = \{\chi \in X : \hat{b}(\chi) \neq 0 \text{ for some } b \in B\}$$

(which necessarily can be identified with $X \cap B$*).*

If $\sup_{\chi \in F}\|\chi\|_B < \infty$ *then* $B = PF_F(G)$.

Proof By Theorem 1, $B = P^{\omega}$ for some ω. Hence, by the Closed Graph Theorem there is a positive constant C such that for all $b \in B$

$$\|b\|_{\omega} \le C\|b\|_{B} ,$$

that is,

$$\sup_{\chi \in F} \omega(\chi)\,|\hat{b}(\chi)| \le C\|b\|_{B} .$$

In particular, putting $b = \eta$ for $\eta \in F$, we obtain

$\omega(\eta) \le C\|\eta\|_{B} \le C \sup_{\chi \in F}\|\chi\|_{B} < \infty$. Thus $\omega \in \ell^{\infty}(F)$ and $B = PF_{F}(G)$.

By putting various growth conditions on ω we can deduce facts about the rate of decay of the Fourier transform of elements of P^{ω}. The most obvious of these is that if $\omega^{-1} \in \ell^{r}(F)$ then $\hat{S} \in \ell^{r}(F)$ for each $S \in P^{\omega}$. In particular, if $\omega^{-1} \in \ell^{1}(F)$ then $P^{\omega} \subseteq A_{F}(G)$, while if $\omega^{-1} \in \ell^{2}(F)$ then $P^{\omega} \subseteq L^{2}_{F}(G)$. In neither case can inclusion be replaced by equality.

Theorem 2 (i) *If* $P^{\omega} \subseteq L^{\infty}_{F}(G)$ *then* $P^{\omega} \subseteq A_{F}(G)$.

(ii) *If* $L^{\infty}_{F}(G) \subseteq P^{\omega}$ *and* $\hat{S}(\chi) = 0$ *for all* $\chi \notin F$ *then* $P^{\omega} = PF_{F}(G)$.

Proof (i) For each $\psi \in \ell^{\infty}(F)$ and $S \in P^{\omega}$, we have $\psi\omega\hat{S} \in c_{0}(F)$, and so the pseudomeasure T, with $\hat{T} = \psi\hat{S}$ on F and $\hat{T} = 0$ otherwise, is in P^{ω}. For a given $S \in P^{\omega}$, take

$$\psi(\chi) = \operatorname{sgn}(\hat{S}(\chi)) ,$$

which is 1 if $\hat{S}(\chi) > 0$, 0 if $\hat{S}(\chi) = 0$ and -1 if $\hat{S}(\chi) < 0$. Then

$$\hat{T}(\chi) = [\operatorname{sgn}(\hat{S}(\chi))]\hat{S}(\chi) = |\hat{S}(\chi)| ,$$

and so T is positive definite. Thus, as $T \in L^{\infty}(G)$, it is a consequence of Bochner's Theorem that

$$\sum_{\chi \in X} \hat{T}(\chi) = \sum_{\chi \in X} \hat{T}(\chi) = \sum_{\chi \in X} |\hat{S}(\chi)| < \infty ,$$

and $S \in A_{F}(G)$.

(ii) It follows from the Closed Graph Theorem that, since $L^\infty_F(G) \subseteq P^\omega$, there exists a positive constant C with $\|\omega\hat{S}\|_{\ell^\infty(F)} \le C\|S\|_\infty$ for all $S \in L^\infty_F(G)$. In particular, taking $S = \chi$, $\omega(\chi) \le C\|\chi\|_\infty = C$ for all $\chi \in F$. Thus $\omega \in \ell^\infty(F)$, and the conclusion follows from Proposition 5.

Note that (ii) remains true if $L^\infty_F(G)$ is replaced by any Banach space B which has the following two properties:

(α) if $F = \{\chi \in X : \hat{b}(\chi) \neq 0$ for some $b \in B\}$, then $\chi \in F \Rightarrow \chi \in B$, and

(β) $\sup\{\|\chi\|_B : \chi \in B \cap F\} < \infty$.

So, in particular, we can replace $L^\infty_F(G)$ by any one of $A_F(G)$, $C_F(G)$, or $L^p(G)$ where $1 \le p < \infty$.

On the other hand, as the next example illustrates there do exist homogeneous Banach spaces of convolution type C which are neither contained in some $A_F(G)$ nor equal to $PF_F(G)$.

EXAMPLE Let $\omega^{-1} \in \ell^r(F) \setminus \ell^1(F)$ for some $r > 1$, and let $\xi \in (0,r-1)$. Define the pseudomeasure S_ξ by

$$\hat{S}_\xi(\chi) = \begin{cases} \omega(\chi)^{-(1+\xi)} & \text{for } \chi \in F \\ 0 & \text{otherwise} \end{cases} .$$

Then $\omega\hat{S}_\xi \in c_0(F)$ so that $S_\xi \in P^\omega$. Further, $\hat{S}_\xi$ belongs to $\ell^{\frac{r}{1+\xi}}(F) \setminus \ell^1(F)$. Hence $S_\xi \notin A_F(G)$. If, however, we assume $r \le 2$ say, then $P^\omega \subseteq L^2_F(G)$.

REFERENCES

[1] Hewitt, E. and Ross, K.A., *Abstract Harmonic Analysis*, 2 vols. Springer Verlag, Berlin, 1963 and 1970.

[2] Larsen, R., *Banach algebras, an introduction*, Dekker, New York, 1973.

[3] Šilov. G.E., *Homogeneous rings of functions,* Uspehi Matematiceskih Nauk (N.S.) **6**, No.1(41), (1951), 91-137. English translation Amer. Math. Soc. Translations, ser 1, **8** (1954), 393-455.

[4] Wang, H-C, *Homogeneous Banach Algebras*, Dekker, New York, 1977.

[5] Ward, J.A., *Closed ideals of homogeneous algebras*, Monat. Math., **96** (1983), 317-324.

[6] Ward, J.A., *Characterisation of homogeneous spaces and their norms*, Pacific J. Math., **114** (2), (1984), 481-495.

School of Mathematical and Physical Sciences
Murdoch University
Perth, Western Australia
AUSTRALIA